COMPLEXITY OF RANDOM SMOOTH FUNCTIONS ON COMPACT MANIFOLDS.

LIVIU I. NICOLAESCU

ABSTRACT. We prove a universality result relating the expected distribution of critical values of a random linear combination of eigenfunctions of the Laplacian on an arbitrary compact Riemann m-dimensional manifold to the expected distribution of eigenvalues of a $(m+1) \times (m+1)$ random symmetric Wigner matrix. We then prove a central limit theorem describing what happens to the expected distribution of critical values when the dimension of the manifold is very large.

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1. OVERVIEW

The goal of this paper is to describe a universal relationship between the distribution of critical values of certain random functions on an arbitrary compact m-dimensional Riemann manifold and the distribution of eigenvalues of certain random symmetric $(m+1) \times (m+1)$-matrices. A special case of this problem concerns the distribution of critical values of the restriction to the unit sphere $S^N \subset \mathbb{R}^{N+1}$ of a random polynomial of very large degree in $(N+1)$-variables.

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1.1. **The setup.** Suppose that (M,g) is a smooth, compact, connected Riemann manifold of dimension $m>1$. We denote by $|dV_g|$ the volume density on M induced by g. We assume that the metric is normalized so that

$$\operatorname{vol}_g(M)=1. \tag{$*$}$$

For any $\boldsymbol{u},\boldsymbol{v}\in C^\infty(M)$ we denote by $(\boldsymbol{u},\boldsymbol{v})_g$ their L^2 inner product defined by the metric g. The L^2-norm of a smooth function $\boldsymbol{u}$ is denoted by $\|\boldsymbol{u}\|$.

Let $\Delta_g: C^\infty(M)\to C^\infty(M)$ denote the scalar Laplacian defined by the metric g. For $L>0$ we set

$$\boldsymbol{U}^L=\boldsymbol{U}^L(M,g):=\bigoplus_{\lambda\in[0,L^2]}\ker(\lambda-\Delta_g),\;\; d(L):=\dim\boldsymbol{U}_L.$$

We equip $\boldsymbol{U}^L$ with the Gaussian probability measure.

$$d\boldsymbol{\gamma}^L(\boldsymbol{u}):=(2\pi)^{-\frac{d(L)}{2}}e^{-\frac{\|\boldsymbol{u}\|^2}{2}}|d\boldsymbol{u}|.$$

Fix an orthonormal Hilbert basis $(\Psi_k)_{k\geq 0}$ of $L^2(M)$ consisting of eigenfunctions of Δ_g,

$$\Delta_g\Psi_k=\lambda_k\Psi_k,\;\; k_0\leq k_1\Rightarrow\lambda_{k_0}\leq\lambda_{k_1}.$$

Then

$$\boldsymbol{U}^L=\operatorname{span}\big\{\,\Psi_k;\;\;\lambda_k\leq L^2\,\big\}.$$

A random (with respect to $d\boldsymbol{\gamma}^L$) function $\boldsymbol{u}\in\boldsymbol{U}^L$ can be viewed as a linear combination

$$\boldsymbol{u}=\sum_{\lambda_k\leq L^2}u_k\Psi_k,$$

where u_k are i.i.d. Gaussian random variables with mean 0 and variance $\sigma^2=1$. We have the following technical result whose proof is contained in Appendix A.

Proposition 1.1. *There exists $L_0>0$ such that for any $L\geq L_0$, a function $\boldsymbol{u}\in\boldsymbol{U}^L$ is almost surely (a.s.) Morse.* □

Remark 1.2. For any $f\in C^\infty(M)$ and any open neighborhood $\mathcal{O}$ of f in $C^\infty(M)$ we can find $L_0\geq 0$ such that for any $L\geq L_0$ we have $\boldsymbol{U}^L\cap\mathcal{O}\neq\emptyset$. Suppose that f is *stable*, i.e., f is Morse and the critical level sets of f contain a single critical point. If $\mathcal{O}$ is sufficiently small, then any $f'\in\mathcal{O}$ is topologically equivalent to f, [17, Prop. III.2.2]. This means that there exists a diffeomorphism Φ of M and a diffeomorphism φ of R such that $f'=\varphi\circ f\circ\Psi^{-1}$. Thus, as $L\to\infty$ the space $\boldsymbol{U}^L$ engulfs all the possible topological types of stable Morse functions. □

For any $\boldsymbol{u}\in C^1(M)$ we denote by $\mathbf{Cr}(\boldsymbol{u})\subset M$ the set of critical points of $\boldsymbol{u}$ and by $\boldsymbol{D}(\boldsymbol{u})$ the set of critical values[1] of $\boldsymbol{u}$. If L is sufficiently large the random set $\boldsymbol{U}^L\ni\boldsymbol{u}\mapsto\mathbf{Cr}(\boldsymbol{u})$ is a.s. finite.

To a Morse function $\boldsymbol{u}$ on M we associate a Borel measure $\mu_{\boldsymbol{u}}$ on M and a Borel measure $\boldsymbol{\sigma}_{\boldsymbol{u}}$ on $\mathbb{R}$ defined by the equalities

$$\mu_{\boldsymbol{u}}:=\sum_{\boldsymbol{p}\in\mathbf{Cr}(\boldsymbol{u})}\delta_{\boldsymbol{p}},\;\;\boldsymbol{\sigma}_{\boldsymbol{u}}:=\boldsymbol{u}_*(\mu_{\boldsymbol{u}})=\sum_{d\boldsymbol{u}(\boldsymbol{p})=0}\delta_{\boldsymbol{u}(\boldsymbol{p})}.$$

Following the terminology on [3, 4] we will refer to $\boldsymbol{\sigma}_{\boldsymbol{u}}$ as the *variational complexity* of $\boldsymbol{u}$. Observe that

$$\operatorname{supp}\mu_{\boldsymbol{u}}=\mathbf{Cr}(\boldsymbol{u}),\;\;\operatorname{supp}\boldsymbol{\sigma}_{\boldsymbol{u}}=\boldsymbol{D}(\boldsymbol{u}).$$

[1] The set $\boldsymbol{D}(\boldsymbol{u})$ is sometime referred to as the *discriminant set* of $\boldsymbol{u}$.

When $\boldsymbol{u} \in \boldsymbol{U}^L$ is not a Morse function we define $\mu_{\boldsymbol{u}}$ and $\boldsymbol{\sigma}_{\boldsymbol{u}}$ arbitrarily. We set

$$s_m := \frac{(4\pi)^{-\frac{m}{2}}}{\Gamma(1+\frac{m}{2})}, \quad d_m := \frac{(4\pi)^{-\frac{m}{2}}}{2\Gamma(2+\frac{m}{2})}, \quad h_m := \frac{(4\pi)^{-\frac{m}{2}}}{4\Gamma(3+\frac{m}{2})}. \tag{1.1}$$

The statistical significance of these numbers is described is Subsection 2.2. We only want to mention here that the Hörmander-Weyl spectral estimates state that

$$\dim \boldsymbol{U}^L = s_m L^m + O(L^{m-1}) \ \text{ as } L \to \infty. \tag{1.2}$$

For $L \gg 0$, the correspondence $\boldsymbol{U}^L \ni \boldsymbol{u} \mapsto \mu_{\boldsymbol{u}}$ is a random measure on M called the *empirical distribution of critical points* of the random function. Its expectation is the measure μ^L on M defined by

$$\int_M f d\mu^L = \int_{\boldsymbol{U}^L} \left(\int_M f d\mu_{\boldsymbol{u}} \right) d\boldsymbol{\gamma}^L(\boldsymbol{u}),$$

for any continuous function $f : M \to \mathbb{R}$. Note that the number

$$\boldsymbol{N}^L := \int_M d\mu^L = \int_{\boldsymbol{U}^L} |\,\mathbf{Cr}(\boldsymbol{u})| d\boldsymbol{\gamma}^L(\boldsymbol{u})$$

is the expected number of critical points of a random function in $\boldsymbol{U}^L$.

In [23] we showed that there exists a universal (explicit) constant C_m that depends only on the dimension m such that

$$\boldsymbol{N}^L \sim C_m \dim \boldsymbol{U}^L \sim C_m s_m^{\boldsymbol{\omega}}(L)^m \ \text{ as } L \to \infty, \tag{1.3}$$

and the normalized measures

$$d\bar{\mu}^L := \frac{1}{\boldsymbol{N}^L} d\mu^L$$

converges weakly to the metric volume measure $|dV_g|$ as $L \to \infty$. This means that for L very large we expect the critical set of a random $\boldsymbol{u} \in \boldsymbol{U}^L$ to be close to uniformly distributed on M.

Similarly, the random measure $\boldsymbol{U}^L \ni \boldsymbol{u} \mapsto \boldsymbol{\sigma}_{\boldsymbol{u}}$ has an expectation $\boldsymbol{\sigma}^L := \boldsymbol{E}_{\boldsymbol{U}^L}(\boldsymbol{\sigma}_{\boldsymbol{u}})$ which is a finite measure on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}} f(\lambda) d\boldsymbol{\sigma}^L(\lambda) = \int_{\boldsymbol{U}^L} \left(\int_{\mathbb{R}} f(\lambda) d\boldsymbol{\sigma}^L_{\boldsymbol{u}}(\lambda) \right) d\boldsymbol{\gamma}^L(\boldsymbol{u}),$$

for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$. Results of Adler-Taylor [1] (see Subection 2.1) show that $\boldsymbol{\sigma}^L$ exists. Note that

$$\int_{\mathbb{R}} \boldsymbol{\sigma}^L(dt) = \boldsymbol{N}^L.$$

1.2. **Statements of the main results.** In this paper we investigate the statistical properties of the measure $\boldsymbol{\sigma}^L$ as $L \to \infty$ and then as $m \to \infty$. To state our results we need a bit of terminology.

For any $t > 0$ we denote by $\mathcal{R}_t : \mathbb{R} \to \mathbb{R}$ the rescaling map $\mathbb{R} \ni x \mapsto tx \in \mathbb{R}$. If μ is a Borel measure on $\mathbb{R}$ we denote by $(\mathcal{R}_t)_*\mu$ its pushforward via the rescaling map $\mathcal{R}_t$. We denote by $\boldsymbol{\gamma}_v$ the Gaussian measure on $\mathbb{R}$ with mean zero and variance $v \geq 0$.

The central result of this paper states that as $L \to \infty$ the probability measures

$$\frac{1}{\boldsymbol{N}^L} \Big(\mathcal{R}_{\frac{1}{\sqrt{\dim \boldsymbol{U}^L}}} \Big)_* \boldsymbol{\sigma}^L$$

converge weakly to a probability measure $\boldsymbol{\sigma}_m$ on $\mathbb{R}$ which can be described explicitly in terms of the statistics of the the eigenvalues of certain random symmetric $(m+1) \times (m+1)$-matrices. Additionally we prove a central limit theorem stating that as $m \to \infty$, the probability measures $\boldsymbol{\sigma}_m$ converge weakly to a Gaussian measure $\boldsymbol{\gamma}_2$. Before we give a more precise description of the measure $\boldsymbol{\sigma}_m$ we need to point out a small annoyance which we will turn to our advantage.

