Topics in topology. Spring 2010.
Pseudo-differential operators and some of their geometric applications[1]

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Contents

Introduction

Notations and terminology

- For any real number c we set
$$\mathbb{Z}_{\geq c} := \{ n \in \mathbb{Z};\ \ n \geq c \}.$$
The sets $\mathbb{Z}_{>c}$, $\mathbb{Z}_{<c}$ etc. are defined similarly.
- For any multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbb{Z}_{\geq 0}$ we set
$$|\alpha| := \sum_{i=1}^{m} \alpha_i,\ \ \alpha! := \alpha_1! \cdot \alpha_2! \cdots \alpha_m!.$$
- For any finite dimensional real vector space V we denote by $\Lambda^k V$ its k-th exterior power and we set
$$\Lambda^k_{\mathbb{C}} V := \Lambda^k V \otimes \mathbb{C}.$$
- If A, B are subset of a topological Hausdorff space, then we write $A \Subset B$ if the closure $\bar{A}$ of A si compact and contained in the interior of B.
- If M is a smooth manifold and E is a finite dimensional vector space we denote by $\underline{E}_M$ the trivial smooth vector bundle $M \times E \to M$.
- We will use the notation $\operatorname{tr} A$ to denote the trace of a *finite dimensional* linear operator, and $\boldsymbol{Tr}\ A$ the trace of an *infinite dimensional* linear operator, whenever this trace is well defined.

Chapter 1

The Fourier transform and Sobolev spaces

1.1. The Fourier transform

In the sequel $\boldsymbol{V}$ will denote real Euclidean space of dimension m. We denote by $(-,-)$ the inner product on $\boldsymbol{V}$, by $|-|$ the Euclidean norm, and by $|dx|$ the Euclidean volume element. We let $\boldsymbol{\omega}_m$ denote the volume of the unit ball in $\boldsymbol{V}$ and by $\boldsymbol{\sigma}_{m-1}$ the "area" of the unit sphere in $\boldsymbol{V}$ so that (see [N, Ex. 9.1.10]

$$\boldsymbol{\omega}_m = \frac{\Gamma(1/2)^m}{\Gamma(1+m/2)} = \begin{cases} \frac{\pi^k}{k!}, & m=2k \\ \frac{2^{2k+1}\pi^k k!}{(2k+1)!}, & m=2k+1, \end{cases}, \quad \boldsymbol{\sigma}_{m-1} = m\boldsymbol{\omega}_m = \frac{2\Gamma(1/2)^m}{\Gamma(m/2)}. \tag{1.1.1}$$

We fix an orthonormal basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ on $\boldsymbol{V}$ and we denote by $(x_1, \dots, x_m)$ the resulting coordinates. For $1 \leq j \leq m$ we define

$$\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}, \quad D_{x_j} := \frac{1}{\boldsymbol{i}}\partial_{x^j} = -\boldsymbol{i}\partial_{x_j}.$$

For every multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$ we set

$$x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha^m}, \quad \partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}, \quad , |\alpha| := \sum_{j=1}^m \alpha_j, \quad D_x^\alpha := \frac{1}{\boldsymbol{i}^{|\alpha|}}\partial_x^\alpha.$$

Finally, for $x \in \boldsymbol{V}$ we set

$$\langle x \rangle = \big(1 + |x|^2\big)^{1/2}.$$

We will sometime need the following classical equality.

Lemma 1.1.1. *For any $s > m/2$ and any $u > 0$ we have*

$$\int_{\boldsymbol{V}} (u^2 + |x|^2)^{-s}\, |dx| = u^{m-2s} \int_{\boldsymbol{V}} (1+|y|^2)^{-s} |dx| = u^{m-2s} \frac{\boldsymbol{\sigma}_{m-1}\Gamma(p)\Gamma(s-p)}{2\Gamma(s)}, \tag{1.1.2}$$

where Γ denotes Euler's Gamma function and $p = \frac{m-2}{2}$.

Proof. The first equality follows by via the change in variables $x = uy$. Next we observe that

$$\int_{\boldsymbol{V}} (1+|y|^2)^{-s}\,|dy| = \boldsymbol{\sigma}_{m-1}\int_0^\infty \frac{r^{m-1}}{(1+r^2)^s}\,dr = \frac{\boldsymbol{\sigma}_{m-1}}{2}\int_0^\infty \frac{t^{(m-2)/2}}{(1+t)^s}dt.$$

The last integral can be described in terms of Euler's Gamma function (see[**WW**, Sec. 12.41])

$$\int_0^\infty \frac{t^{(m-2)/2}}{(1+t)^s}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(s)},\ \ p=\frac{m-2}{2},\ \ q=s-p.$$

□

We have For any smooth function $f:\boldsymbol{V}\to\mathbb{C}$, and any non-negative integer s we set

$$\boldsymbol{p}_s(f) = \sup_{x\in\boldsymbol{V},\ 0\leq|\alpha|\leq s}\langle x\rangle^s|D_x^\alpha f(x)|.$$

A smooth function $f:\boldsymbol{V}\to\mathbb{C}$ is said to have *fast decay* if

$$\boldsymbol{p}_s(f)<\infty,\ \ \text{for any } s\in\mathbb{Z}_{\geq 0}.$$

We denote by $\mathcal{S}(\boldsymbol{V})$ the vector space of smooth functions $\boldsymbol{V}\to\mathbb{C}$ with fast decay. Note that

$$f\in\mathcal{S}(\boldsymbol{V}) \Longleftrightarrow \forall\alpha,\beta\in\mathbb{Z}^m_{\geq 0},\ \ \sup_{x\in\boldsymbol{V}}\Big|\,x^\alpha D_x^\beta f(x)\,\Big|<\infty. \tag{1.1.3}$$

The space $\mathcal{S}(\boldsymbol{V})$ is equipped with a natural locally convex[1] topology. A set $\mathcal{N}\subset\mathcal{S}(\boldsymbol{V})$ is a neighborhood of 0 in this topology if and only if there exists $s\in\mathbb{Z}_{\geq 0}$ and $\varepsilon>0$ such that $\mathcal{N}$ contains all the functions $f\in\mathcal{S}(\boldsymbol{V})$ satisfying $\boldsymbol{p}_s(f)_s<\varepsilon$.

A sequence of functions $f_n\in\mathcal{S}(\boldsymbol{V})$ converges to $f\in\mathcal{S}$ in this topology if and only if

$$\forall\varepsilon>0,\ \forall s\geq 0,\ \exists N>0:\ \ \boldsymbol{p}_s(f_n-f)\leq\varepsilon,\ \ \forall n\geq N. \tag{1.1.4}$$

We will refer to this topology as the *natural topology* of $\mathcal{S}(\boldsymbol{V})$.

For any vector $\boldsymbol{v}\in\boldsymbol{V}$ and any multi-index α we define $E_{\boldsymbol{v}}, T_{\boldsymbol{v}}, M_{x^\alpha}:\mathcal{S}(\boldsymbol{V})\to\mathcal{S}(\boldsymbol{V})$

$$E_{\boldsymbol{v}}f(x) := e^{\boldsymbol{i}(\boldsymbol{v},x)}f(x),\ \ T_{\boldsymbol{v}}f(x):=f(x+\boldsymbol{v}),\ \ M_{x^\alpha}f(x)=x^\alpha f(x).$$

For every $j=1,\dots,m$ and any $h\in\mathbb{R}$ we define

$$T_j^h := T_{h\boldsymbol{e}_j},\ \ \Delta_j^h := T_j^h f - f.$$

The proof of the following elementary fact is left as an exercise.

Proposition 1.1.2. *(a) For any $p\in[1,\infty]$ we have*

$$\mathcal{S}(\boldsymbol{V})\subset L^p(\boldsymbol{V},|dx|).$$

(b) For any $j=1,\dots,m$ the linear operators

$$M_{x_j},\partial_j:\mathcal{S}(\boldsymbol{V})\to\mathcal{S}(\boldsymbol{V}),\ \ f\mapsto\partial_j f=\frac{\partial f}{\partial x^j}$$

are continuous with respect to the natural topology on $\mathcal{S}$.

(c) For any $j=1,\dots,m$ and any $f\in\mathcal{S}(\boldsymbol{V})$. we have

$$\lim_{h\to 0}\frac{1}{h}\Delta_j^h f = \partial_j f,\ \ \textit{in the topology of } \mathcal{S}(\boldsymbol{V}).$$

□

[1] A topological vector space is called *locally convex* if any neighborhood of 0 contains a convex neighborhood.

