Aequationes Mathematicae **48** (1994) 143–162
University of Waterloo

0001-9054/94/030143-20 \$1.50 + 0.20/0
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Rigidity of generalized laplacians and some geometric applications

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Summary. Every generalized laplacian L defined on a manifold M determines a sheaf of "L-harmonic" sections namely the sheaf of local solutions of $Lu = 0$. We study the converse problem: to what extent this sheaf determines the operator. Our main result states that the sheaf of L-harmonic sections determines the operator up to a conformal factor. Moreover, when the operator is a covariant laplacian and the dimension of M is greater than 2, the sheaf determines L up to a multiplicative constant. An interesting consequence is the following: if two Riemann metrics on a smooth manifold of dimension greater than 2 have the same sheaves of harmonic functions then they are homothetic.

0. Introduction

Let (M, g) be an N-dimensional Riemann manifold and consider the associated Laplace–Beltrami operator

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} g^{ij} \partial_j), \qquad \text{with } \partial_i = \partial/\partial x_i. \tag{1}$$

Here $(g^{ij}(x))$ is the inverse matrix of $(g_{ij}(x))$ while $|g| = \det(g_{ij}(x))$. We now call a function g-*harmonic* if it satisfies the equation

$$\Delta_g u = 0. \tag{2}$$

The g harmonic functions form a sheaf $\mathscr{H}_g$ on M. This paper deals with the following

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AMS (1991) subject classification: Primary 35J25, secondary 58G30.

Manuscript received February 14, 1992 *and, in final form, August* 3, 1993.

The generalized laplacians satisfy the unique continuation property ([Ar], [J]). This shows that the above sheaves have a built-in "rigidity". Namely, if U is a connected open subset of M and V is open $V \subset U$ then the restriction map

$$r^U_V : \mathcal{H}_L(U) \to \mathcal{H}_L(V)$$

is injective (roughly speaking they are far from being flabby).

Definition 1.2. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two sheaves over a topological space X we say that $\mathcal{S}_2$ contains $\mathcal{S}_1$ (and write $\mathcal{S}_1 \subset \mathcal{S}_2$) if for any open subset U of X we have

$$\mathcal{S}_1(U) \subset \mathcal{S}_2(U).$$

We can now state the main results of this paper.

Theorem 1.1. *Let $L_1, L_2 \in L(\mathcal{E})$. The following are equivalent:*
(i) $[L_1]_c = [L_2]_c$
(ii) $\mathcal{H}_{L_1} \subset \mathcal{H}_{L_2}$.

We call the above property of generalized laplacians *rigidity*.

Theorem 1.2. *Assume $N \geq 3$. Let $L_1, L_2 \in \mathcal{L}_{geom}(\mathcal{E})$. The following are equivalent:*
(i) $[L_1]_h = [L_2]_h$
(ii) $\mathcal{H}_{L_1} \subset \mathcal{H}_{L_2}$.

We call the above property of geometric laplacians *strong rigidity*. As a consequence of Theorem 1.2 we have

Corollary 1.1. *If two Riemann metrics on a connected smooth manifold of dimension greater than 2 have the same sheaves of harmonic functions then there exists a positive constant μ such that*

$$g = \mu h.$$

Proof. By Theorem 1.2 we have

$$\Delta_g = \lambda^2 \Delta_h \qquad \text{for some } \lambda > 0.$$

Since the principal part of Δ_g is g^{ij} and the principal part of Δ_h is h^{ij} we get the desired result. □

REMARK 1.1. The above result is not true in dimension 2. Indeed consider the euclidian laplacian in $\mathbf{R}^2$, $\Delta = \partial_x^2 + \partial_y^2$. Let $g = (f^2(x, y)\delta_{ij})$ be a conformal metric. Using formula (1) in the introduction, we see that $\Delta_g = -(1/f)\Delta$. Thus $\mathscr{H}_\Delta = \mathscr{H}_{\Delta_g}$. However, clearly the two operators are not homothetically equivalent.