Observe that if $\boldsymbol{u} : M \to \mathbb{R}$ is a fixed Morse function and c is a constant, then

$$\mathbf{Cr}(c+\boldsymbol{u}) = \mathbf{Cr}(\boldsymbol{u}),\ \ \mu_{c+\boldsymbol{u}} = \mu_{\boldsymbol{u}},$$

but

$$\boldsymbol{D}(\boldsymbol{u}+c) = c+\boldsymbol{D}(\boldsymbol{u}),\ \ \boldsymbol{\sigma}_{\boldsymbol{u}+c} = \delta_c * \boldsymbol{\sigma}_{\boldsymbol{u}},$$

where $*$ denotes the convolution of two finite measures on $\mathbb{R}$. More generally, if X is a scalar random variable with probability distribution ν_X, then the expected variational complexity of the random function $X+\boldsymbol{u}$ is the measure $\nu_X * \boldsymbol{\sigma}_u$, where $*$ denotes the operation of convolution. In particular, if the distribution ν_X is a Gaussian, then the measure $\boldsymbol{\sigma}_{\boldsymbol{u}}$ is uniquely determined by the measure $\nu_X * \boldsymbol{\sigma}_u$ since the convolution with a Gaussian is an injective operation.

It turns out that it is easier to understand the statistics of the variational complexity of the perturbation of a random $\boldsymbol{u}\in \boldsymbol{U}_L$ by an independent Gaussian variable of cleverly chosen variance.

We consider random functions of the form

$$\boldsymbol{u}_{\boldsymbol{\omega}} = X_{\boldsymbol{\omega}} + \boldsymbol{u} = X_{\boldsymbol{\omega}} + \sum_{\lambda_k\leq L^2} u_k\Psi_k,$$

where the Fourier coefficients u_k are i.i.d. standard Gaussians, and $X_{\boldsymbol{\omega}}\in \boldsymbol{N}(0,\boldsymbol{\omega})$ is a scalar random variable independent of the u_k's. In applications $\boldsymbol{\omega}$ will depend on m and L.

Since $X_{\boldsymbol{\omega}}$ is independent of $\boldsymbol{u}$ we deduce that the expected variational complexity of $X_{\boldsymbol{\omega}}+\boldsymbol{u}$ is the measure $\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}$ on $\mathbb{R}$ given by

$$\boldsymbol{\sigma}^L_{\boldsymbol{\omega}} = \boldsymbol{E}(\,\boldsymbol{\sigma}_{X_{\boldsymbol{\omega}}+\boldsymbol{u}}) = \gamma_{\boldsymbol{\omega}} * \boldsymbol{\sigma}^L. \tag{1.4}$$

Note that

$$\boldsymbol{N}^L = \int_{\mathbb{R}} d\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}(t) = \int_{\mathbb{R}} d\boldsymbol{\sigma}^L(t).$$

The first goal of this paper is to investigate the behavior of the probability measures $\frac{1}{\boldsymbol{N}^L}\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}$ as $L\to\infty$ for certain very special $\boldsymbol{\omega}$'s. For reasons that we will shortly, we choose $\boldsymbol{\omega}$ of the form

$$\boldsymbol{\omega} = \boldsymbol{\omega}(m,L,r) = \bar{\boldsymbol{\omega}}_{m,r}L^m \tag{1.5}$$

where $r>0$ and the positive quantity $\bar{\boldsymbol{\omega}}_{m,r}$ are uniquely determined by the equality

$$s_m + \bar{\boldsymbol{\omega}}_{m,r} = r\frac{d_m^2}{h_m} =: s_m^{\boldsymbol{\omega}}. \tag{1.6}$$

Observe that as $L\to\infty$ we have $\boldsymbol{\omega}(m,L,r)\to\infty$ so the random variable $X_{\boldsymbol{\omega}}$ is more and more diffuse. From the elementary identity

$$s_m = h_m(m+2)(m+4),\ \ d_m = (m+4)h_m \tag{1.7}$$

we deduce that

$$s_m^{\boldsymbol{\omega}} = r\frac{m+4}{m+2}s_m,\ \ \bar{\boldsymbol{\omega}}_{m,r} = \left(\frac{r(m+4)}{m+2)} - 1\right)s_m. \tag{1.8}$$

The inequality $s_m^{\boldsymbol{\omega}}\geq s_m$ shows that the parameter r must satisfy the m-dependent constraint

$$r\geq \frac{m+2}{m+4}. \tag{C_m}$$

For $v \in (0, \infty)$ and N a positive integer we denote by GOE_N^v the space $\mathcal{S}_N$ of real, symmetric $N \times N$ matrices A equipped with a Gaussian measure such that the entries a_{ij} are independent, zero-mean, normal random variables with variances

$$\boldsymbol{var}(a_{ii}) = 2v, \ \ \boldsymbol{var}(a_{ij}) = v, \ \ \forall 1 \leq i < j \leq N.$$

We denote by $\rho_{N,v}(\lambda)$ the *normalized correlation* function of GOE_N^v. It is uniquely determined by the equality

$$\int_{\mathbb{R}} f(\lambda)\rho_{N,v}(\lambda)d\lambda = \frac{1}{N}\boldsymbol{E}_{\mathrm{GOE}_N^v}\big(\operatorname{tr} f(A)\big),$$

for any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. The function $\rho_{N,v}(\lambda)$ also has a probabilistic interpretation. For any Borel set $B \subset \mathbb{R}$ the expected number of eigenvalues in B of a random $A \in \mathrm{GOE}_N^v$ is equal to

$$N \int_B \rho_{N,v}(\lambda)d\lambda.$$

The celebrated Wigner semicircle theorem, [2, Thm. 2.1.1], [20, Eq.(7.2.33)], states that as $N \to \infty$ the rescaled probability measures

$$\big(\mathcal{R}_{\frac{1}{\sqrt{N}}}\big)_*\big(\rho_{N,v}(\lambda)d\lambda\big)$$

converge weakly to the semicircle measure given by the density

$$\rho_{\infty,v}(\lambda) := \frac{1}{2\pi v} \times \begin{cases} \sqrt{4v - \lambda^2}, & |\lambda| \leq \sqrt{4v} \\ 0, & |\lambda| > \sqrt{4v}. \end{cases}$$

We can now explain what we gain by working with the perturbed function $\boldsymbol{u}_{\boldsymbol{\omega}} = X_{\boldsymbol{\omega}} + \boldsymbol{u}$. The computation of $\boldsymbol{\sigma}^L$ uses the (conditioned) Gaussian random matrix

$$Z^{L,x} = \Big(L^{-\frac{m+4}{2}} \operatorname{Hess}(\boldsymbol{u}_{\boldsymbol{\omega}}, \boldsymbol{p}) \,\Big|\, d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p}) = 0, \ \ \boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p}) = L^{\frac{m}{2}} x\Big), \tag{1.9}$$

where $\operatorname{Hess}(\boldsymbol{u}_{\boldsymbol{\omega}}, \boldsymbol{p})$ denotes the hessian of $\boldsymbol{u}_{\boldsymbol{\omega}}$ at $\boldsymbol{p}$ and x is a fixed real number. The probability distribution of this random matrix depends on the choice of $\boldsymbol{\omega}$.

In Lemma 2.3 we show that if $\boldsymbol{\omega}$ is chosen according to the prescriptions (1.5), (1.8) where $r \geq 1$, then the random $m \times m$ matrix (1.9) converges as $L \to \infty$ to a sum between a $\mathrm{GOE}_m^{h_m}$-distributed matrix and an independent normally distributed multiple of the identity; see especially (2.12). Once this happens we can use a simple trick of Fyodorov [15] to express the limit as $L \to \infty$ of the expectation of $|\det Z^{L,x}|$ in terms of statistics of the ensemble $\mathrm{GOE}_{m+1}^{h_m}$; see Lemmas C.1, C.2. For example, in the extreme case $r = 1$, the random matrix $Z^{L,x} + \frac{x}{m+4}\mathbb{1}_m$ converges as $L \to \infty$ to a random matrix in the ensemble $\mathrm{GOE}_m^{h_m}$.

None of the above nice accidents would take place if we did not perturb the function by the carefully chosen random variable $X_{\boldsymbol{\omega}}$. The choice of $X_{\boldsymbol{\omega}}$ is essentially forced upon us by (2.12) and the discussion in Appendix C, especially the equality (C.1).

We can now state the main technical result of this paper.

Theorem 1.3. *Fix a positive real number satisfying $r \geq 1$. Let $\boldsymbol{\omega} = \boldsymbol{\omega}(m, L, r)$ be defined by the equalities (1.5) and (1.6). Then as $L \to \infty$ the rescaled measures*

$$\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m^{\boldsymbol{\omega}} L^m}}}\Big)_* \boldsymbol{\sigma}_{\boldsymbol{\omega}}^L$$

converge weakly to a probability measure $\boldsymbol{\sigma}_{m,r}$ on $\mathbb{R}$ satisfying the equality

$$\boldsymbol{\sigma}_{m,r} \propto \gamma_{\frac{(r-1)}{r}} * \big(e^{-\frac{r\lambda^2}{4}}\rho_{m+1,r^{-1}}(\lambda)d\lambda\big), \tag{1.10}$$

where $\propto$ denotes the relation of proportionality of two finite measures. In particular, when $r=1$ we have

$$\boldsymbol{\sigma}_{m,1} \propto e^{-\frac{\lambda^2}{4}}\rho_{m+1,1}(\lambda)d\lambda.$$

The above result has several interesting consequences.

Corollary 1.4 (Universality.)**.** *As $L\to\infty$ the rescaled measures*

$$\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{\dim \boldsymbol{U}^L}}}\Big)_*\boldsymbol{\sigma}^L$$

converge weakly to a probability measure $\boldsymbol{\sigma}_m$ on $\mathbb{R}$ uniquely determined by the convolution equation

$$\boldsymbol{\gamma}_{\frac{2}{m+2}} * \boldsymbol{\sigma}_m = \Big(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\Big)_*\boldsymbol{\sigma}_{m,1}.$$

The fact that the convolution equation

$$\boldsymbol{\gamma}_{\frac{2}{m+2}} * \mu = \Big(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\Big)_*\boldsymbol{\sigma}_{m,1}$$

has a *unique* solution μ can be seen easily by taking Fourier transforms. The above result shows that the large L behavior of the average complexity $\boldsymbol{\sigma}^L$ is independent of the background manifold M and of the metric g. We do not have a more explicit and simpler description of $\boldsymbol{\sigma}_m$ and we doubt that such a description exists.

Corollary 1.5. *As $m\to\infty$, the measures $\boldsymbol{\sigma}_m$ converge weakly to the Gaussian measure $\boldsymbol{\gamma}_2$.*

1.3. **Related results.** The scaling limit of various statistical quantities associated to Gaussian random fields in the high frequency limit has been studied in detail over the past fifteen years. In [27] S. Zelditch investigates the volume of the zero set of such a random field and proves a related universality result. The zero set of a random section of a large power of an ample line bundle over a Kähler manifold displays a similar universal scaling behavior; see [8] and the references therein.

The distribution of critical points (or energy landscape) of isotropic random functions on $\mathbb{R}^m$ was investigated by Fyodorov [15, 16] who also relates this problem to the staistics of the eigenvalues in the ensemble GOE_{m+1}. Recently A. Auffinger [3, 4] has investigated the distributions of critical values of certain isotropic random fields on a round sphere S^m, where $m\to\infty$, and described a connection with the distribution of eigenvalues of symmetric matrices in the ensemble GOE_{m+1}.

The scaling limit of the distribution of critical points of random holomorphic sections of a large power of an ample line bundle on a Kähler manifold was investigated by M. Douglas, B. Schiffman and S. Zelditch, [11, 12, 13]. The dimensional dependence of the number of critical points of such a random holomorphic section was described by B. Baugher, [6].

The universality result described in Theorem 1.3 is not an isolated phenomenon and fits a more general pattern. To explain this, fix a measurable function $w:[0,\infty)\to[0,\infty)$ called *weight*. For $L>0$ we define the rescaled weight

$$w_L:[0,\infty)\to[0,\infty),\;\; w_L(t)=w\Big(\frac{t}{L}\Big)$$

and consider the random function on M

$$\boldsymbol{u}^L=\sum_{k\geq 0}\sqrt{w_L\big(\lambda_k^{\frac{1}{2}}\big)}C_k\Psi_k, \tag{1.11}$$

where C_k are independent normally distributed random variables. If w decays sufficiently fast as $t \to \infty$, then $\boldsymbol{u}^L$ is a.s. smooth.