Proposition 1.1.3 (Integration by parts)**.** *Let $u : \boldsymbol{V} \to \mathbb{C}$ be a smooth function. Suppose that there exists $C, k > 0$ such that*

$$|\partial_j u(x)| \leq C(1 + |x|^k), \ \ \forall x \in \boldsymbol{V}, \ \ j = 1, \dots, m.$$

Then for any $f \in \mathcal{S}(\boldsymbol{V})$ and any $j = 1, \dots, m$ the functions $\partial_j uf$ and $u\partial_j f$ are integrable and moreover

$$\int_{\boldsymbol{V}} \partial_j u(x) f(x)\, |dx| = -\int u(x) \partial_j f(x)\, |dx| \tag{1.1.5}$$

Proof. Note that the growth condition on the partial derivatives of u implies via the mean value theorem that for some constant $C_0 > 0$ we have

$$|u(x)| \leq C_0(1 + |x|^{k+1}), \ \ \forall x \in \boldsymbol{V}.$$

The integrability of $(\partial_j u)f$ and $u(\partial_j f)$ follows from the growth properties of u, f and their derivatives.

From the divergence formula we deduce

$$\int_{|x|\leq R} \partial_{x_j} u(x) f(x)\, |dx| = \int_{|x|=R} u(x) f(x) (\boldsymbol{n}_x, \boldsymbol{e}_j) d\sigma_R(x) - \int_{|x|\leq R} u(x) \partial_{x_j} f(x)\, |dx|, \tag{1.1.6}$$

where $d\sigma_R$ denotes the "area" element on the sphere $\{|x| = R\}$, $\boldsymbol{n}$ denotes the unit outer normal vector field along this sphere, while the inner product $(\boldsymbol{n}_x, \boldsymbol{e}_j)$ is equal to $\frac{x_j}{R}$. Now observe that

$$\left| \int_{|x|=R} u(x) f(x) (\boldsymbol{n}_x, \boldsymbol{e}_j) d\sigma_R(x) \right| = \left| \frac{1}{R} \int_{|x|=R} u(x) f(x) x_j d\sigma_R(x) \right|$$

$$\leq \frac{C_0 \boldsymbol{p}_s(f)(1 + R^{k+1})}{R^s} \int_{|x|=R} d\sigma_R(x) = \frac{\boldsymbol{\sigma}_{m-1} \boldsymbol{p}_s(f) C_0 (1 + R^{k+1})}{R^{s-m+1}}.$$

If we let $s > m + k$ we deduce

$$\lim_{R\to\infty} \frac{\boldsymbol{\sigma}_{m-1} \boldsymbol{p}_s(f) C_0 (1 + R^{k+1})}{R^{s-m+1}} = 0.$$

The equality (1.1.5) now follows by letting $R \to \infty$ in (1.1.6). □

For simplicity we set

$$|dx|_* := (2\pi)^{-m/2} |dx|. \tag{$*$}$$

Definition 1.1.4. The *Fourier transform* of a function $f \in \mathcal{S}(\boldsymbol{V})$ is the function $\widehat{f} : \boldsymbol{V} \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) := \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(\xi, x)} f(x) |dx|_*.$$ □

Proposition 1.1.5. *If $f \in \mathcal{S}(\boldsymbol{V})$ then $\widehat{f} \in \mathcal{S}(\boldsymbol{V})$. Moreover, for any $j = 1, \dots, m$, and any $bv \in \boldsymbol{V}$ we have*

$$\widehat{T_{\boldsymbol{v}} f} = E_{\boldsymbol{v}} \widehat{f}, \ \ \widehat{E_{\boldsymbol{v}} f} = T_{-\boldsymbol{v}} \widehat{f}, \tag{1.1.7}$$

$$\widehat{D_{x_j} f} = M_{\xi_j} \widehat{f}, \tag{1.1.8}$$

$$\widehat{M_{x_j} f} = -D_{\xi_j} \widehat{f}. \tag{1.1.9}$$

Proof. The equalities (1.1.7) follow by direct computation.

Let us first observe that $\widehat{f}$ is smooth. For any $j = 1, \dots, m$ we have

$$\partial_{\xi_j}\Big(e^{-\boldsymbol{i}(\xi,x)} f(x) \Big) = -\boldsymbol{i} x_j e^{-\boldsymbol{i}(\xi,x)} f(x) \in \mathcal{S}(\boldsymbol{V}).$$

Invoking classical theorems on the differentiability of integrals depending on parameters we deduce that $\widehat{f}$ is smooth and

$$\partial_{\xi_j}\widehat{f} = -\boldsymbol{i}\int_{\boldsymbol{V}} x_j e^{-\boldsymbol{i}(\xi,x)} f(x)|dx|_* = -\boldsymbol{i}\,\widehat{M_{x_j}f}$$

which proves (1.1.9). Observe that (1.1.8) follows from the integration by parts formula (1.1.5).

Let us prove that $\widehat{f} \in \mathcal{S}(\boldsymbol{V})$. From (1.1.9) we deduce that for any multi-indices α and β we have

$$\xi^\alpha D_\xi^\beta \widehat{f}(\xi) = (-1)^{|\beta|}\widehat{D_x^\alpha M_x^\beta f}.$$

The smooth function $g = D_x^\alpha M_x^\beta f$ has fast decay so it suffices to show that for any $g \in \mathcal{S}(\boldsymbol{V})$ the Fourier transform $\widehat{g}$ is bounded. We have

$$|\widehat{g}(\xi)| \leq \int_{\boldsymbol{V}} |g(x)|\,|dx| < \infty$$

since the functions in $\mathcal{S}(\boldsymbol{V})$ are Lebesgue integrable. Hence

$$\sup_{\xi\in\boldsymbol{V}}\Big|\,\xi^\alpha D_\xi^\beta \widehat{f}(\xi)\,\Big| \leq \|D_x^\alpha M_x^\beta f\|_{L^1(E)}.$$

Using (1.1.3) we deduce $\widehat{f} \in \mathcal{S}(\boldsymbol{V})$. □

The Fourier transform thus defines a linear map $\boldsymbol{\mathcal{F}} : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ which is also continuous (Exercise 1.4).

Example 1.1.6. Consider the gaussian function $\Gamma_{\boldsymbol{V}} \in \mathcal{S}(\boldsymbol{V})$ given by $\Gamma_{\boldsymbol{V}}(x) = e^{-|x|^2/2}$. We want to prove that

$$\boldsymbol{\mathcal{F}}[\Gamma_{\boldsymbol{V}}] = \Gamma_{\boldsymbol{V}}. \tag{1.1.10}$$

We follow the elegant approach of L. Hörmander [**H1**, §7.1]. Observe first that

$$(x_j + \boldsymbol{i}D_{x_j})\Gamma_{\boldsymbol{V}} = \big(x_j + \partial_{x_j}\big)\Gamma_{\boldsymbol{V}} = 0, \;\; \text{so that} \;\; \big(-D_{\xi_j} + \boldsymbol{i}\xi_j\big)\widehat{\Gamma}_{\boldsymbol{V}} = 0.$$

This implies that $\widehat{\Gamma}_{\boldsymbol{V}}(\xi) = ce^{-|\xi|^2/2}$, where

$$(2\pi)^{m/2}c = (2\pi)^{m/2}\widehat{\Gamma}_{\boldsymbol{V}}(0) = \int_{\boldsymbol{V}} e^{-|x|^2/2}\,|dx| = \prod_{j=1}^m\left(\int_{-\infty}^{\infty} e^{-x_j^2/2}\,|dx_j|\right) = (2\pi)^{m/2}.$$

This proves (1.1.10). □

Theorem 1.1.7 (Fourier Inversion Formula)**.** *The Fourier transform $\boldsymbol{\mathcal{F}} : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ is bijective and its inverse is given by*

$$\boldsymbol{\mathcal{F}}^{-1} = R\circ\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}\circ R,$$

where $R : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ is the reflection operator

$$(Rf)\,(x) = f(-x), \;\; \forall f \in \mathcal{S}(\boldsymbol{V}), x \in \boldsymbol{V}.$$

In other words, $f \in \mathcal{S}(\boldsymbol{V})$ *can be recovered from* $\widehat{f}$ *via the* Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^{m/2}} \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} \widehat{f}(\xi)\, |d\xi| = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} \widehat{f}(\xi)\, |d\xi|_*. \tag{1.1.11}$$