2. Some technical results

We have gathered in this section the main estimates needed in the proof of Theorem 1.1. The preferred functional framework will be that of Sobolev spaces $L^{k,p}$ (functions k-times "differentiable" with derivatives in L^p; see [GT], [M]). Theorem 1.1, 2 have a local nature so we loose no generality if we assume that $M = \mathbf{R}^N$ and $\mathscr{E} \cong M \times \mathbf{R}^p$. We will denote by B_r the open ball of radius r of $\mathbf{R}^N$ (in the euclidian metric) centered at the origin. All the Sobolev or sup norms we will use are defined in the euclidian context. Δ will denote the euclidian laplacian and $\lambda_1(r)$ will denote the first eigenvalue of Δ on B_r with homogeneous Dirichlet conditions. $\lambda_1(r) = \lambda_1(1)/r^2$. Set

$$Du = \sum \partial_i u \, dx^i \quad \text{and} \quad D^2u = \sum \partial_{ij}^2 u \, dx^i \otimes dx^j$$

$L_0^{1,2}(B_r)$ is the completion of $C_0^\infty(B_r, \mathbf{R}^p)$ in the $L^{1,2}$ norm. The norm in $L_0^{1,2}(B_r)$ is

$$\|u\|_{L_0^{1,2}(B_r)}^2 = \int_{B_r} |Du|^2.$$

We will frequently use the Poincaré inequality in the form (see [M])

$$\|u\|_{L^2(B_r)}^2 \le r^2/\lambda_1 \|Du\|_{L^2(B_r)}^2, \qquad \forall u \in L_0^{1,2}(B_r, \mathbf{R}^p). \tag{1}$$

Instead of the usual $C^{2,\alpha}(B_r)$-norm $(0 < \alpha < 1)$ we will use the conformal invariant one

$$|||u|||_r = r^{2+\alpha}\|D^2u\|_{C^\alpha(B_r)} + r^2\|D^2u\|_{C^0(B_r)} + r\|Du\|_{C^0(B_r)} + \|u\|_{C^0(B_r)}.$$

All differential operators we will deal with have smooth coefficients. To any metric $g = (g_{ij})$ on $\mathbf{R}^N$ one can associate a second order operator

$$G: C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$$

by

$$G = \sum g^{ij} \partial_{ij}^2.$$

The standard estimates and existence results for scalar second order elliptic operators continue to hold for vectorial operators G as above. In particular we will use the following

PROPOSITION 2.1. *For any* $f \in L^2(\overline{B_r}, \mathscr{E})$ *the problem*

$$\begin{cases} Gu = f & \text{in } B_r \\ u = 0 & \text{on } \partial B_r \end{cases} \tag{2}$$

has a unique weak solution $u = T_r(f) \in L^{2,2} \cap L_0^{1,2}(B_r)$ *which satisfies*

$$\|D(T_r f)\|_{L^2(B_r)}^2 \leq 1/\mu_1(r) \|f\|_{L^2(B_r)}^2 \tag{3}$$

$$\|T_r f\|_{L^2(B_r)} \leq Cr^2 \|f\|_{L^2(B_r)} \tag{4}$$

where $0 < \mu_1(r) \leq Cr^{-2}$ *(for r small) is the first eigenvalue of (G + Dirichlet conditions) on* B_r. *If moreover* $f \in C^\alpha(\overline{B_r}, \mathscr{E})$ *then*

$$|\!|\!| T_r f |\!|\!|_r \leq Cr^2 \|f\|_{C^\alpha(\overline{B_r})} \tag{5}$$

Here as throughout the paper C will denote various constants independent of r and f.

For proof of this result we refer to [GT]. Using standard perturbation techniques (as in [GT] or [M]) we can extend the above result to more general classes of elliptic systems (with scalar principal symbol)

LEMMA 2.1 (Key Estimates). *Let* $L \in \mathscr{L}(\mathscr{E})$, $L = G + F + H$ *(see* (1.1)). *Then there exists* $\varrho = \varrho(L) > 0$ *such that* $\forall r < \varrho$ *and* $\forall f \in C^\alpha(\overline{B_r})$ $(0 < \alpha < 1)$ *the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } B_r \\ u = 0 & \text{on } \partial B_r \end{cases} \tag{6}$$