The correlation kernel of the random function (1.11) can be identified with the Schwartz kernel of the smoothing operator $w_L(\sqrt{\Delta})$. We can then ask about the behavior as $L \to \infty$ of the expected variational complexity of the random function (1.11). This is in turn conditioned by the regularity of w: the more regular is w the more detailed is the information about this behavior. This paper covers the "worst" situation, when $w = \boldsymbol{I}_{[0,1]}$ is obviously discontinuous. Above and throughout this paper we use the notation $\boldsymbol{I}_A$ to denote the indicator function of a subset A of a given set S.

In [24], a sequel to this paper, we investigate the random functions defined by weights w which are smooth. We show that Theorem 1.3 has a suitable counterpart in this case. Moreover, the smoothness of w allows us to provide additional nontrivial information that is not available in the singular case $w = \boldsymbol{I}_{[0,1]}$.

1.4. **Organization of the paper.**

Let us briefly describe the principles hiding behind the above results. Theorem 1.3 follows from a Kac-Rice type formula [1, 11] aided by the refined spectral estimates of the spectral function of the Laplacian on a compact Riemann manifold, [7, 14, 18, 26]. Corollary 1.4 is a rather immediate consequence of Theorem 1.3 while Corollary 1.5 follows from Corollary 1.4 via a refined version of Wigner's semicircle theorem.

The basic facts coverning the Kac-Rice formula are presented in Subsection 2.1 while the proofs of the above results are presented in Subsections 2.2, 2.3, 2.4. Appendix A is devoted to the proof of Proposition 1.1. To aid the reader with a more geometric bias we have included two probabilistic appendices. In Appendix B we have collected a few basic facts about Gaussian measures used throughout the paper. In the more exotic Appendix C we discuss a family of symmetric random matrices and some of their properties needed in the main body of the paper.

Aknowledgments. I would like to thank the anonymous referee for the many constructive suggestions that helped me improve the quality of the present paper.

2. Proofs

2.1. **A Kac-Rice type formula.**

As we have already mentioned, the key result behind Theorem 1.3 is a Kac-Rice type result which we intend to discuss in some detail in this section. This result gives an explicit, yet quite complicated description of the measure $\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}$. More precisely, for any Borel subset $B \subset \mathbb{R}$ the Kac-Rice formula provides an integral representation of $\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}(B)$ of the form

$$\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}(B) = \int_M f_{L,\boldsymbol{\omega},B}(\boldsymbol{p})\,|dV_g(\boldsymbol{p})|,$$

for some integrable function $f_{L,\boldsymbol{\omega},B} : M \to \mathbb{R}$. The core of the Kac-Rice formula is an explicit probabilistic description of the density $f_{L,\boldsymbol{\omega},B}$.

Fix a point $\boldsymbol{p} \in M$. This determines three Gaussian random variables.

$$\begin{gathered}(\boldsymbol{U}^L, \boldsymbol{\gamma}^L_{\boldsymbol{\omega}}) \ni \boldsymbol{u}_{\boldsymbol{\omega}} \mapsto \boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p}) \in \mathbb{R},\\ (\boldsymbol{U}^L, \boldsymbol{\gamma}^L_{\boldsymbol{\omega}}) \ni \boldsymbol{u}_{\boldsymbol{\omega}} \mapsto d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p}) \in T^*_{\boldsymbol{p}}M,\\ (\boldsymbol{U}^L, \boldsymbol{\gamma}^L_{\boldsymbol{\omega}}) \ni \boldsymbol{u}_{\boldsymbol{\omega}} \mapsto \mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u}_{\boldsymbol{\omega}}) \in \mathcal{S}(T_{\boldsymbol{p}}M),\end{gathered} \tag{$RV_{\boldsymbol{\omega}}$}$$

where $\mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u}_{\boldsymbol{\omega}}) : T_{\boldsymbol{p}}M \times T_{\boldsymbol{p}}M \to \mathbb{R}$ is the Hessian of $\boldsymbol{u}_{\boldsymbol{\omega}}$ at $\boldsymbol{p}$ defined in terms of the Levi-Civita connection of g and then identified with a symmetric endomorphism of $T_{\boldsymbol{p}}M$ using again the metric

g. More concretely, if $(x^i)_{1\leq i\leq m}$ are g-normal coordinates at $\boldsymbol{p}$, then

$$\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}_{\boldsymbol{\omega}})\partial_{x^j} = \sum_{i=1}^m \partial^2_{x^ix^j}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\partial_{x^i}.$$

As shown in the proof of Proposition 1.1, for L very large the map $\boldsymbol{U}^L \ni \boldsymbol{u} \mapsto d\boldsymbol{u}(\boldsymbol{p}) \in T^*_{\boldsymbol{p}}M$ is surjective which implies that the covariance form of the Gaussian random vector $d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})$ is positive definite. We can identify it with a symmetric, positive definite linear operator

$$\boldsymbol{S}\big(\,d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) : T_{\boldsymbol{p}}M \to T_{\boldsymbol{p}}M.$$

More concretely, if $(x^i)_{1\leq i\leq m}$ are g-normal coordinates at $\boldsymbol{p}$, then we can identify $\boldsymbol{S}\big(\,d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big)$ with a $m\times m$ real symmetric matrix whose (i,j)-entry is given by

$$\boldsymbol{S}_{ij}\big(\,d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) = \boldsymbol{E}\big(\,\partial_{x_i}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\cdot\partial_{x^j}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big).$$

Theorem 2.1. *Fix a Borel subset $B\subset\mathbb{R}$. For any $\boldsymbol{p}\in M$ define*

$$f_{L,\boldsymbol{\omega},B}(\boldsymbol{p}) := \Big(\det\big(2\pi\boldsymbol{S}(\,\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,)\,\big)^{-\frac{1}{2}}\,\boldsymbol{E}\Big(\,|\det\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}_{\boldsymbol{\omega}})|\,\cdot\,\boldsymbol{I}_B(\,\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,)\;\Big|\,d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})=0\,\Big),$$

where $\boldsymbol{E}\big(\,\mathbf{var}\;\;|\,\mathbf{cons}\,\big)$ stands for the conditional expectation of the variable **var** *given the constraint* **cons**. *Then*

$$\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}(B) = \int_M f_{L,\boldsymbol{\omega},B}(\boldsymbol{p})\,|dV_g(\boldsymbol{p})|. \tag{2.1}$$

□

This theorem is a special case of a general result of Adler-Taylor, [1, Thm. 11.2.1]. The many technical assumptions in Adler-Taylor Theorem are trivially satisfied in this case. In [23] we proved this theorem in the case $B=\mathbb{R}$ and $\boldsymbol{\omega}=0$. The strategy used there can be modified to yield the more general Theorem 2.1.

For the above theorem to be of any use we need to have some concrete information about the Gaussian random variables ($RV_{\boldsymbol{\omega}}$). All the relevant statistical invariants of these variables can be extracted from the covariance kernel of the random function $\boldsymbol{u}_{\boldsymbol{\omega}}$. This is the function

$$\begin{gathered}\mathcal{E}^L_{\boldsymbol{\omega}} : M\times M\to\mathbb{R},\\ \mathcal{E}^L_{\boldsymbol{\omega}}(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{E}\big(\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{q})\,\big) = \boldsymbol{E}\big((X+\boldsymbol{u}(\boldsymbol{p})\,)\,\cdot\,(X+\boldsymbol{u}(\boldsymbol{q})\,)\,\big)\\ = \boldsymbol{\omega} + \sum_{\lambda_k\leq L^2}\Psi_k(\boldsymbol{p})\Psi_k(\boldsymbol{q}) =: \boldsymbol{\omega} + \mathcal{E}^L(\boldsymbol{p},\boldsymbol{q}).\end{gathered}$$

The function $\mathcal{E}^L$ is the spectral function of the Laplacian, i.e., the Schwartz kernel of P_L, the orthogonal projection onto $\boldsymbol{U}^L$. Fortunately, a lot is known about the behavior of $\mathcal{E}^L$ as $L\to\infty$, [7, 14, 18, 26, 27].

2.2. **Proof of Theorem 1.3.** Fix $L\gg 0$. For any $\boldsymbol{p}\in M$ we have the centered Gaussian vector ($RV_{\boldsymbol{\omega}}$), $\boldsymbol{\omega}=0$,

$$(\boldsymbol{U}^L,\boldsymbol{\gamma}^L)\ni u\mapsto\big(\,\boldsymbol{u}(\boldsymbol{p}), d\boldsymbol{u}(\boldsymbol{p}), \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})\,\big)\in\mathbb{R}\oplus T^*_{\boldsymbol{p}}M\oplus\mathcal{S}(T_{\boldsymbol{p}}M).$$

We fix normal coordinates $(x^i)_{1\leq i\leq m}$ at $\boldsymbol{p}$ and we can identify the above Gaussian vector with the centered Gaussian vector

$$\Big(\,\boldsymbol{u}(\boldsymbol{p}),\;\big(\,\partial_{x^i}\boldsymbol{u}(p)\,\big)_{1\leq i\leq m},\;\big(\,\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p})\,\big)_{1\leq i,j\leq m}\,\Big)\in\mathbb{R}\oplus\mathbb{R}^m\oplus\mathcal{S}_m.$$

In [23, §3] we showed that the spectral estimates of Bin-Hörmander [7, 18] imply the following asymptotic estimates.

Lemma 2.2. *For any $1 \leq i,j,k,\ell \leq m$ we have the uniform in $\boldsymbol{p}$ asymptotic estimates as $L \to \infty$*

$$\boldsymbol{E}\big(\,\boldsymbol{u}(p)^2\,\big) = s_m^{\boldsymbol{\omega}} L^m\big(1+O(L^{-1})\,\big), \tag{2.2a}$$

$$\boldsymbol{E}\big(\,\partial_{x^i}\boldsymbol{u}(\boldsymbol{p})\partial_{x^j}\boldsymbol{u}(\boldsymbol{p})\,\big) = d_m L^{m+2}\delta_{ij}\big(\,1+O(L^{-1})\,\big), \tag{2.2b}$$

$$\boldsymbol{E}\big(\,\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p})\partial^2_{x^kx^\ell}\boldsymbol{u}(\boldsymbol{p})\,\big) = h_m L^{m+4}(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\big(\,1+O(L^{-1})\,\big), \tag{2.2c}$$

$$\boldsymbol{E}\big(\,\boldsymbol{u}(\boldsymbol{p})\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p})\,\,\big) = -d_m L^{m+2}\delta_{ij}\big(\,1+O(L^{-1})\,\big), \tag{2.2d}$$

$$\boldsymbol{E}\big(\,\boldsymbol{u}(p)\partial_{x^i}\boldsymbol{u}(\boldsymbol{p})\,\big) = O(L^m),\;\; \boldsymbol{E}\big(\,\partial_{x^i}\boldsymbol{u}(\boldsymbol{p})\partial^2_{x^jx^k}\boldsymbol{u}(\boldsymbol{p})\,\big) = O(L^{m+2}), \tag{2.2e}$$

where the constants s_m, d_m, h_m are defined by (1.1). □

Now let $\boldsymbol{\omega} = \boldsymbol{\omega}(m,L,r)$ be defined as in (1.5), (1.6). Using the notation (1.8) we deduce from the above that in the case of the random function $\boldsymbol{u}_{\boldsymbol{\omega}}$ we have the estimates

$$\boldsymbol{E}\big(\,\boldsymbol{u}_{\boldsymbol{\omega}}(p)^2\,\big) = s_m^{\boldsymbol{\omega}} L^m\big(1+O(L^{-1})\,\big), \tag{2.3a}$$

$$\boldsymbol{E}\big(\,\partial_{x^i}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\partial_{x^j}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) = d_m L^{m+2}\delta_{ij}\big(\,1+O(L^{-1})\,\big), \tag{2.3b}$$

$$\boldsymbol{E}\big(\,\partial^2_{x^ix^j}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\partial^2_{x^kx^\ell}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) = h_m L^{m+4}(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\big(\,1+O(L^{-1})\,\big), \tag{2.3c}$$

$$\boldsymbol{E}\big(\,\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\partial^2_{x^ix^j}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\,\big) = -d_m L^{m+2}\delta_{ij}\big(\,1+O(L^{-1})\,\big), \tag{2.3d}$$

$$\boldsymbol{E}\big(\,\boldsymbol{u}_{\boldsymbol{\omega}}(p)\partial_{x^i}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) = O(L^m),\;\; \boldsymbol{E}\big(\,\partial_{x^i}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\partial^2_{x^jx^k}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) = O(L^{m+2}). \tag{2.3e}$$