Proof. We consider the operator $\mathcal{J} = R \circ \boldsymbol{\mathfrak{F}}^2 : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ and we will prove that $\mathcal{J} = \mathbb{1}$. From the equalities (1.1.7), (1.1.9) and (1.1.10) we deduce

$$\mathcal{J} \circ M_{x_j} = M_{x_j} \circ \mathcal{J},\;\; \mathcal{J} \circ T_{\boldsymbol{v}} = T_{\boldsymbol{v}} \circ \mathcal{J},\;\; \forall j = 1, \dots, m,\;\; \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{1.1.12a}$$

$$\mathcal{J}[\Gamma_{\boldsymbol{V}}] = \Gamma_{\boldsymbol{V}}. \tag{1.1.12b}$$

For $f \in \mathcal{S}(\boldsymbol{V})$ we have[2]

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(\,tx\,)\, dt = \sum_{j=1}^m x_j \underbrace{\int_0^1 (\partial_j f)\,(\,tx\,)\, dt}_{=:\widetilde{f}_j(x)}$$

Clearly the functions $\widetilde{f}_j$ are smooth and have moderate growth. We have

$$f(x) - f(0) = \sum_{j=1}^m M_{x_j} \widetilde{f}_j$$

If $f(0) = 0$ then we deduce from (1.1.12a) that

$$\mathcal{J}[f] = \sum_{j=1}^n M_{x_j} \mathcal{J}[\widetilde{f}_j]$$

so that $\mathcal{J}[f](0) = 0$. If now $g \in \mathcal{S}(\boldsymbol{V})$, $c = g(0)$, then the function $f = g - c\Gamma_{\boldsymbol{V}}$ vanishes at 0 which shows that

$$\mathcal{J}[g](0) = c \cdot \mathcal{J}[\Gamma_{\boldsymbol{V}}](0) = g(0).$$

Using the translation invariance of $\mathcal{J}$ we deduce that for any $\boldsymbol{v} \in \boldsymbol{V}$ we have

$$\mathcal{J}[f](\boldsymbol{v}) = (T_{\boldsymbol{v}}\mathcal{J})[f](0) = \mathcal{J}[T_{\boldsymbol{v}} f](0) = (T_{\boldsymbol{v}} f)\,(0) = f(\boldsymbol{v}).$$

In a similar fashion we conclude that $\mathcal{J} = \boldsymbol{\mathfrak{F}} \circ R \circ \boldsymbol{\mathfrak{F}} = \mathbb{1}$ so that $\boldsymbol{\mathfrak{F}}^{-1} = R \circ \boldsymbol{\mathfrak{F}} = \boldsymbol{\mathfrak{F}} \circ R$. □

The Fourier inversion formula has several important consequences. Let $(-,-)_{L^2}$ denote the inner product in $L^2(\boldsymbol{V}, |dx|)$

$$(f,g)_{L^2} := \int_{\boldsymbol{V}} f(x)\overline{g(x)}\, |dx|,\;\; ; \forall f,g \in L^2(\boldsymbol{V}, |dx|).$$

Corollary 1.1.8 (Parseval formula)**.** *For any* $f, g \in \mathcal{S}(\boldsymbol{V})$ *we have*

$$(\widehat{f}, \widehat{g})_{L^2} = (f,g)_{L^2}.$$

[2]This trick is sometimes called Hadamard's lemma.

Proof. If $f, g \in \mathcal{S}(\boldsymbol{V})$ then

$$(\widehat{f}, \widehat{g})_{L^2} = \int_{\boldsymbol{V}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,|d\xi| = \int_{\boldsymbol{V}} \left(\widehat{f}(\xi) \int_{\boldsymbol{V}} e^{\boldsymbol{i}(\xi,x)}\,\overline{g(x)}\,|dx|_* \right) |d\xi|$$

$$= \int_{\boldsymbol{V}} \left(\int_{\boldsymbol{V}} e^{\boldsymbol{i}(\xi,x)}\,\widehat{f}(\xi)\,|d\xi|_* \right) \overline{g(x)}\,|dx| \stackrel{(1.1.11)}{=} \int_{\boldsymbol{V}} f(x)\,\overline{g(x)}\,|dx| = (f,g)_{L^2}.$$

□

Corollary 1.1.9. *For every $f, g \in \mathcal{S}(\boldsymbol{V})$ we have*

$$\int_{\boldsymbol{V}} f(x)g(x)|dx| = \int_{\boldsymbol{V}} \widehat{f}(\xi)\widehat{g}(-\xi)|d\xi|. \tag{1.1.13}$$

Proof. This follows from Parseval formula since

$$\int_{\boldsymbol{V}} f(x)g(x)|dx| = (f, \overline{g})_{L^2} = (\widehat{f}, \widehat{\overline{g}})_{L^2} = \int_{\boldsymbol{V}} \widehat{f}(\xi)\widehat{g}(-\xi)|d\xi|.$$

□

A final operation we want to discuss is the *convolution*. Given $f, g \in \mathcal{S}(\boldsymbol{V})$ we define $f*g : \boldsymbol{V} \to \mathbb{C}$ via the integral formula

$$f * g(x) = \int_{\boldsymbol{V}} f(x-y)g(y)\,|dy| \int_{\boldsymbol{V}} f(-z)g(z+x)|dz| = \int_{\boldsymbol{V}} Rf(z)\,T_x g(z)\,|dz|. \tag{1.1.14}$$

Lemma 1.1.10. *$f * g \in \mathcal{S}(\boldsymbol{V})$ for any $f, g \in \mathcal{S}(\boldsymbol{V})$.*

Proof. It is not hard to see that $f * g \in C^\infty(\boldsymbol{V})$. To prove that $f * g$ has fast decay at ∞ we will rely the following elementary inequality known as *Peetre's inequality*

$$\langle u+v \rangle^s \leq 2^{|s|/2} \langle u \rangle^s \langle v \rangle^{|s|}, \quad \forall u, v \in \boldsymbol{V}, \quad s \in \mathbb{R} \tag{1.1.15}$$

We will present a proof of this inequality a bit later.

Observe that

$$D^\alpha(f * g) = (f * D^\alpha g)$$

For any integers $N, \nu > 0$ there exists a constant $C > 0$ such that

$$|f(-z)| \leq C\langle z \rangle^{-N},$$

$$|D^\alpha g(-z+x)| \leq C\langle x - z\rangle^{-\nu} \stackrel{(1.1.15)}{\leq} C\langle x \rangle^{-\nu}\langle z \rangle^{\nu}$$

so that

$$|f(-z)D^\alpha g(-z+x)| \leq C\langle z \rangle^{\nu - N}\langle x \rangle^{-\nu}.$$

If we choose $N > \nu + m$ then the function $\langle z \rangle^{\nu - N}$ is integrable on $\boldsymbol{V}$ and we deduce

$$|D_x^\alpha (f * g)(x)| \leq \int_{\boldsymbol{V}} |f(-z)D^\alpha g(-z+x)|\,|dz| \leq C\langle x \rangle^{-\nu} \left(\int_{\boldsymbol{V}} \langle z \rangle^{\nu - N}\,|dz| \right).$$

□

Proof of Peetre's inequality. We have

$$(1+|u+v|^2) \leq 2(1+|u|^2)(1+|v|^2),$$

so that, if $s \geq 0$ we have

$$\langle u+v\rangle^s \leq 2^{s/2}\langle u\rangle^s\langle v\rangle^s.$$

In particular, if $t \geq 0$ we have

$$\langle u\rangle^t = \langle v-(v+u)\rangle \leq 2^{t/2}\langle v\rangle^t\langle u+v\rangle^t$$

so that

$$\langle v\rangle^s\langle u+v\rangle^{-t} \leq 2^{t/2}\langle u\rangle^{-t}\langle v\rangle^t$$

which proves (1.1.15) for $s=-t\leq 0$. □

The Fourier transform interacts nicely with this operation. More precisely, we have

$$\widehat{f*g}(\xi) = (2\pi)^{m/2}\widehat{f}(\xi)\widehat{g}(\xi),\ \ \forall f,g\in\mathcal{S}(\boldsymbol{V}),\ \ \xi\in\boldsymbol{V}. \tag{1.1.16}$$

Indeed, if we denote by $|dxdy|$ the volume element on $\boldsymbol{V}\times\boldsymbol{V}$ we have

$$\widehat{f*g}(\xi) = \frac{1}{(2\pi)^{m/2}}\int_{\boldsymbol{V}\times\boldsymbol{V}}\left(e^{-\boldsymbol{i}(\xi,x)}f(x-y)g(y)\right)|dxdy|$$

$(z=x-y)$

$$= \frac{1}{(2\pi)^{m/2}}\int_{\boldsymbol{V}\times\boldsymbol{V}}\left(e^{-\boldsymbol{i}(\xi,y+z)}f(z)g(y)\right)|dzdy|$$

$$\stackrel{\text{Fubini}}{=} (2\pi)^{m/2}\left(\int_{\boldsymbol{V}}e^{-\boldsymbol{i}(\xi,z)}f(z)\,|dz|_*\right)\cdot\left(\int_{\boldsymbol{V}}e^{-\boldsymbol{i}(\xi,y)}g(y)\,|dy|_*\right) = (2\pi)^{m/2}\widehat{f}(\xi)\widehat{g}(\xi).$$

From the Fourier inversion formula we deduce

$$\widehat{(fg)}(\xi) = (2\pi)^{-m/2}\big(\widehat{f}*\widehat{g}\big)(-\xi),\ \ \forall f,g\in\mathcal{S}(\boldsymbol{V}),\ \ \xi\in\boldsymbol{V}. \tag{1.1.17}$$

1.2. Temperate distributions

A *temperate* or *tempered distribution* is a continuous, $\mathbb{C}$-linear map $u:\mathcal{S}(\boldsymbol{V})\to\mathbb{C}$. Observe that a linear function $u:\mathcal{S}(\boldsymbol{V})\to\mathbb{C}$ is continuous if and only if

$$\exists s\in\mathbb{Z}_{\geq 0},\ \ \exists C>0:\ \ |u(f)|\leq C\boldsymbol{p}_s(f),\ \ \forall f\in\mathcal{S}(\boldsymbol{V}).$$

We denote by $\mathcal{S}(\boldsymbol{V})^\mathsf{v}$ the vector space of temperate distributions on $\boldsymbol{V}$. We have a natural bilinear map

$$\langle -,-\rangle:\mathcal{S}(\boldsymbol{V})^\mathsf{v}\times\mathcal{S}(\boldsymbol{V})\to\mathbb{C},\ \ \mathcal{S}(\boldsymbol{V})^\mathsf{v}\times\mathcal{S}(\boldsymbol{V})\ni(u,f)\mapsto\langle u,f\rangle:=u(f).$$

The space $\mathcal{S}(\boldsymbol{V})^\mathsf{v}$ is equipped with a natural topology, namely the smallest topology such that for any $f\in\mathcal{S}(\boldsymbol{V})$ the maps

$$\mathcal{S}(\boldsymbol{V})^\mathsf{v}\ni u\mapsto\langle u,f\rangle\in\mathbb{C}$$

are continuous. The open sets of this topology are unions of polyhedra $\mathcal{P}(A)$, where A an arbitrary *finite* subset $A\subset\mathcal{S}(\boldsymbol{V})$ and

$$\mathcal{P}(A) = \big\{\,u\in\mathcal{S}(\boldsymbol{V})^\mathsf{v};\ \ \big|\langle u,\alpha\rangle\big|<1,\ \ \forall\alpha\in A\,\big\}. \tag{1.2.1}$$

We will refer to this topology as the *weak topology* on $\mathcal{S}(\boldsymbol{V})^\mathsf{v}$.

Example 1.2.1. (a) If $p \in [1, \infty]$, then any function $\varphi \in L^p(\boldsymbol{V}, |dx|)$ defines a temperate distribution

$$u_\varphi : \mathcal{S}(\boldsymbol{V}) \to \mathbb{C}, \;\; \langle u_\varphi, f \rangle = \langle\!\langle \varphi, f \rangle\!\rangle := \int_{\boldsymbol{V}} \varphi(x) f(x)\, |dx|, \;\; \forall f \in \mathcal{S}(\boldsymbol{V}).$$

The functional u_φ uniquely determines φ so that the space $L^p(\boldsymbol{V}, |dx|)$ is naturally a subspace of $\mathcal{S}(\boldsymbol{V})^{\mathsf{v}}$.

(b) Suppose $\varphi : \boldsymbol{V} \setminus \to \mathbb{C}$ is a locally integrable function with polynomial growth, i.e., there exists an integer $k > 0$ and $R > 0$ such that

$$\sup_{|x| \geq R} \langle x \rangle^{-k} |\varphi(x)\, |dx| < \infty.$$

We get a continuous linear functional $u_\varphi : \mathcal{S}(\boldsymbol{V}) \to \mathbb{C}$,

$$\langle u_\varphi, f \rangle = \langle\!\langle \varphi, f \rangle\!\rangle = \int_{\boldsymbol{V}} \varphi(x) f(x)\, |dx|.$$

This shows that the locally integrable functions with polynomial growth can be viewed as temperate distributions. The functions $\varphi(x) = |x|^\lambda$, $\lambda > -\dim V$ have this property and thus they define temperate distributions. □

Example 1.2.2. (a) For any $\boldsymbol{x}_0 \in \boldsymbol{V}$ we define the *Dirac distribution* concentrated at $\boldsymbol{x}_0$ to be the temperate distribution $\delta_{\boldsymbol{x}_0}$ defined by the linear map

$$\delta_{\boldsymbol{x}_0} : \mathcal{S}(\boldsymbol{V}) \to \mathbb{C}, \;\; \langle \delta_{\boldsymbol{x}_0}, f \rangle = f(\boldsymbol{x}_0).$$

One can verify easily that $\delta_{\boldsymbol{x}_0}$ is indeed continuous. Often, in the physics literature, the distribution δ_0 is viewed as a function $\delta(x)$ that is identically 0 outside the origin, it has the value ∞ at the origin and

$$\int_{\boldsymbol{V}} \delta(x)\, |dx| = 1.$$

In this notation we have $\delta_{\boldsymbol{x}_0} = \delta(x - \boldsymbol{x}_0)$.

(b) Suppose M is a submanifold of $\boldsymbol{V}$ such that the embedding $M \hookrightarrow E$ is proper. The metric on $\boldsymbol{V}$ induces a volume density $|dv_M|$ on $\boldsymbol{V}$. Assume the $|dv_M|$ has polynomial growth, i.e.,

$$\int_{M \cap \{|x| \leq R\}} |dv_M| = O(R^k) \;\text{ as } R \to \infty.$$

Then we can define a tempered distribution

$$\delta_M : \mathcal{S}(\boldsymbol{V}) \to \mathbb{C}, \;\; \langle \delta_M, f \rangle = \int_M f(x) |dv_M(x)|, \;\; \forall f \in \mathcal{S}(\boldsymbol{V}).$$

When $M = \{\boldsymbol{x}_0\}$ the distribution δ_M coincides with the Dirac distribution at $\boldsymbol{x}_0$. Other interesting case is when M is a linear subspace of E. For example when $E = \mathbb{R}^2$, and Δ is the diagonal subspace

$$\Delta = \big\{ (x, y) \in \mathbb{R}^2; \;\; x = y \big\},$$

then

$$\langle \delta_\Delta, f \rangle = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} f(x, x)\, |dx|, \;\; \forall f \in \mathcal{S}_{\mathbb{R}^2}.$$ □

Example 1.2.3. Let $\varphi \in \mathcal{S}(\boldsymbol{V})$ be a nonnegative function such that

$$\int_{\boldsymbol{V}} \varphi(x)\,|dx| = 1.$$

For any $\varepsilon > 0$ we define $\varphi_\varepsilon \in \mathcal{S}(\boldsymbol{V})$ by

$$\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi(x/\varepsilon).$$

Then

$$\lim_{\varepsilon\to} \varphi_\varepsilon = \delta_0 \;\text{ in the weak topology of } \mathcal{S}(\boldsymbol{V})^\vee.$$

In other words, we have to prove that for any $f \in \mathcal{S}(\boldsymbol{V})$ we have

$$\lim_{\varepsilon\searrow 0} \langle\!\langle \varphi_\varepsilon, f\rangle\!\rangle = f(0).$$

To see this note first that

$$\int_{\boldsymbol{V}} \varphi_\varepsilon\,|dx| = 1$$

so that

$$\langle\!\langle \varphi_\varepsilon, f\rangle\!\rangle - f(0) = \int_{\boldsymbol{V}} \varphi_\varepsilon f\,|dx| - f(0)\int_{\boldsymbol{V}} \varphi_\varepsilon\,|dx|$$

$$= \varepsilon^{-m}\int_{\boldsymbol{V}} \varphi(x/\varepsilon)(f(x)-f(0))\,|dx| = \int_{\boldsymbol{V}} \varphi(x)\big(\,f(\varepsilon x) - f(0)\,\big)\,|dx|$$