From the estimate (2.3b) we deduce that

$$\boldsymbol{S}\big(\,d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big) = d_m L^{m+2}\big(\,\mathbb{1}_m + O(L^{-1})\,\big),$$

so that

$$\sqrt{|\det \boldsymbol{S}(\boldsymbol{u}_{\boldsymbol{\omega}}(p))|} = (d_m)^{\frac{m}{2}} L^{\frac{m(m+2)}{2}}\big(\,1+O(L^{-1})\,\big)\;\text{ as } L\to\infty. \tag{2.4}$$

Consider the rescaled random vector

$$\begin{aligned}(s^L, v^L, H^L) =&(s^{L,\boldsymbol{\omega},\boldsymbol{p}}, v^{L,\boldsymbol{\omega},.\boldsymbol{p}}, H^{L,\boldsymbol{\omega},\boldsymbol{p}})\\ &:= \big(L^{-\frac{m}{2}}\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p}), L^{-\frac{m+2}{2}}d\boldsymbol{u}_{\boldsymbol{\omega}}(p),\;\; L^{-\frac{m+4}{2}}\operatorname{Hess}_{\boldsymbol{p}}\boldsymbol{u}_{\boldsymbol{\omega}}\,\big).\end{aligned}$$

Form the above we deduce the following uniform in $\boldsymbol{p}$ estimates as $L\to\infty$.

$$\boldsymbol{E}\big(\,(s^L)^2\,\big) = s_m^{\omega}\big(1+O(L^{-1})\,\big), \tag{2.5a}$$

$$\boldsymbol{E}\big(\,v_i^L v_j^L\,\big) = d_m\delta_{ij}\big(\,1+O(L^{-1})\,\big), \tag{2.5b}$$

$$\boldsymbol{E}\big(\,H_{ij}^L H_{kl}^L\,\big) = h_m(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\big(\,1+O(L^{-1})\,\big), \tag{2.5c}$$

$$\boldsymbol{E}\big(\,s^L H_{ij}^L\,\big) = -d_m\delta_{ij}\big(\,1+O(L^{-1})\,\big), \tag{2.5d}$$

$$\boldsymbol{E}\big(\,s^L v_i^L\,\big) = O(L^{-1}),\;\; \boldsymbol{E}\big(\,v_i^L H_{jk}^L\,\big) = O(L^{-1}). \tag{2.5e}$$

The probability distribution of the variable s^L is

$$d\gamma_{s^L}(x) = \frac{1}{\sqrt{2\pi\bar{s}_m^{\boldsymbol{\omega}}(L)}} e^{-\frac{x^2}{2\bar{s}_m^{\boldsymbol{\omega}}(L)}}|dx|,$$

where its variance $\bar{s}_m^{\boldsymbol{\omega}}(L)$ satisfies the estimate

$$\bar{s}_m^{\boldsymbol{\omega}}(L) = s_m^{\boldsymbol{\omega}} + O(L^{-1}).$$

Fix a Borel set $B \subset \mathbb{R}$. We have

$$\begin{aligned}\boldsymbol{E}\big(\,|\det \operatorname{Hess} \boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})|\boldsymbol{I}_B\big(\,\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})\,\big)\;\big|\; d\boldsymbol{u}_{\boldsymbol{\omega}}(\boldsymbol{p})=0\,\big) &= L^{\frac{m(m+4)}{2}}\boldsymbol{E}\big(\,|\det H^L|\boldsymbol{I}_{L^{-\frac{m}{2}}B}(s^L)\;\big|\; v^L=0\,\big)\\ &= L^{\frac{m(m+4)}{2}}\underbrace{\int_{L^{-\frac{m}{2}}B}\boldsymbol{E}\big(\,|\det H^L|\;\big|\; s^L=x,\; v^L=0\,\big)\frac{e^{-\frac{x^2}{2\bar{s}_m^{\boldsymbol{\omega}}(L)}}}{\sqrt{2\pi\bar{s}_m^{\boldsymbol{\omega}}(L)}}|dx|}_{=:q_{L,\boldsymbol{p}}(L^{-\frac{m}{2}}B)}.\end{aligned} \tag{2.6}$$

Using (2.4) and (2.6) we deduce from Theorem 2.1 that

$$\boldsymbol{\sigma}_{\boldsymbol{\omega}}^L(B) = L^m\left(\frac{1}{2\pi d_m}\right)^{\frac{m}{2}}\int_M q_{L,\boldsymbol{p}}(L^{-\frac{m}{2}}B)\rho_L(\boldsymbol{p})|dV_g(\boldsymbol{p})|,$$

where $\rho_L : M \to \mathbb{R}$ is a function that satisfies the uniform in $\boldsymbol{p}$ estimate

$$\rho_L(p) = 1 + O(L^{-1}) \;\text{ as } L \to \infty. \tag{2.7}$$

Hence

$$\frac{1}{L^m}\left(\mathcal{R}_{L^{-\frac{m}{2}}}\right)_*\boldsymbol{\sigma}_{\boldsymbol{\omega}}^L(B) = \left(\frac{1}{2\pi d_m}\right)^{\frac{m}{2}}\int_M q_{L,\boldsymbol{p}}(B)\rho_L(\boldsymbol{p})|dV_g(\boldsymbol{p})|. \tag{2.8}$$

To continue the computation we need to investigate the behavior of $q_{L,\boldsymbol{p}}(B)$ as $L \to \infty$. More concretely, we need to elucidate the nature of the Gaussian matrix

$$Z^{L,x} := \big(\,H^L \;\big|\; s^L = x,\; v^L = 0\,\big).$$

Lemma 2.3. *Set*

$$\kappa = \kappa(r) := \frac{(r-1)}{2r},\;\; r \geq 1.$$

The Gaussian random matrix $Z^{L,x}$ converges uniformly in $\boldsymbol{p}$ as $L \to \infty$ to the random matrix $A - \frac{x}{r(m+4)}\mathbb{1}_m$, where A belongs to the Gaussian ensemble $\mathcal{S}_m^{2\kappa h_m, h_m}$ described in Appendix C.

Proof. We will use the regression formula (B.3). For simplicity we set

$$Y^L := (s^L, v^L) \in \mathbb{R} \oplus \mathbb{R}^m.$$

The components of Y are

$$Y_0^L = s^L,\;\; Y_i^L = v_i^L,\;\; 1 \leq i \leq m.$$

Using (2.5a), (2.5b) and (2.5e) we deduce that for any $1 \leq i, j \leq m$ we have

$$\boldsymbol{E}(Y_0^L Y_i^L) = s_m^{\boldsymbol{\omega}}\delta_{0i} + O(L^{-1}),\;\; \boldsymbol{E}(Y_i^L Y_j^L) = d_m\delta_{ij} + O(L^{-1}).$$

If $\boldsymbol{S}(Y^L)$ denotes the covariance operator of Y^L, then we deduce that

$$\boldsymbol{S}(Y^L)^{-1}_{0,i} = \frac{1}{s_m^{\boldsymbol{\omega}}}\delta_{0i} + O(L^{-1}),\;\; \boldsymbol{S}(Y^L)^{-1}_{ij} = \frac{1}{d_m}\delta_{ij} + O(L^{-1}). \tag{2.9}$$

We now need to compute the covariance operator $\boldsymbol{Cov}(H^L, Y^L)$. To do so we equip $\mathcal{S}_m$ with the inner product

$$(A, B) = \operatorname{tr}(AB),\;\; A, B \in \mathcal{S}_m$$

The space $\mathcal{S}_m$ has a canonical orthonormal basis $\widehat{E}_{ij}$, $1 \leq i \leq j \leq m$, where

$$\widehat{\boldsymbol{E}}_{ij} = \begin{cases} \boldsymbol{E}_{ij}, & i = j\\ \frac{1}{\sqrt{2}}\boldsymbol{E}_{ij}, & i < j\end{cases}$$

and $\boldsymbol{E}_{ij}$ denotes the symmetric matrix nonzero entries only at locations (i,j) and (j,i) and these entries are equal to 1. Thus a matrix $A \in \mathcal{S}_m$ can be written as

$$A = \sum_{i \leq j} a_{ij}\boldsymbol{E}_{ij} = \sum_{i \leq j} \hat{a}_{ij}\widehat{\boldsymbol{E}}_{ij},$$

where

$$\widehat{a}_{ij} = \begin{cases} a_{ij}, & i = j, \\ \sqrt{2}a_{ij}, & i < j. \end{cases}$$

The covariance operator $\boldsymbol{Cov}(H^L, Y^L)$ is a linear map

$$\boldsymbol{Cov}(H^L, Y^L) : \mathbb{R} \oplus \mathbb{R}^m \to \mathcal{S}_m$$

given by

$$\boldsymbol{Cov}(H^L, Y^L)\left(\sum_{\alpha=0}^{m} y_\alpha \boldsymbol{e}_\alpha\right) = \sum_{i<j,\alpha} \boldsymbol{E}(\widehat{H}_{ij}^L Y_\alpha^L) y_\alpha \widehat{\boldsymbol{E}}_{ij} = \sum_{i<j,\alpha} \boldsymbol{E}(H_{ij}^L Y_\alpha^L) y_\alpha \boldsymbol{E}_{ij},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ denotes the canonical orthonormal basis in $\mathbb{R} \oplus \mathbb{R}^m$. Using (2.5d) and (2.5e) we deduce that

$$\boldsymbol{Cov}(H^L, Y^L)\left(\sum_{\alpha=0}^{m} y_\alpha \boldsymbol{e}_\alpha\right) = -y_0 d_m \mathbb{1}_m + O(L^{-1}). \tag{2.10}$$

We deduce that the transpose $\boldsymbol{Cov}(H^\varepsilon, Y\varepsilon)^\vee$ satisfies

$$\boldsymbol{Cov}(H^L, Y^L)^\vee \left(\sum_{i \leq j} \hat{a}_{ij}\widehat{\boldsymbol{E}}_{ij}\right) = -d_m \operatorname{tr}(A)\boldsymbol{e}_0 + O(L^{-1}). \tag{2.11}$$

The covariance operator of the random symmetric matrix $Z^L = XZ^{L,x}$ is then

$$\boldsymbol{S}(Z^L) = \boldsymbol{S}(H^L) - \boldsymbol{Cov}(H^L, Y^L)\boldsymbol{S}(Y^L)^{-1}\boldsymbol{Cov}(H^L, Y^L)^\vee.$$

This means that

$$\boldsymbol{E}\big(\,\widehat{z}_{ij}^L \cdot \widehat{z}_{k\ell}^L\,\big) = (\widehat{\boldsymbol{E}}_{ij}, \boldsymbol{S}(Z^L)\widehat{\boldsymbol{E}}_{k\ell}).$$