Observe that

$$\sup_{x\in\boldsymbol{V}} |f(\varepsilon x) - f(0)| \leq 2\sup_{x\in\boldsymbol{V}} |f(x)|,$$

and $\lim_{\varepsilon\to 0} f(\varepsilon x) = f(0)$, $\forall x \in \boldsymbol{V}$. The dominated convergence theorem now implies that the last integral above converges to 0 as $\varepsilon \searrow 0$. □

The continuous linear map

$$M_{x^\alpha} : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V}).$$

extends by to a continuous linear map

$$M_{x^\alpha} : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee,\;\; \langle M_{x^\alpha}u, f\rangle = \langle u, M_{x^\alpha}f\rangle,\;\; \forall (u,f) \in \mathcal{S}(\boldsymbol{V})^\vee \times \mathcal{S}(\boldsymbol{V}).$$

For any $\lambda > 0$ we have a rescaling map

$$S_\lambda : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V}),\;\; (S_\lambda f)(x) = f(\lambda x).$$

Observe that for any $f, g \in \mathcal{S}(\boldsymbol{V})$, and any $\lambda > 0$ we have

$$\langle u_{S_\lambda f}, g\rangle = \langle\!\langle S_\lambda f, g\rangle\!\rangle = \int_{\boldsymbol{V}} f(\lambda x) g(x)\,|dx|$$

$$\overset{y=\lambda x}{=} \lambda^{-m}\int_{\boldsymbol{V}} f(y) S_{\lambda^{-1}} g(y)\,|dy| = \langle u_f, \lambda^{-m} S_{\lambda^{-1}} g\rangle.$$

This allows us to define $S_\lambda : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$ by

$$\langle S_\lambda u, g\rangle = \langle u, \lambda^{-m} S_{\lambda^{-1}} g\rangle,\;\; \forall u \in \mathcal{S}(\boldsymbol{V})^\vee,\;\; g \in \mathcal{S}(\boldsymbol{V}).$$

Similarly, the reflection operator $R : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ and the translation operators $T_{\boldsymbol{v}}$, $\boldsymbol{v} \in \boldsymbol{V}$, extend to operators $R, T_{\boldsymbol{v}} : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$

$$\langle Ru, g\rangle := \langle u, Rg\rangle,\;\; \langle T_{\boldsymbol{v}} u, f\rangle := \langle u, T_{-\boldsymbol{v}} g\rangle \;\; \forall u \in \mathcal{S}(\boldsymbol{V})^\vee,\;\; g \in \mathcal{S}(\boldsymbol{V}).$$

Let us observe that if $\varphi \in \mathcal{S}(\boldsymbol{V})$, then for any $j = 1, \dots, m$ and any $f \in \mathcal{S}(\boldsymbol{V})$ we have

$$\langle u_{\partial_j \varphi}, f\rangle = \langle\!\langle \partial_j \varphi, f\rangle\!\rangle = \int_{\boldsymbol{V}} \partial_j \varphi f\,|dx| \stackrel{(1.1.5)}{=} -\int_{\boldsymbol{V}} \varphi \partial_j f\,|dx| = -\langle u_\varphi, \partial_j f\rangle.$$

Using this as inspiration we define the *weak* or *distributional derivative* $\partial_j u$ of a temperate distribution u to be the linear functional $\partial_j u : \mathcal{S}(\boldsymbol{V}) \to \mathbb{C}$ determined by

$$\langle \partial_j u, f\rangle := -\langle u, \partial_j f\rangle, \ \ \forall f \in \mathcal{S}(\boldsymbol{V}).$$

Example 1.2.4. (a) Observe that

$$\langle D_x^\alpha \delta_0, f\rangle = (-1)^{|\alpha|} D_x^\alpha f(0), \ \ \forall f \in \mathcal{S}(\boldsymbol{V}). \tag{1.2.2}$$

(b) Consider the Heaviside function $\boldsymbol{\theta} : \mathbb{R} \to \mathbb{R}$,

$$\boldsymbol{\theta}(t) := \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Then $\boldsymbol{\theta} \in L^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^\vee$ and its distributional derivative $\partial_t \boldsymbol{\theta}$ is the Dirac distribution δ_0. □

We can also define the Fourier transform of a distribution. Observe that if $f \in \mathcal{S}(\boldsymbol{V})$ then for any $g \in \mathcal{S}(\boldsymbol{V})$ we have

$$\langle\!\langle \widehat{f}, g\rangle\!\rangle = \int_{\boldsymbol{V}} \widehat{f}(\xi) g(\xi)\,|d\xi| = (2\pi)^{m/2} \int_{\boldsymbol{V}} f(x) \left(\int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\xi)} g(\xi) |d\xi|_* \right) |dx|_*$$

$$= \int_{\boldsymbol{V}} f(x)\,\widehat{g}(x)\,|dx| = \langle\!\langle f, \widehat{g}\rangle\!\rangle = \langle\, u_f, \widehat{g}\,\rangle.$$

Following this pattern we define the *Fourier transform* of a temperate distribution $u \in \mathcal{S}(\boldsymbol{V})^\vee$ to be the linear functional $\widehat{u} : \mathcal{S}(\boldsymbol{V}) \to \mathbb{R}$ given by

$$\langle \widehat{u}, f\rangle := \langle u, \widehat{f}\rangle, \ \ \forall f \in \mathcal{S}(\boldsymbol{V}).$$

In other words, the extension $\widetilde{\boldsymbol{\mathfrak{F}}}$ of $\boldsymbol{\mathfrak{F}}$ to $\mathcal{S}(\boldsymbol{V})^\vee$ is none other than the dual of the map

$$\boldsymbol{\mathfrak{F}} : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V}),$$

i.e.,

$$\langle \widetilde{\boldsymbol{\mathfrak{F}}}[u], f\rangle = \langle u, \boldsymbol{\mathfrak{F}}[f]\rangle, \ \ \forall u \in \mathcal{S}(\boldsymbol{V})^\vee, \ \ f \in \mathcal{S}(\boldsymbol{V}).$$

Example 1.2.5. (a) Consider the distribution given by the constant function 1. Then

$$\langle \widehat{1}, f\rangle = \langle 1, \widehat{f}\rangle = \int_{\boldsymbol{V}} f(\xi)\,|d\xi|$$

$$= (2\pi)^{m/2} \int_{\boldsymbol{V}} e^{\boldsymbol{i}(0,\xi)} \widehat{f}(\xi)\,|d\xi|_* = (2\pi)^{m/2} f(0) = (2\pi)^{m/2} \langle \delta_0, f\rangle.$$

Hence

$$\widehat{1} = (2\pi)^{m/2} \delta_0. \tag{1.2.3}$$

A simple computation shows

$$\widehat{\delta_0} = (2\pi)^{-m/2} 1. \tag{1.2.4}$$

More generally for any $v \in \boldsymbol{V}$ we have

$$\langle \widehat{\delta_v}, f\rangle = \langle \delta_v, \widehat{f}\rangle = \widehat{f}(v) = \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(v,x)} f(x)\,|dx|_*$$

so that

$$\widehat{\delta_v}(\xi) = \frac{1}{(2\pi)^{/2}} e^{-\boldsymbol{i}(v,\xi)} \in L^\infty(\boldsymbol{V}) \cap C^0(\boldsymbol{V}). \tag{1.2.5}$$

(b) For any multi-index α the monomial x^α is a function with polynomial growth and thus can be viewed as temperate distribution. For any $f \in \mathcal{S}(\boldsymbol{V})$ we have

$$\langle \widehat{x^\alpha}, f \rangle = \langle\!\langle x^\alpha, \widehat{f} \rangle\!\rangle = \int_{\boldsymbol{V}} \xi^\alpha \widehat{f}(\xi)\,|d\xi| \stackrel{(1.1.8)}{=} \int_{\boldsymbol{V}} \widehat{D_x^\alpha f}(\xi)\,|d\xi| = (2\pi)^{m/2} D_x^\alpha f(0).$$

Using (1.2.2) we deduce

$$\widehat{x^\alpha} = (-1)^{|\alpha|}(2\pi)^{m/2} D_x^\alpha \delta_0. \tag{1.2.6}$$

□

The Fourier transform thus defines a linear map $\boldsymbol{\mathfrak{F}} : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$. We leave the proof of the following result as an exercise to the reader.

Proposition 1.2.6. *The Fourier transform $\widetilde{\boldsymbol{\mathfrak{F}}} : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$ is a continuous linear, bijective map. Moreover $\boldsymbol{\mathfrak{F}}^{-1} = \widetilde{\boldsymbol{\mathfrak{F}}}\widetilde{R}$, where $\widetilde{R} = R^\vee : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$ is the extension to $\boldsymbol{\mathcal{S}}(\boldsymbol{V})^\vee$ of the reflection operator.*

Proof. For any $u \in \mathcal{S}(\boldsymbol{V})^\vee$ and $\alpha \in \mathcal{S}(\boldsymbol{V})$ we have

$$\big\langle\, \widetilde{\boldsymbol{\mathfrak{F}}}(u), \alpha \,\big\rangle = \big\langle\, u, \boldsymbol{\mathfrak{F}}(\alpha) \,\big\rangle.$$

This shows that if A is a fine subset of $\mathcal{S}(\boldsymbol{V})$ and u belongs to the neighborhood $\mathcal{P}\big(\boldsymbol{\mathfrak{F}}(A)\,\big)$ defined as in (1.2.1) then $\widetilde{\boldsymbol{\mathfrak{F}}}(u) \in \mathcal{P}(A)$. This proves the continuity of the map $\widetilde{\boldsymbol{\mathfrak{F}}} : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$.

The Fourier inversion formula implies that

$$R \circ \boldsymbol{\mathfrak{F}} \circ \boldsymbol{\mathfrak{F}} = \boldsymbol{\mathfrak{F}} \circ R \circ \boldsymbol{\mathfrak{F}} = \mathbb{1}_{\boldsymbol{\mathcal{S}}(\boldsymbol{V})}.$$

Passing to duals we deduce

$$\boldsymbol{\mathfrak{F}}^\vee \circ \boldsymbol{\mathfrak{F}}^\vee \circ R^\vee = \boldsymbol{\mathfrak{F}}^\vee \circ R^\vee \circ \boldsymbol{\mathfrak{F}}^\vee = \mathbb{1}_{\mathcal{S}(\boldsymbol{V})^\vee}.$$

Since $\boldsymbol{\mathfrak{F}}^\vee = \widetilde{\boldsymbol{\mathfrak{F}}}$ the above equalities prove that $\widetilde{\boldsymbol{\mathfrak{F}}}$ is bijective with inverse $\boldsymbol{\mathfrak{F}}^\vee \circ R^\vee$. □

In the sequel we will continue to denote by $\boldsymbol{\mathfrak{F}}$ the extension of $\boldsymbol{\mathfrak{F}}$ to $\boldsymbol{\mathcal{S}}(\boldsymbol{V})$.

We have seen that $L^2(\boldsymbol{V}, |dx|)$ can be identified with a subspace of $\mathcal{S}(\boldsymbol{V})^\vee$. It behaves rather nicely with respect to the Fourier transform. More precisely, we have the following result.

Proposition 1.2.7 (Plancherel)**.** *The Fourier transform $\boldsymbol{\mathfrak{F}} : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$ maps $L^2(\boldsymbol{V}, |dx|) \subset \mathcal{S}(\boldsymbol{V})^\vee$ into $L^2(\boldsymbol{V}, |dx|)$ and the resulting map $\boldsymbol{\mathfrak{F}} : L^2(\boldsymbol{V}, |dx|) \to L^2(\boldsymbol{V}, |dx|)$ is an isomorphism of Hilbert spaces.*

Proof. We know that $\mathcal{S}(\boldsymbol{V})$ is dense in $L^2(\boldsymbol{V}, |dx|)$ and $\boldsymbol{\mathfrak{F}}$ maps $\mathcal{S}(\boldsymbol{V})$ bijectively onto itself and

$$\big\|\, \boldsymbol{\mathfrak{F}}(u) - \boldsymbol{\mathfrak{F}}(v) \,\big\|_{L^2} = \|u - v\|_{L^2}, \;\; \forall u, v \in \mathcal{S}(\boldsymbol{V}).$$

Let us first show that

$$\boldsymbol{\mathfrak{F}}\big(\, L^2(\boldsymbol{V}) \,\big) \subset L^2(\boldsymbol{V}).$$

Let $f \in L^2(\boldsymbol{V}, |dx|)$ then there exist functions $f_\in \mathcal{S}(\boldsymbol{V})$ such that $f_n \to f$ in L^2, as $n \to \infty$. Then

$$\lim_{j,k\to\infty} \|f_j - f_k\|_{L^2} = 0$$

and (see Exercise 1.5)

$$f_n \to f \ \text{ in } \mathcal{S}(\boldsymbol{V})^{\vee}. \tag{1.2.7}$$

From the Parseval formula we deduce

$$\|\widehat{f}_j - \widehat{f}_k\|_{L^2} = \|f_j - f_k\|_{L^2}.$$

This proves that the sequence $(\widehat{f}_n)_{n\geq 0} \subset L^2(\boldsymbol{V})$ is Cauchy. Since $L^2(\boldsymbol{V}, |dx|)$ is a complete space, there exists $g \in L^2(\boldsymbol{V}, |dx|)$ such that $\widehat{f}_n \to g$ in $L^2(\boldsymbol{V}, |dx|)$ as $n \to \infty$. In other words, $\boldsymbol{\mathfrak{F}}(f_n) \to g$ in $L^2(\boldsymbol{V}, |dx|)$ as $n \to \infty$. Invoking Exercise 1.5 again we deduce

$$\boldsymbol{\mathfrak{F}}(f_n) \to g \in \mathcal{S}(\boldsymbol{V})^{\vee} \ \text{ as } n \to \infty.$$

On the other hand, using Proposition 1.2.6 and (1.2.7) we deduce that

$$\lim_{n\to\infty} \boldsymbol{\mathfrak{F}}(f_n) = \boldsymbol{\mathfrak{F}}\big(\lim_{n\to\infty} f_n\big) \ \text{ in } \mathcal{S}(\boldsymbol{V})^{\vee}.$$

Hence $\boldsymbol{\mathfrak{F}}(f) = g \in L^2(\boldsymbol{V}, |dx|)$.

Conversely, let us show that

$$L^2(\boldsymbol{V}) \subset \boldsymbol{\mathfrak{F}}\big(\, L^2(\boldsymbol{V})\,\big).$$

Let $g \in L^2(\boldsymbol{V})$. Then there exist $g_n \in \mathcal{S}(\boldsymbol{V})$ such that $g_n \to g$ in L^2. Set $f_n = \boldsymbol{\mathfrak{F}}^{-1}(g_n)$. Since $\boldsymbol{\mathfrak{F}}^{-1}$ is an L^2-isometry and the sequence (g_n) is Cauchy in the norm L^2, we deduce that the sequence (f_n) is Cauchy in the same norm and thus there exists $f \in L^2(\boldsymbol{V})$ such that $f_n \to f$ in L^2. We conclude as above that $g = \boldsymbol{\mathfrak{F}}(f) \in \boldsymbol{\mathfrak{F}}\big(\, L^2(\boldsymbol{V})\,\big)$. □

Finally we want to define the operation of convolution of a temperate distribution u with a function $\varphi \in \mathcal{S}(\boldsymbol{V})$ using (1.1.14) as a guide. Note that we can rewrite (1.1.14) as

$$f * g(x) = \langle Ru_f, T_x g\rangle.$$

If $u \in \mathcal{S}(\boldsymbol{V})^{\vee}$ and $g \in \mathcal{S}(\boldsymbol{V})$ then we define $u * g : \boldsymbol{V} \to \mathbb{C}$ by

$$u * g(x) = \langle Ru, T_x g\rangle.$$

The convolution formulæ (1.1.16) and (1.1.17) and generalize to temperate distributions

$$\widehat{\varphi * u} = (2\pi)^{m/2} M_{\widehat{\varphi}}\widehat{u}, \ \ \widehat{M_{\varphi} u} = (2\pi)^{-m/2} R\widehat{\varphi} * \widehat{u}, \ \ \forall \varphi \in \mathcal{S}(\boldsymbol{V}), \ \ u \in \mathcal{S}(\boldsymbol{V})^{\vee}. \tag{1.2.8}$$

1.3. Other spaces of distributions

Let Ω be an open subset of $\boldsymbol{V}$. We denote by $\mathcal{E}(\Omega)$ the vector space of complex valued smooth functions on Ω. For every nonnegative integer ν and every compact subset $K \subset \Omega$

$$\boldsymbol{p}_{\nu,K} : \mathcal{E}(\Omega) \to [0,\infty), \ \ \boldsymbol{p}_{\nu,K}(f) = \sup_{x\in K, \ |\alpha|\leq \nu} |D_x^{\alpha} f(x)|.$$