Using (2.9), (2.10) and (2.11) we deduce that

$$\boldsymbol{Cov}(H^L, Y^L)\boldsymbol{S}(Y^L)^{-1}\boldsymbol{Cov}(H^L, Y^L)^\vee \left(\sum_{i \leq j} \hat{a}_{ij}\widehat{\boldsymbol{E}}_{ij}\right) = \frac{d_m^2}{s_m^{\boldsymbol{\omega}}} \operatorname{tr}(A)\mathbb{1}_m + O(L^{-1})$$

$$\boldsymbol{E}\big(\,(z_{ij}^L)^2\,\big) = h_m + O(L^{-1}),\ \ \boldsymbol{E}(z_{ii}^L z_{jj}^L) = h_m - \frac{d_m^2}{s_m^{\boldsymbol{\omega}}} + O(L^{-1}),\ \ \forall i < j,$$

$$\boldsymbol{E}\big(\,(z_{ii}^L)^2\,\big) = 3h_m - \frac{d_m^2}{s_m^{\boldsymbol{\omega}}} + O(L^{-1}),\ \ \forall i,$$

and

$$\boldsymbol{E}(z_{ij}^L z_{k\ell}^L) = O(L^{-1}),\ \ \forall i < j,\ \ k \leq \ell,\ \ (i,j) \neq (k,\ell).$$

We can rewrite these equalities in the compact form

$$\boldsymbol{E}(z_{ij}^L z_{k\ell}^L) = \left(h_m - \frac{d_m^2}{s_m^{\boldsymbol{\omega}}}\right)\delta_{ij}\delta_{k\ell} + h_m(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + O(L^{-1}). \tag{2.12}$$

The last equality is ultimately the main reason for our choices (1.5) and (1.8) in defining $X_{\boldsymbol{\omega}}$. Note that with r defined as in (1.6) we have

$$h_m - \frac{d_m^2}{s_m^{\boldsymbol{\omega}}} \stackrel{(1.7)}{=} \frac{r-1}{r} h_m.$$

Hence

$$\boldsymbol{E}(z_{ij}^L z_{k\ell}^L) = 2\kappa h_m \delta_{ij}\delta_{k\ell} + h_m(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + O(L^{-1}). \tag{2.13}$$

Using (B.4) we deduce that the expectation of Z^L is

$$\boldsymbol{E}(Z^L) = \boldsymbol{Cov}(H^L, Y^L)\boldsymbol{S}(Y^L)^{-1}(x\boldsymbol{e}_0) = -\frac{x}{r(m+4)}\mathbb{1}_m + O(L^{-1}). \tag{2.14}$$

This completes the proof of Lemma 2.3. □

We deduce that

$$\lim_{L\to\infty} q_{L,\boldsymbol{p}}(B) = q_\infty(B) := \int_B \boldsymbol{E}_{\mathcal{S}_m^{2\kappa h_m, h_m}}\Big(\big|\det\Big(A - \frac{x}{r(m+4)}\mathbb{1}_m\Big)\big|\Big)\frac{e^{-\frac{x^2}{2s_m^{\boldsymbol{\omega}}}}}{\sqrt{2\pi s_m^{\boldsymbol{\omega}}}}dx$$

$$= (h_m)^{\frac{m}{2}}\int_B \boldsymbol{E}_{\mathcal{S}_m^{2\kappa,1}}\Big(\big|\det\Big(A - \frac{x}{r(m+4)\sqrt{h_m}}\mathbb{1}_m\Big)\big|\Big)\frac{e^{-\frac{x^2}{2s_m^{\boldsymbol{\omega}}}}}{\sqrt{2\pi s_m^{\boldsymbol{\omega}}}}dx$$

$$= (h_m)^{\frac{m}{2}}\int_{(s_m^{\boldsymbol{\omega}})^{-\frac{1}{2}}B} \boldsymbol{E}_{\mathcal{S}_m^{2\kappa,1}}\Big(\big|\det\big(A - \alpha_m y\mathbb{1}_m\big)\big|\Big)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy,$$

where

$$\alpha_m = \frac{\sqrt{s_m^{\boldsymbol{\omega}}}}{r(m+4)\sqrt{h_m}} \stackrel{(1.7)}{=} \frac{1}{\sqrt{r}}.$$

This proves that

$$\lim_{L\to\infty}\Big(\mathcal{R}_{(s_m^{\boldsymbol{\omega}})^{-\frac{1}{2}}}\Big)_* q_{L,\boldsymbol{p}}(B) = (h_m)^{\frac{m}{2}}\underbrace{\int_B \boldsymbol{E}_{\mathcal{S}_m^{2\kappa,1}}\left(\left|\det\left(A - \frac{y}{\sqrt{r}}\mathbb{1}_m\right)\right|\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy}_{=:\mu_m(B)}.$$

Using the last equality, the normalization (∗) and the estimate (2.7) in (2.8) we conclude

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{s_m^{\boldsymbol{\omega}}L^m}\big(\mathcal{R}_{(s_m^{\boldsymbol{\omega}}L^m)^{-\frac{1}{2}}}\big)_*\boldsymbol{\sigma}_{\boldsymbol{\omega}}^L(B) &= \frac{1}{s_m}\left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}}\mu_m(B)\\ &\stackrel{(1.7)}{=}\left(\frac{2}{m+4}\right)^{\frac{m}{2}}\Gamma\left(1+\frac{m}{2}\right)\mu_m(B).\end{aligned} \tag{2.15}$$

In particular, this shows that

$$\boldsymbol{N}^L \sim s_m^{\boldsymbol{\omega}}L^m\left(\frac{2}{m+4}\right)^{\frac{m}{2}}\Gamma\left(1+\frac{m}{2}\right)\mu_m(\mathbb{R}).$$

Observe that the probability density of μ_m is

$$\frac{d\mu_m}{dy} = \boldsymbol{E}_{\mathcal{S}_m^{2\kappa,1}}\left(\left|\det\left(A - \frac{y}{\sqrt{r}}\mathbb{1}_m\right)\right|\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}. \tag{2.16}$$

We now distinguish two cases.

Case 1. $r > 1$ From Lemma C.2 we deduce that

$$\begin{aligned}
&\boldsymbol{E}_{\mathbb{S}_m^{2\kappa,1}}\Big(\Big|\det\Big(A - \frac{y}{\sqrt{r}}\mathbb{1}_m\Big)\Big|\Big) \\
&= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi\kappa}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4\tau^2}(\lambda-(\tau^2+1)\frac{y}{\sqrt{r}})^2+\frac{(\tau^2+1)y^2}{4r}}d\lambda,
\end{aligned}\tag{2.17}$$

where

$$\tau^2 := \frac{\kappa}{\kappa-1} = \frac{r-1}{r+1}.$$

Thus

$$\begin{aligned}
\frac{d\mu_m}{dy} &= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{2\pi\sqrt{\kappa}}e^{\frac{(\tau^2+1-2r)y^2}{4r}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4\tau^2}(\lambda-(\tau^2+1)\frac{y}{\sqrt{r}})^2}d\lambda \\
&= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{2\pi\sqrt{\kappa}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4\tau^2}(\lambda-(\tau^2+1)\frac{y}{\sqrt{r}})^2-\frac{ry^2}{2(r+1)}}d\lambda.
\end{aligned}$$

An elementary computation shows that

$$-\frac{1}{4\tau^2}\left(\lambda-(\tau^2+1)\frac{y}{\sqrt{r}}\right)^2-\frac{ry^2}{2(r+1)} = -\frac{1}{4}\lambda^2-\left(\sqrt{\frac{1}{2(r-1)}}\lambda - y\sqrt{\frac{r}{2(r-1)}}\right)^2.$$

Now set

$$\beta = \beta(r) := \frac{1}{(r-1)}.$$

We deduce

$$\frac{d\mu_m}{dy} = 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{2\pi\sqrt{\kappa}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4}\lambda^2}e^{-\frac{\beta}{2}(\lambda-\sqrt{r}y)^2}d\lambda$$

$(\lambda := \sqrt{r}\lambda)$

$$= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{2\pi\sqrt{\kappa}}\int_{\mathbb{R}}\sqrt{r}\rho_{m+1,1}(\sqrt{r}\lambda)e^{-\frac{r}{4}\lambda^2}e^{-\frac{r\beta}{2}(\lambda-y)^2}d\lambda$$

$$\overset{(C.6)}{=} 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi\kappa r\beta}}\int_{\mathbb{R}}\rho_{m+1,1/r}(\lambda)e^{-\frac{r}{4}\lambda^2}d\gamma_{\frac{1}{\beta r}}(y-\lambda)d\lambda.$$

Using the last equality in (2.15) and then invoking the estimate (1.3) we obtain the case $r > 1$ of Theorem 1.3.

Case 2. $r = 1$. The proof of Theorem 1.3 in this case follows a similar pattern. Note first that in this case $\kappa = 0$ so invoking Lemma C.1 we obtain the following counterpart of (2.17)

$$\boldsymbol{E}_{\mathrm{GOE}_m^1}\Big(\Big|\det\Big(A - y\mathbb{1}_m\Big)\Big|\Big) = 2^{\frac{m+4}{2}}\Gamma\left(\frac{m+3}{2}\right)e^{\frac{y^2}{4}}\rho_{m+1,1}(y).$$

Using this in (2.16) we deduce immediately (1.10) in the case $r = 1$. This completes the proof of Theorem 1.3.

∎

2.3. **Proof of Corollary 1.4.** We use Theorem 1.3 in the case $r=1$. Using (1.4), (1.5) and (1.8) we deduce that when $r=1$ we have

$$s_m = s_m^{\boldsymbol{\omega}}\frac{m+2}{m+4},\ \ \boldsymbol{\sigma}^L_{\boldsymbol{\omega}} = \gamma_{\frac{2s_m L^m}{m+2}} * \boldsymbol{\sigma}^L. \tag{2.18}$$

We deduce

$$\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}}\Big)_*\boldsymbol{\sigma}^L_{\boldsymbol{\omega}} = \Big(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\Big)_*\left(\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m^{\boldsymbol{\omega}} L^m}}}\Big)_*\boldsymbol{\sigma}^L_{\boldsymbol{\omega}}\right). \tag{2.19}$$

Using (2.18) we deduce that

$$\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}}\Big)_*\boldsymbol{\sigma}^L_{\boldsymbol{\omega}} = \gamma_{\frac{2}{m+2}} * \left(\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}}\Big)_*\boldsymbol{\sigma}^L\right). \tag{2.20}$$

Using the spectral estimates (1.2), the equality (2.19) and Theorem 1.3 with $r=1$ we deduce

$$\lim_{L\to\infty}\gamma_{\frac{2}{m+2}} * \left(\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}}\Big)_*\boldsymbol{\sigma}^L\right) = \lim_{L\to\infty}\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m^{\boldsymbol{\omega}} L^m}}}\Big)_*\boldsymbol{\sigma}^L_{\boldsymbol{\omega}} = \Big(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\Big)_*\boldsymbol{\sigma}_{m+1,1}.$$

We can now conclude by invoking Lévy's continuity theorem [19, Thm. 15.23(ii)]. Here are the details.