We define a linear topology on $\mathcal{E}(\Omega)$ such that a basis of open neighborhoods of $0 \in \mathcal{E}(\Omega)$ is given by the collection

$$\mathcal{N}_{\nu,\varepsilon,K} = \big\{ f \in \mathcal{E}(\Omega); \ \ \boldsymbol{p}_{\nu,K}(f) < \varepsilon \,\big\}, \ \ \nu \in \mathbb{Z}_{\geq 0}, \ \ \varepsilon > 0, \ \ K \subset \Omega \text{ compact}.$$

Observe that we have a canonical continuous linear map

$$\mathcal{S}(\boldsymbol{V}) \to \mathcal{E}(\Omega), \ \ \mathcal{S}(\boldsymbol{V}) \ni f \mapsto f|_{\Omega}.$$

Denote by $\mathcal{D}(\Omega)$ the subspace of $\mathcal{E}(\Omega)$ consisting of smooth functions with compact support. For any compact subset $K \subset \Omega$ denote by $\mathcal{D}_K(\Omega)$ the subspace of $\mathcal{D}(\Omega)$ consisting of functions with support in K. Note that

$$\mathcal{D}(\Omega) = \bigcup_K \mathcal{D}_K(\Omega).$$

The space $\mathcal{D}_K(\Omega)$ admits a natural linear topology such that a basis of open neighborhoods of the origin in $\mathcal{D}_K(\Omega)$ is given by the sets

$$\mathcal{O}_{\nu,\varepsilon,K} = \mathcal{N}_{\nu,\varepsilon,K} \cap \mathcal{D}_K(\Omega) := \left\{ f \in \mathcal{D}_K(\Omega);\;\; \boldsymbol{p}_{\nu,K}(f) < \varepsilon \right\},\;\; \nu \in \mathbb{Z}_{\geq 0}.$$

The natural topology on $\mathcal{D}(\Omega)$ is the largest locally convex topology such that all the inclusion maps $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$ are continuous.

For a proof of the following result we refer to [**Schw**, §III.1,2] or [**Tr**, Ch.13,14].

Theorem 1.3.1. *(a) If F is a locally convex topological vector space and $L : \mathcal{D}(\Omega) \to F$ is a linear map, then L is continuous if and only if for any compact set $K \subset \Omega$ the restriction $L : \mathcal{D}_K(\Omega) \to F$ is continuous.*

(b) A sequence $(f_n) \subset \mathcal{D}(\Omega)$ converges in the topology of $\mathcal{D}(\Omega)$ to $f \in \mathcal{D}(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that

$$\operatorname{supp} f \subset K,\;\; \operatorname{supp} f_n \subset K,\;\; \forall n \;\; \textit{and} \;\; f_n \to f \;\; \textit{in}\; \mathcal{D}_K(\Omega). \qquad \square$$

Observe that if $\Omega_1 \subset \Omega_2$ then $\mathcal{D}(\Omega_1) \subset \mathcal{D}(\Omega_2)$ and the canonical inclusion $\mathcal{D}(\Omega_1) \hookrightarrow \mathcal{D}(\Omega_2)$ is continuous. Note also that the natural inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{S}(\boldsymbol{V})$ is also continuous.

We now denote by $\mathcal{D}(\Omega)^{\mathsf{v}}$ the vector space of continuous linear functionals $u : \mathcal{D}(\Omega) \to \mathbb{C}$. We will refer to the elements in $\mathcal{D}(\Omega)^{\mathsf{v}}$ as *distributions*. on Ω.

From Theorem 1.3.1(a) we deduce that a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is continuous if and only if for any compact set $K \subset \Omega$ there exists an integer $\nu = \nu_K \geq 0$ and a constant $C_K > 0$ such that

$$|u(f)| \leq C_K p_{\nu_K,K}(f),\;\; \forall f \in \mathcal{D}_K(\Omega).$$

Again we have a natural pairing

$$\langle -,- \rangle : \mathcal{D}(\Omega)^{\mathsf{v}} \times \mathcal{D}(\Omega) \to \mathbb{C},\;\; \langle u, f \rangle := u(f),\;\; \forall (u,f) \in \mathcal{D}(\Omega)^{\mathsf{v}} \times \mathcal{D}(\Omega).$$

Just like the space of temperate distributions we can equip $\mathcal{D}(\Omega)^{\mathsf{v}}$ with a *weak topology*. This is the smallest topology on $\mathcal{D}(\Omega)^{\mathsf{v}}$ such for any $f \in \mathcal{D}(\Omega)$ that the linear map

$$\mathcal{D}(\Omega)^{\mathsf{v}} \to \mathbb{C},\;\; u \mapsto \langle u, f \rangle$$

is continuous. The open sets of this topology are unions of polyhedra $\mathcal{P}(F)$, where F an arbitrary *finite* subset $F \subset \mathcal{D}(\Omega)$ and

$$\mathcal{P}(F) = \left\{ u \in \mathcal{D}(\Omega)^{\mathsf{v}};\;\; \left| \langle u, f \rangle \right| < 1,\;\; \forall f \in F \right\}.$$

Example 1.3.2. Any smooth function $f \in \mathcal{E}(\Omega)$ defines a distribution $u_f \in \mathcal{D}(\Omega)^{\mathsf{v}}$ by setting

$$\langle u_f, g \rangle = \langle\!\langle f, g \rangle\!\rangle = \int_\Omega fg\,|dx|,\;\; \forall g \in \mathcal{D}(\omega).$$

The above integral is well defined since the integrand fg is continuous and has compact support. Thus we have a natural embedding

$$\mathcal{E}(\Omega) \hookrightarrow \mathcal{D}(\Omega)^{\mathsf{v}}$$

and the resulting map is continuous with respect to the natural topology on $\mathcal{E}(\Omega)$ and the weak topology on $\mathcal{D}(\Omega)^\vee$. For this reason the distributions are sometime called *generalized functions*. □

The distributional derivatives of a generalized function $u \in \mathcal{D}(\Omega)^\vee$ are defined as before

$$\langle \partial_{x_j} u, \varphi \rangle := -\langle u, \partial_{x_j}\varphi \rangle, \;\; \forall \varphi \in \mathcal{D}(\Omega).$$

Example 1.3.3. Observe that if $f \in C^\infty(\Omega)$ then

$$\partial_{x_j} u_f = u_{\partial_{x_j} f} \;\text{ in }\; \mathcal{D}(\omega)^\vee.$$ □

Note that for any open subset $\mathcal{O} \to \Omega$ we have an inclusion $\mathcal{D}(\mathcal{O}) \to \mathcal{D}(\Omega)$ and by duality, a map $\mathcal{D}(\Omega)^\vee \to \mathcal{D}(\mathcal{O})^\vee$ called the *restriction to* $\mathcal{O}$ of a distribution on Ω. We say that a distribution $u \in \mathcal{D}(\Omega)^\vee$ *vanishes on the open set* $\mathcal{O} \subset \Omega$ if it has a trivial restriction to $\mathcal{O}$. Equivalently, this means that

$$\langle u, f \rangle = 0, \;\; \forall f \in \mathcal{D}(\Omega), \;\; \operatorname{supp} f \subset \mathcal{O}.$$

Lemma 1.3.4. *Suppose $u \in \mathcal{D}(\Omega)^\vee$ and $(\mathcal{O}_i)_{i \in I}$ is a family of open subsets of Ω such that vanishes on $\mathcal{O}_i$, $\forall i \in I$. Then u vanishes on the union of the open sets $\mathcal{O}_i$.*

Proof. Set $\mathcal{O} := \bigcup_{i \in I} \mathcal{O}_i$. We need to show that

$$\langle u, f \rangle = 0, \;\; \forall f \in \mathcal{D}(\mathcal{O}).$$

Let $f \in \mathcal{D}(\mathcal{O})$. Since $\operatorname{supp} f$ is compact there exists a finite subset $J \subset I$ such that

$$\operatorname{supp} f \subset \mathcal{O}_J := \bigcup_{j \in J} \mathcal{O}_j.$$

We can now choose a partition of unity subordinated to the cover $(\mathcal{O}_j)_{j \in J}$, that is, a collection of functions $\{\, \varphi_j \in C^\infty(\mathcal{O}_J) \,\}_{j \in J}$ such that

$$\operatorname{supp} \varphi_j \subset \mathcal{O}_j, \;\; \forall j \in J \;\text{ and }\; \sum_{j \in J} \varphi_j = 1.$$

We set $f_j := \varphi_j f$. Then $f_j \in \mathcal{D}(\mathcal{O}_j)$, so that $\langle u, f_j \rangle = 0$. From the equality $f = \sum_j f_j$ we deduce

$$\langle u, f \rangle = \sum_j \langle u, f_j \rangle = 0.$$

□

For any $u \in \mathcal{D}(\Omega)^\vee$ we denote by $\mathcal{O}_u$ the union of all the open subsets $\mathcal{O} \subset \Omega$ such that u vanishes on $\mathcal{O}$. Then $\mathcal{O}_u$ is an open subset of Ω, and u vanishes on $\mathcal{O}_u$. The complement $\Omega \setminus \mathcal{O}_u$ is called the *support* of u and it is denoted by $\operatorname{supp} u$. Clearly, $\operatorname{supp} u$ is a closed subset of Ω.