Denote by $\mu(\xi)$ and respectively $\mu^L_{\boldsymbol{\omega}}(\xi)$ the Fourier transforms of the measures

$$\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m^{\boldsymbol{\omega}} L^m}}}\Big)_*\boldsymbol{\sigma}^L_{\boldsymbol{\omega}} \ \text{ and respectively } \ \frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}}\Big)_*\boldsymbol{\sigma}^L$$

Observe that the Fourier transform of the Gaussian measure $\gamma_{\frac{2}{m+2}}$ is $e^{-\frac{1}{(m+2)}|\xi|^2}$. Then (2.20) implies

$$\mu^L(\xi) = e^{\frac{1}{(m+2)}|\xi|^2}\mu^L_{\boldsymbol{\omega}}(\xi). \tag{2.21}$$

Theorem 1.3 coupled with Levy's theorem imply that the family of functions $\mu^L_{\boldsymbol{\omega}}(\xi)$ has a limit $\mu^\infty_{\boldsymbol{\omega}}(\xi)$ as $L\to\infty$. Hence the family $\mu^L(\xi)$ has a limit $\mu^\infty(\xi)$ as $L\to\infty$ satisfying

$$\mu^\infty(\xi) = e^{\frac{1}{(m+2)}|\xi|^2}\mu^\infty_{\boldsymbol{\omega}}(\xi).$$

The limit $\mu^\infty_{\boldsymbol{\omega}}(\xi)$ is the Fourier transform of

$$\Big(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\Big)_*\boldsymbol{\sigma}_{m+1,1}.$$

Invoking Levy's theorem again, we deduce from (2.21) that the measures

$$\frac{1}{\boldsymbol{N}^L}\Big(\mathcal{R}_{\frac{1}{\sqrt{s_m^{\boldsymbol{\omega}}(L)}}}\Big)_*\boldsymbol{\sigma}^L$$

converge as $L\to\infty$ to a measure $\boldsymbol{\sigma}_m$ whose Fourier transform is $\mu^L(\xi)$. The equality

$$\boldsymbol{\gamma}_{\frac{2}{m+2}} * \boldsymbol{\sigma}_m = \Big(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\Big)_*\boldsymbol{\sigma}_{m,1},$$

is the Fourier inverse of the equality (2.21). □

2.4. **Proof of Corollary 1.5.** By invoking Levy's continuity theorem and Corollary 1.4 we see that is suffices to show that the probability measures $\boldsymbol{\sigma}_{m,1}$ converge weakly to the Gaussian measure γ_2.

Set

$$\bar{R}_m(x) := \sqrt{m}\rho_{m+1,1}(\sqrt{m}\,x) = \rho_{m+1,\frac{1}{m}}(x),$$

$$R_\infty(x) = \frac{1}{2\pi}\boldsymbol{I}_{\{|x|\leq 2\}}\sqrt{4-x^2}.$$

Fix $c \in (0,2)$. In [23, §4.2]. we proved that

$$\lim_{m\to\infty}\sup_{|x|\leq c}|\bar{R}_m(x) - R_\infty(x)| = 0, \tag{2.22a}$$

$$\sup_{|x|\geq c}|\bar{R}_m(x) - R_\infty(x)| = O(1) \;\text{ as } m\to\infty. \tag{2.22b}$$

We deduce that

$$\rho_{m+1,1}(\lambda)e^{-\frac{\lambda^2}{4}} = \sqrt{\frac{4\pi}{m}}\bar{R}_m\left(\frac{\lambda}{\sqrt{m}}\right)\frac{1}{\sqrt{4\pi}}e^{-\frac{\lambda^2}{4}}, \tag{2.23}$$

and

$$I_m := \int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{\lambda^2}{4}}d\lambda = \sqrt{\frac{4\pi}{m}}\int_{\mathbb{R}}\bar{R}_m(x)\sqrt{\frac{m}{4\pi}}e^{-\frac{mx^2}{4}}dx = \sqrt{\frac{4\pi}{m}}\int_{\mathbb{R}}\bar{R}_m(x)d\gamma_{\frac{2}{m}}(x).$$

The estimates (2.22a), (2.22b) imply that

$$I_m \sim \sqrt{4\pi}R_\infty(0)m^{-\frac{1}{2}} \;\text{ as } m\to\infty.$$

To prove that the probability measures

$$\frac{1}{I_m}\rho_{m+1,1}(\lambda)e^{-\frac{\lambda^2}{4}}d\lambda$$

converges weakly to $\boldsymbol{\gamma}_2$ it suffices to show that the finite measures

$$\nu_m := m^{\frac{1}{2}}\rho_{m+1,1}(\lambda)e^{-\frac{\lambda^2}{4}}d\lambda$$

converge weakly to the finite measure $\nu_\infty := R_\infty(0)e^{-\frac{\lambda^2}{4}}d\lambda$.

Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded continuous function. Using (2.23) we deduce that

$$\int_{\mathbb{R}}f(\lambda)d\nu_m(\lambda) = \int_{\mathbb{R}}f(\lambda)\bar{R}_m\big(m^{-\frac{1}{2}}\lambda\big)e^{-\frac{\lambda^2}{4}}d\lambda.$$

We deduce that

$$\begin{aligned}\int_{\mathbb{R}}f(\lambda)d\nu_m(\lambda) - \int_{\mathbb{R}}f(\lambda)d\nu_\infty(\lambda) &= \int_{\mathbb{R}}f(\lambda)\left(\bar{R}_m\big(m^{-\frac{1}{2}}x\big) - R_\infty(0)\right)e^{-\frac{\lambda^2}{4}}d\lambda \\ &= \underbrace{\int_{\mathbb{R}}f(\lambda)\boldsymbol{I}_{\{|\lambda|\leq c\sqrt{m}\}}\left(\bar{R}_m\big(m^{-\frac{1}{2}}x\big) - R_\infty(0)\right)e^{-\frac{\lambda^2}{4}}d\lambda}_{A_m} \\ &\quad + \underbrace{\int_{\mathbb{R}}f(\lambda)\boldsymbol{I}_{\{|\lambda|\geq c\sqrt{m}\}}\left(\bar{R}_m\big(m^{-\frac{1}{2}}x\big) - R_\infty(0)\right)e^{-\frac{\lambda^2}{4}}d\lambda}_{B_m}.\end{aligned}$$

The estimate (2.22a) coupled with the dominated convergence theorem imply that $A_m\to 0$ as $m\to\infty$. The estimate (2.22b) and the dominated convergence theorem imply that $B_m\to\infty$ as $m\to\infty$.

□

APPENDIX A. PROOF OF PROPOSITION 1.1

We will prove that there exists $L_0 > 0$ such that for any $L \geq L_0$ the space $\boldsymbol{U}^L$ is *ample*, i.e., for any $\boldsymbol{p} \in M$ and any $\xi \in T^*_{\boldsymbol{p}}M$ there exists $\boldsymbol{u} \in \boldsymbol{U}^L$ such that $d\boldsymbol{u}(\boldsymbol{p}) = \xi$. We can then invoke [22, Cor. 1.26] to conclude that the functions in $\boldsymbol{U}^L$ are a.s. Morse.

Choose smooth functions $f_1, \dots, f_N : M \to \mathbb{R}$ such that the map

$$M \ni \boldsymbol{p} \mapsto \big(f_1(\boldsymbol{p}), \dots, f_N(\boldsymbol{p}) \big) \in \mathbb{R}^N$$

is a smooth embedding. Denote by $\boldsymbol{F}$ the subspace of $C^\infty(M)$ spanned by the functions $f_1, \dots, f_N$ and by $P_L : L^2(M) \to \boldsymbol{U}^L$ the L^2-orthogonal projection onto $\boldsymbol{U}^L$.

Lemma A.1.

$$\lim_{L\to\infty} \sup_{f \in \boldsymbol{F}\setminus 0} \frac{\|f - P_L f\|_{C^2}}{\|f\|_{C^2}} = 0.$$

Proof. Fix a basis $\varphi_1, \dots, \varphi_\nu$ of $\boldsymbol{F}$, $\nu = \dim \boldsymbol{F}$ so that any $f \in \boldsymbol{F}$ has a unique decomposition

$$f = \sum_{i=1}^{\nu} x_i(f)\varphi_i, \quad x_i(f) \in \mathbb{R}.$$

Since $\dim \boldsymbol{F} < \infty$ the C^2-norm on $\boldsymbol{F}$ is equivalent with the norm

$$\|f\|^* := \sum_{i=1}^{\nu} |x_i(f)|.$$

We have

$$\|f - P_L f\|_{C^2} \leq \sum_{i=1}^{\nu} |x_i(f)| \|\varphi_i - P_L\varphi_i\|_{C^2} \leq \|f\|^* \max_{1\leq i\leq\nu} \|\varphi_i - P_L\varphi_i\|_{C^2}$$

$$\leq C\Big(\max_{1\leq i\leq\nu} \|\varphi_i - P_L\varphi_i\|_{C^2}\Big) \|f\|_{C^2},$$

for some constant $C > 0$. Now observe that

$$\max_{1\leq i\leq\nu} \|\varphi_i - P_L\varphi_i\|_{C^2} \to 0 \ \text{ as } \ L \to \infty.$$

□

To prove the ampleness of $\boldsymbol{U}^L$ for L large we argue by contradiction. Thus, we assume that for any positive integer n we can find $\boldsymbol{p}_n \in M$ and a tangent vector $X_n \in T_{\boldsymbol{p}_n}M$ such that

$$|X_n|_g = 1, \ \ d\boldsymbol{u}(X_n) = 0, \ \ \forall u \in \boldsymbol{U}^n.$$

Upon extracting a subsequence we can assume that $\boldsymbol{p}_n \to \boldsymbol{p}_\infty$ and $X_n \to X_\infty \in T_{\boldsymbol{p}_\infty}M$ as $n \to \infty$. Since the space $\boldsymbol{F}$ is obviously ample we can find $f_\infty \in \boldsymbol{F}$ such that $df_\infty(X_\infty) = 1$. Set $\boldsymbol{u}_n := P_n f_\infty$. Then $d\boldsymbol{u}_n(X_n) = 0$ for any n and

$$|df_\infty(X_n)| = \big| d\big(f_\infty(X_n) - \boldsymbol{u}_n(X_n) \big) \big| \leq \|f_\infty - P_n f_\infty\|_{C^2} \leq \varepsilon_n \|f_\infty\|_{C^2},$$

where $\varepsilon_n \to 0$ as $n \to \infty$ according to Lemma A.1. On the other hand

$$df_\infty(X_n) \to df_\infty(X_\infty) = 1.$$

This contradiction completes the proof of Proposition 1.1. □

APPENDIX B. GAUSSIAN MEASURES AND GAUSSIAN VECTORS

For the reader's convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [9]. A *Gaussian measure* on $\mathbb{R}$ is a Borel measure $\boldsymbol{\gamma}_{\mu,v}$, $v \geq 0$, $m \in \mathbb{R}$, of the form

$$\boldsymbol{\gamma}_{\mu,v}(dx) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(x-\mu)^2}{2v}} dx.$$

The scalar μ is called the *mean*, while v is called the *variance*. We allow v to be zero in which case

$$\boldsymbol{\gamma}_{\mu,0} = \delta_\mu = \text{the Dirac measure on } \mathbb{R} \text{ concentrated at } \mu.$$

For a real valued random variable X we write

$$X \in \boldsymbol{N}(\mu, v) \tag{B.1}$$

if the probability measure of X is $\boldsymbol{\gamma}_{\mu,v}$.

Suppose that $\boldsymbol{V}$ is a finite dimensional vector space with dual $\boldsymbol{V}^\vee$. A *Gaussian measure* on $\boldsymbol{V}$ is a Borel measure γ on $\boldsymbol{V}$ such that, for any $\xi \in \boldsymbol{V}^\vee$, the pushforward $\xi_*(\gamma)$ is a Gaussian measure on $\mathbb{R}$,

$$\xi_*(\gamma) = \boldsymbol{\gamma}_{\mu(\xi),\sigma(\xi)}.$$

One can show that the map $\boldsymbol{V}^\vee \ni \xi \mapsto \mu(\xi) \in \mathbb{R}$ is linear, and thus can be identified with a vector $\boldsymbol{\mu}_\gamma \in \boldsymbol{V}$ called the *barycenter* or *expectation* of γ that can be alternatively defined by the equality

$$\boldsymbol{\mu}_\gamma = \int_{\boldsymbol{V}} \boldsymbol{v} d\gamma(\boldsymbol{v}).$$

Moreover, there exists a nonnegative definite, symmetric bilinear map

$$\boldsymbol{\Sigma} : \boldsymbol{V}^\vee \times \boldsymbol{V}^\vee \to \mathbb{R} \ \text{ such that } \ \sigma(\xi)^2 = \boldsymbol{\Sigma}(\xi, \xi), \ \ \forall \xi \in \boldsymbol{V}^\vee.$$

The form $\boldsymbol{\Sigma}$ is called the *covariance form* and can be identified with a linear operator $\boldsymbol{S} : \boldsymbol{V}^\vee \to \boldsymbol{V}$ such that

$$\boldsymbol{\Sigma}(\xi, \eta) = \langle \xi, \boldsymbol{S}\eta \rangle, \ \ \forall \xi, \eta \in \boldsymbol{V}^\vee,$$

where $\langle -, - \rangle : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \mathbb{R}$ denotes the natural bilinear pairing between a vector space and its dual. The operator $\boldsymbol{S}$ is called the *covariance operator* and it is explicitly described by the integral formula

$$\langle \xi, \boldsymbol{S}\eta \rangle = \Lambda(\xi, \eta) = \int_{\boldsymbol{V}} \langle \xi, \boldsymbol{v} - \boldsymbol{\mu}_\gamma \rangle \langle \eta, \boldsymbol{v} - \boldsymbol{\mu}_\gamma \rangle d\gamma(\boldsymbol{v}).$$

The Gaussian measure is said to be *nondegenerate* if $\boldsymbol{\Sigma}$ is nondegenerate, and it is called *centered* if $\boldsymbol{\mu} = 0$. A nondegenerate Gaussian measure on $\boldsymbol{V}$ is uniquely determined by its covariance form and its barycenter.

Example B.1. Suppose that $\boldsymbol{U}$ is an n-dimensional Euclidean space with inner product $(-, -)$. We use the inner product to identify $\boldsymbol{U}$ with its dual $\boldsymbol{U}^\vee$. If $A : \boldsymbol{U} \to \boldsymbol{U}$ is a symmetric, positive definite operator, then

$$d\boldsymbol{\gamma}_A(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det A}} e^{-\frac{1}{2}(A^{-1}\boldsymbol{u}, \boldsymbol{u})} \, |d\boldsymbol{u}| \tag{B.2}$$

is a centered Gaussian measure on $\boldsymbol{U}$ with covariance form described by the operator A. □

If $\boldsymbol{V}$ is a finite dimensional vector space equipped with a Gaussian measure γ and $\boldsymbol{L} : \boldsymbol{V} \to \boldsymbol{U}$ is a linear map, then the pushforward $\boldsymbol{L}_*\gamma$ is a Gaussian measure on $\boldsymbol{U}$ with barycenter

$$\boldsymbol{\mu}_{\boldsymbol{L}_*\gamma} = \boldsymbol{L}(\boldsymbol{\mu}_\gamma)$$

and covariance form

$$\boldsymbol{\Sigma}_{\boldsymbol{L}_*\gamma} : \boldsymbol{U}^\vee \times \boldsymbol{U}^\vee \to \mathbb{R}, \;\; \boldsymbol{\Sigma}_{\boldsymbol{L}_*\gamma}(\eta,\eta) = \boldsymbol{\Sigma}_\gamma(\boldsymbol{L}^\vee\eta, \boldsymbol{L}^\vee\eta), \;\; \forall \eta \in \boldsymbol{U}^\vee,$$

where $\boldsymbol{L}^\vee : \boldsymbol{U}^\vee \to \boldsymbol{V}^\vee$ is the dual (transpose) of the linear map $\boldsymbol{L}$. Observe that if γ is nondegenerate and $\boldsymbol{L}$ is surjective, then $\boldsymbol{L}_*\gamma$ is also nondegenerate.

Suppose $(\mathcal{S}, \mu)$ is a probability space. A *Gaussian* random vector on $(\mathcal{S}, \mu)$ is a (Borel) measurable map

$$X : \mathcal{S} \to \boldsymbol{V}, \;\; \boldsymbol{V} \text{ finite dimensional vector space}$$

such that $X_*\mu$ is a Gaussian measure on $\boldsymbol{V}$. We will refer to this measure as the *associated Gaussian measure*, we denote it by γ_X and we denote by $\boldsymbol{\Sigma}_X$ (respectively $\boldsymbol{S}(X)$) its covariance form (respectively operator),

$$\boldsymbol{\Sigma}_X(\xi_1,\xi_2) = \boldsymbol{E}\big(\, \langle \xi_1, X - \boldsymbol{E}(X)\,\rangle\, \langle \xi_2, X - \boldsymbol{E}(X)\,\rangle\,\big).$$

Note that the expectation of γ_X is precisely the expectation of X. The random vector is called *nondegenerate*, respectively *centered*, if the Gaussian measure γ_X is such.

Let us point out that if $X : \mathcal{S} \to \boldsymbol{U}$ is a Gaussian random vector and $\boldsymbol{L} : \boldsymbol{U} \to \boldsymbol{V}$ is a linear map, then the random vector $\boldsymbol{L}X : \mathcal{S} \to \boldsymbol{V}$ is also Gaussian. Moreover

$$\boldsymbol{E}(\boldsymbol{L}X) = \boldsymbol{L}\boldsymbol{E}(X), \;\; \boldsymbol{\Sigma}_{\boldsymbol{L}X}(\xi,\xi) = \boldsymbol{\Sigma}_X(\boldsymbol{L}^\vee\xi, \boldsymbol{L}^\vee\xi), \;\; \forall \xi \in \boldsymbol{V}^\vee,$$

where $\boldsymbol{L}^\vee : \boldsymbol{V}^\vee \to \boldsymbol{U}^\vee$ is the linear map dual to $\boldsymbol{L}$. Equivalently, $\boldsymbol{S}(\boldsymbol{L}X) = \boldsymbol{L}\boldsymbol{S}(X)\boldsymbol{L}^\vee$.

Suppose that $X_j : \mathcal{S} \to \boldsymbol{V}_1$, $j = 1, 2$, are two *centered* Gaussian random vectors such that the direct sum $X_1 \oplus X_2 : \mathcal{S} \to \boldsymbol{V}_1 \oplus \boldsymbol{V}_2$ is also a centered Gaussian random vector with associated Gaussian measure

$$\gamma_{X_1 \oplus X_2} = p_{X_1 \oplus X_2}(\boldsymbol{x}_1, \boldsymbol{x}_2)|d\boldsymbol{x}_1 d\boldsymbol{x}_2|.$$

We obtain a bilinear form

$$\boldsymbol{cov}(X_1, X_2) : \boldsymbol{V}_1^\vee \times \boldsymbol{V}_2^\vee \to \mathbb{R}, \;\; \boldsymbol{cov}(X_1,X_2)(\xi_1,\xi_2) = \boldsymbol{\Sigma}(\xi_1,\xi_2),$$

called the *covariance form*. The random vectors X_1 and X_2 are independent if and only if they are uncorrelated, i.e.,

$$\boldsymbol{cov}(X_1, X_2) = 0.$$

We can then identify $\boldsymbol{cov}(X_1, X_2)$ with a linear operator $\boldsymbol{Cov}(X_1, X_2) : \boldsymbol{V}_2 \to \boldsymbol{V}_1$, via the equality

$$\begin{aligned} \boldsymbol{E}\big(\, \langle \xi_1, X_1 \rangle \langle \xi_2, X_2 \rangle\,\big) &= \boldsymbol{cov}(X_1,X_2)(\xi_1,\xi_2) \\ &= \big\langle\, \xi_1, \boldsymbol{Cov}(X_1,X_2)\xi_2^\dagger\,\big\rangle, \;\; \forall \xi_1 \in \boldsymbol{V}_1^\vee, \;\; \xi_2 \in \boldsymbol{V}_2^\vee, \end{aligned}$$

where $\xi_2^\dagger \in \boldsymbol{V}_2$ denotes the vector metric dual to ξ_2. The operator $\boldsymbol{Cov}(X_1, X_2)$ is called the *covariance operator* of X_1, X_2.

The conditional random variable $(X_1 | X_2 = x_2)$ has probability density

$$p_{(X_1|X_2=\boldsymbol{x}_2)}(\boldsymbol{x}_1) = \frac{p_{X_1\oplus X_2}(\boldsymbol{x}_1,\boldsymbol{x}_2)}{\int_{\boldsymbol{V}_1} p_{X_1\oplus X_2}(\boldsymbol{x}_1,\boldsymbol{x}_2)|d\boldsymbol{x}_1|}.$$

For a measurable function $f : \boldsymbol{V}_1 \to \mathbb{R}$ the conditional expectation $\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2)$ is the (deterministic) scalar

$$\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2) = \int_{\boldsymbol{V}_1} f(\boldsymbol{x}_1) p_{(X_1|X_2=\boldsymbol{x}_2)}(\boldsymbol{x}_1)|d\boldsymbol{x}_1|.$$

If X_2 is nondegenerate, the *regression formula*, [5], implies that the random vector $(X_1|X_2 = x_2)$ is a Gaussian vector with covariance operator

$$\boldsymbol{S}(X_1|X_2 = x_2) = \boldsymbol{S}(X_1) - \boldsymbol{Cov}(X_1,X_2)\boldsymbol{S}(X_2)^{-1}\boldsymbol{Cov}(X_2,X_1), \tag{B.3}$$

and expectation

$$\boldsymbol{E}(X_1|X_2 = x_2) = Cx_2, \tag{B.4}$$

where C is given by

$$C = \boldsymbol{Cov}(X_1, X_2)\boldsymbol{S}(X_2)^{-1}. \tag{B.5}$$

APPENDIX C. A CLASS OF RANDOM SYMMETRIC MATRICES

We denote by $\mathcal{S}_m$ the space of real symmetric $m \times m$ matrices. This is an Euclidean space with respect to the inner product $(A, B) := \operatorname{tr}(AB)$. This inner product is invariant with respect to the action of $\mathrm{SO}(m)$ on $\mathcal{S}_m$. We set

$$\widehat{\boldsymbol{E}}_{ij} := \begin{cases} \boldsymbol{E}_{ij}, & i = j \\ \frac{1}{\sqrt{2}}E_{ij}, & i < j. \end{cases}.$$

The collection $(\widehat{\boldsymbol{E}}_{ij})_{i\leq j}$ is a basis of $\mathcal{S}_m$ orthonormal with respect to the above inner product. We set

$$\hat{a}_{ij} := \begin{cases} a_{ij}, & i = j \\ \sqrt{2}a_{ij}, & i < j. \end{cases}$$

The collection $(\hat{a}_{ij})_{i\leq j}$ the orthonormal basis of $\mathcal{S}_m^\vee$ dual to $(\widehat{\boldsymbol{E}}_{ij})$. The volume density induced by this metric is

$$|dA| := \prod_{i\leq j} d\widehat{a}_{ij} = 2^{\frac{1}{2}\binom{m}{2}} \prod_{i\leq j} da_{ij}.$$

Throughout the paper we encountered a 2-parameter family of Gaussian probability measures on $\mathcal{S}_m$. More precisely for any real numbers u, v such that

$$v > 0, mu + 2v > 0,$$

we denote by $\mathcal{S}_m^{u,v}$ the space $\mathcal{S}_m$ equipped with the centered Gaussian measure $d\boldsymbol{\Gamma}_{u,v}(A)$ uniquely determined by the covariance equalities

$$\boldsymbol{E}(a_{ij}a_{k\ell}) = u\delta_{ij}\delta_{k\ell} + v(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \;\; \forall 1 \leq i, j, .k, \ell \leq m.$$

In particular we have

$$\boldsymbol{E}(a_{ii}^2) = u + 2v, \;\; \boldsymbol{E}(a_{ii}a_{jj}) = u, \quad \boldsymbol{E}(a_{ij}^2) = v, \;\; \forall 1 \leq i \neq j \leq m,$$

while all other covariances are trivial. The ensemble $\mathcal{S}_m^{0,v}$ is a rescaled version of the Gaussian Orthogonal Ensemble (GOE) and we will refer to it as GOE_m^v.

For $u > 0$ the ensemble $\mathcal{S}_m^{u,v}$ can be given an alternate description. More precisely a random $A \in \mathcal{S}_m^{u,v}$ can be described as a sum

$$A = B + \; X\mathbb{1}_m, \;\; B \in \mathrm{GOE}_m^v, \;\; X \in \boldsymbol{N}(0, u), \quad B \text{ and } X \text{ independent.}$$

We write this

$$\mathcal{S}_m^{u,v} = \mathrm{GOE}_m^v \,\hat{+} \boldsymbol{N}(0, u)\mathbb{1}_m, \tag{C.1}$$

where $\hat{+}$ indicates a sum of *independent* variables.

The Gaussian measure $d\boldsymbol{\Gamma}_{u,v}$ coincides with the Gaussian measure $d\boldsymbol{\Gamma}_{u+2v,u,v}$ defined in [23, App. B]. We recall a few facts from [23, App. B].

The probability density $d\boldsymbol{\Gamma}_{u,v}$ has the explicit description

$$d\boldsymbol{\Gamma}_{u,v}(A) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}\sqrt{D(u,v)}} e^{-\frac{1}{4v}\operatorname{tr} A^2 - \frac{u'}{2}(\operatorname{tr} A)^2}|dA|,$$

where

$$D(u,v)=(2v)^{(m-1)+\binom{m}{2}}\big(mu+2v\big),$$

and

$$u'=\frac{1}{m}\left(\frac{1}{mu+2v}-\frac{1}{2v}\right)=-\frac{u}{2v(mu+2v)}.$$

In the special case GOE^v_m we have $u=u'=0$ and

$$d\mathbf{\Gamma}_{0,v}(A)=\frac{1}{(2\pi v)^{\frac{m(m+1)}{4}}}e^{-\frac{1}{4v}\operatorname{tr}A^2}|dA|. \tag{C.2}$$

We have a *Weyl integration formula* [2] which states that if $f:\mathcal{S}_m\to\mathbb{R}$ is a measurable function which is invariant under conjugation, then the value $f(A)$ at $A\in\mathcal{S}_m$ depends only on the eigenvalues $\lambda_1(A)\leq\cdots\leq\lambda_n(A)$ of A and we have

$$\boldsymbol{E}_{\mathrm{GOE}^v_m}\big(f(X)\big)=\frac{1}{\boldsymbol{Z}_m(v)}\int_{\mathbb{R}^m}f(\lambda_1,\dots,\lambda_m)\underbrace{\left(\prod_{1\leq i<j\leq m}|\lambda_i-\lambda_j|\right)\prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}}_{=:Q_{m,v}(\lambda)}|d\lambda_1\cdots d\lambda_m|, \tag{C.3}$$

where the normalization constant $\boldsymbol{Z}_m(v)$ is defined by

$$\boldsymbol{Z}_m(v)=\int_{\mathbb{R}^m}\prod_{1\leq i<j\leq m}|\lambda_i-\lambda_j|\prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}|d\lambda_1\cdots d\lambda_m|$$

$$=(2v)^{\frac{m(m+1)}{4}}\underbrace{\int_{\mathbb{R}^m}\prod_{1\leq i<j\leq m}|\lambda_i-\lambda_j|\prod_{i=1}^m e^{-\frac{\lambda_i^2}{2}}|d\lambda_1\cdots d\lambda_m|}_{=:\boldsymbol{Z}_m}.$$

The precise value of $\boldsymbol{Z}_m$ can be computed via Selberg integrals, [2, Eq. (2.5.11)], and we have

$$\boldsymbol{Z}_m=(2\pi)^{\frac{m}{2}}m!\prod_{j=1}^m\frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1}{2})}=2^{\frac{m}{2}}m!\prod_{j=1}^m\Gamma\left(\frac{j}{2}\right). \tag{C.4}$$

For any positive integer n we define the *normalized* 1-point corelation function $\rho_{n,v}(x)$ of GOE^v_n to be

$$\rho_{n,v}(x)=\frac{1}{\boldsymbol{Z}_n(v)}\int_{\mathbb{R}^{n-1}}Q_{n,v}(x,\lambda_2,\dots,\lambda_n)d\lambda_1\cdots d\lambda_n.$$

For any Borel measurable function $f:\mathbb{R}\to\mathbb{R}$ we have [10, §4.4]

$$\frac{1}{n}\boldsymbol{E}_{\mathrm{GOE}^v_n}\big(\operatorname{tr}f(X)\big)=\int_{\mathbb{R}}f(\lambda)\rho_{n,v}(\lambda)d\lambda. \tag{C.5}$$

The equality (C.5) characterizes $\rho_{n,v}$. Let us observe that for any constant $c>0$, if

$$A\in\mathrm{GOE}^v_n\Longleftrightarrow cA\in\mathrm{GOE}^{c^2v}_n.$$

Hence for any Borel set $B\subset\mathbb{R}$ we have

$$\int_{cB}\rho_{n,c^2v}(x)dx=\int_B\rho_{n,v}(y)dy.$$

We conclude that

$$c\rho_{n,c^2v}(cy)=\rho_{n,v}(y),\ \ \forall n,c,y. \tag{C.6}$$

The behavior of the 1-point correlation function $\rho_{n,v}(x)$ for n large is described by *Wigner semicircle law* which states that for any $v>0$ the sequence of measures on $\mathbb{R}$

$$\rho_{n,vn^{-1}}(x)dx = n^{\frac{1}{2}}\rho_{n,v}(n^{\frac{1}{2}}x)dx$$

converges weakly as $n\to\infty$ to the semicircle distribution

$$\rho_{\infty,v}(x)|dx| = \boldsymbol{I}_{\{|x|\leq 2\sqrt{v}\}}\frac{1}{2\pi v}\sqrt{4v-x^2}|dx|.$$

The expected value of the absolute value of the determinant of of a random $A\in \mathrm{GOE}_m^v$ can be expressed neatly in terms of the correlation function $\rho_{m+1,v}$. More precisely, we have the following result first observed by Y.V. Fyodorov [15] in a context related to ours.

Lemma C.1. *Suppose $v>0$. Then for any $c\in\mathbb{R}$ we have*

$$\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\,|\det(A-c\mathbb{1}_m)|\,\big) = 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}}\Gamma\left(\frac{m+3}{2}\right)e^{\frac{c^2}{4v}}\rho_{m+1,v}(c).$$

Proof. Using the Weyl integration formula we deduce

$$\begin{aligned}
\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\,|\det(A-c\mathbb{1}_m)|\,\big) &= \frac{1}{\boldsymbol{Z}_m(v)}\int_{\mathbb{R}^m}\prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}|c-\lambda_i|\prod_{i\leq j}|\lambda_i-\lambda_j|d\lambda_1\cdots d\lambda_m\\
&= \frac{e^{\frac{c^2}{4v}}}{\boldsymbol{Z}_m(v)}\int_{\mathbb{R}^m}e^{-\frac{c^2}{4v}}\prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}|c-\lambda_i|\prod_{i\leq j}|\lambda_i-\lambda_j|d\lambda_1\cdots d\lambda_m\\
&= \frac{e^{\frac{c^2}{4v}}\boldsymbol{Z}_{m+1}(v)}{\boldsymbol{Z}_m(v)}\frac{1}{\boldsymbol{Z}_{m+1}(v)}\int_{\mathbb{R}^m}Q_{m+1,v}(c,\lambda_1,\dots,\lambda_m)d\lambda_1\cdots d\lambda_m\\
&= \frac{e^{\frac{c^2}{4v}}\boldsymbol{Z}_{m+1}(v)}{\boldsymbol{Z}_m(v)}\rho_{m+1,v}(c) = v^{\frac{m+1}{2}}\frac{e^{\frac{c^2}{4v}}\boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m}\rho_{m+1,v}(c)\\
&= (m+1)\sqrt{2}(2v)^{\frac{m+1}{2}}e^{\frac{c^2}{4v}}\Gamma\left(\frac{m+1}{2}\right)\rho_{m+1,v}(c) = 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}}\Gamma\left(\frac{m+3}{2}\right)e^{\frac{c^2}{4v}}\rho_{m+1,v}(c).
\end{aligned}$$

□

The above result admits the following generalization, [3, Lemma 3.2.3].

Lemma C.2. *Let $u>0$. Then*

$$\boldsymbol{E}_{\mathbb{S}_m^{u,v}}\big(\,|\det(A-c\mathbb{1}_m)|\,\big) = 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}}\rho_{m+1,v}(c-x)e^{\frac{(c-x)^2}{4v}-\frac{x^2}{2u}}dx.$$

In particular, if $u=2kv$, $k<1$ we have

$$\boldsymbol{E}_{\mathbb{S}_m^{2kv,v}}\big(\,|\det(A-c\mathbb{1}_m)|\,\big) = 2^{\frac{3}{2}}(2v)^{\frac{m}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi k}}\int_{\mathbb{R}}\rho_{m+1,v}(c-x)e^{-\frac{1}{4vt_k^2}(x+t_k^2c)^2+\frac{(t_k^2+1)c^2}{4v}}dx,$$

$(\lambda := c-x)$

$$= 2^{\frac{3}{2}}(2v)^{\frac{m}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi k}}\int_{\mathbb{R}}\rho_{m+1,v}(\lambda)e^{-\frac{1}{4vt_k^2}(\lambda-(t_k^2+1)c)^2+\frac{(t_k^2-1)c^2}{4v}}d\lambda,$$

where

$$t_k^2 := \frac{1}{\frac{1}{k}-1} = \frac{k}{1-k}.$$

Proof. Recall the equality (C.1) $\mathcal{S}_m^{u,v} = \mathrm{GOE}_m^v \hat{+} \boldsymbol{N}(0,u)\mathbb{1}_m$. We deduce that

$$\boldsymbol{E}_{\mathcal{S}_m^{u,v}}\big(\,|\det(A-c\mathbb{1}_m)|\,\big) = \boldsymbol{E}\big(\det(B+(X-c)\mathbb{1})|\,\big)$$

$$= \frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}} \boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\,|\det(B-(c-X)\mathbb{1}_m)|\;\big|\;X=x\big)e^{-\frac{x^2}{2u}}dx$$

$$= \frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}} \boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\,|\det(B-(c-x)\mathbb{1}_m)|\,\big)e^{-\frac{x^2}{2u}}dx$$

$$= 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}}\rho_{m+1,v}(c-x)e^{\frac{(c-x)^2}{4v}-\frac{x^2}{2u}}dx.$$

Now observe that if $u = 2kv$ then

$$\frac{(c-x)^2}{4v} - \frac{x^2}{2u} = -\frac{x^2}{4kv} + \frac{1}{4v}(x^2-2cx+c^2)$$

$$= \frac{1}{4v}\left(-\frac{1}{t_k^2}x^2 - 2cx - c^2t_k^2\right) + \frac{c^2(1+t_k^2)}{4v} = -\frac{1}{4vt_k^2}(x+t_k^2c)^2 + \frac{c^2(1+t_k^2)}{4v}.$$

□

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
E-mail address: nicolaescu.1@nd.edu
URL: http://www.nd.edu/~lnicolae/