We define $\mathcal{E}(\Omega)^\vee$ as the space of continuous linear functionals $u : \mathcal{E}(\Omega) \to \mathbb{C}$, that is, linear functions $u : \mathcal{E}(\Omega) \to \mathbb{C}$ such that there exists a compact set $K \subset \Omega$, and integer $\nu \geq 0$ and a constant $C >$ so that

$$|u(f)| \leq C \boldsymbol{p}_{\nu,K}(f), \;\; \forall f \in \mathcal{E}(\Omega). \tag{1.3.1}$$

Note that the inclusion $\mathcal{D}(\omega) \hookrightarrow \mathcal{E}(\Omega)$ induces a continuous map $\mathcal{E}(\Omega)^\vee \to \mathcal{D}(\Omega)^\vee$.

Theorem 1.3.5. *The natural map $\mathcal{E}(\Omega)^\vee \to \mathcal{D}(\Omega)^\vee$ is injective and its image coincides with the space of distributions with compact support.*

Proof. Let $u \in \mathcal{E}(\Omega)^{\vee}$. We want to prove first that u has compact support when viewed as a distribution in $\mathcal{D}(\Omega)^{\vee}$. We know that there exists a compact set $K \subset \Omega$, an integer $\nu \geq 0$ and a constant $C > 0$ such that (1.3.1) holds. This shows that if $f \in \mathcal{D}(\Omega)$ and $\operatorname{supp} f \cap K = \emptyset$ the $u(f) = 0$. This proves that $\operatorname{supp} u \subset K$, and thus u has compact support.

To prove the injectivity of the map $\mathcal{E}(\Omega)^{\vee} \to \mathcal{D}(\Omega)^{\vee}$ we consider $u \in \mathcal{E}(\Omega)^{\vee}$ such that

$$\langle u, f \rangle = 0, \forall f \in \mathcal{D}(\Omega). \tag{1.3.2}$$

and we have to prove that $\langle u, g \rangle = 0$, $\forall g \in \mathcal{E}(\Omega)$. Choose a compact set $K \subset \Omega$, an integer $\nu \geq 0$ and $C > 0$ such that (1.3.1) holds. This proves that

$$\langle u, g \rangle = 0, \ \ \forall g \in \mathcal{E}(\Omega), \ \ \operatorname{supp} g \cap K = \emptyset. \tag{1.3.3}$$

Next fix $\varphi \in \mathcal{D}(\Omega)$, such that $\varphi \equiv 1$ on K. Then, $\forall g \in \mathcal{E}(\Omega)$ we have

$$\varphi g \in \mathcal{D}(\Omega), \ \ \operatorname{supp}(1-\varphi) g \cap K = \emptyset.$$

Thus

$$\langle u, g \rangle = \langle u, \varphi g \rangle + \langle u, (1-\varphi) g \rangle \overset{(1.3.2),(1.3.3)}{=} 0.$$

□

In view of the above proposition, and Example 1.3.2 we will introduce the notations

$$C^{-\infty}(\Omega) := \mathcal{D}(\Omega)^{\vee}, \ \ C_0^{-\infty}(\Omega) := \mathcal{E}(\Omega)^{\vee}.$$

The natural inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{S}(\boldsymbol{V})$ induces a continuous 'restriction' map

$$\mathcal{S}(\boldsymbol{V})^{\vee} \to C^{-\infty}(\Omega)$$

This restriction is injective if and only if $\Omega = \boldsymbol{V}$. Also we have a natural restriction map $\mathcal{S}(\boldsymbol{V}) \to \mathcal{E}(\Omega)$ that and we obtain by duality an "extension" map

$$C_0^{-\infty}(\Omega) \to \mathcal{S}(\boldsymbol{V})^{\vee}.$$

Arguing as in the proof of Theorem 1.3.5 we deduce that this map is injective. In particular, we have a sequence of inclusions

$$C_0^{-\infty}(\boldsymbol{V}) \hookrightarrow \mathcal{S}(\boldsymbol{V})^{\vee} \hookrightarrow C^{-\infty}(\boldsymbol{V}).$$

A diffeomorphism $F : \Omega_1 \to \Omega_2$ induces a continuous linear map

$$F^* : C^{\infty}(\Omega_2) \to C^{\infty}(\Omega_1), \ \ C^{\infty}(\Omega_2) \ni v \mapsto u \circ F \in C^{\infty}(\Omega_1).$$

By duality we get a continuous linear map

$$F_* := (F^*)^{\vee} : C_0^{-\infty}(\Omega_1) \to C_0^{-\infty}(\Omega_2),$$

called *push-forward* given by

$$\langle F_* u, f \rangle = \langle u, F^* f \rangle, \ \ \forall f \in C^{\infty}(\Omega_2). \tag{1.3.4}$$

The restriction of the push-forward operation to $C_0^{\infty}(\Omega_1)$ is more subtle than it looks. One might think that $F_* u = (F^{-1})^* u$, for $u \in C_0^{\infty}(\Omega_1)$. This is far from the truth.

Suppose $u \in C_0^{\infty}(\Omega_1)$ is a genuine smooth compactly supported function. We fix Euclidean coordinates $y = (y_1, \ldots, y_m)$ on Ω_2 and Euclidean coordinates $x = (x_1, \ldots, x_m)$ on Ω_1. Then the diffeomorphism F is described by a collection of m smooth functions

$$y_i = y_i(x_1, \ldots, x_m), \ \ 1 \leq i \leq m,$$

while its inverse is described by m smooth functions

$$x_j = x_j(y_1, , \dots, y_m), \;\; 1 \leq j \leq m.$$

We set

$$\left|\frac{\partial x}{\partial y}\right| := \left|\det\Big(\frac{\partial x_j}{\partial y_i}\Big)_{1\leq i,j\leq m}\right|.$$

Set $v := F_*u$, Then $v \in C_0^\infty(\Omega)$ and for every $f \in C^\infty(\Omega_2)$ we have

$$\langle\!\langle v, f\rangle\!\rangle = \int_{\Omega_2} v(y)f(y), |dy| = \int_{\Omega_1} u(x)f(y(x))\,|dx| = \int_{\Omega_2} u(x(y))f(y)\left|\frac{\partial x}{\partial y}\right|\,|dy|$$

$$= \int_{\Omega_2}(F^{-1})^*u(y)\left|\frac{\partial x}{\partial y}\right| f(y)\,|dy|.$$

Hence

$$(F_*u)(y) = (F^{-1})^*u(y)\cdot\left|\frac{\partial x}{\partial y}\right|, \;\; \forall u \in C_0^\infty(\Omega), \;\; y \in \Omega_2. \tag{1.3.5}$$

Remark 1.3.6. To give another interpretation to the operation

$$F_* : C_0^\infty(\Omega_1) \to C_0^\infty(\Omega_2)$$

we consider the compactly supported measure μ_u on Ω_1 defined by

$$\mu_u(B) = \int_B u(x)\,|dx|,$$

for any borelian subset $B \subset \Omega_1$. We get a new measure $F_*\mu_u$ on Ω_2 defined by

$$F_*\mu_u(B') = \mu_u\big(\,F^{-1}(B')\,\big),$$

for any borelian subset $B' \subset \Omega_2$. The equality (1.3.5) implies that

$$F_*\mu_u = \mu_{F_*u},$$

i.e., for any borelian $B' \subset \Omega_2$ we have

$$F_*\mu_u(B') = \int_{B'}(F_*u)(y)\,|dy|.$$

In particular, for any $u \in C_0^\infty(\Omega_1)$ we have

$$\int_{\Omega_1} u(x)\,|dx| = \int_{\Omega_2}(F_*u)(y)\,|dy|. \qquad \square$$

The definition of the pushforward implies immediately the following result.

Proposition 1.3.7. *If $F : \Omega_1 \to \Omega_2$ is a diffeomorphism, then the push-forward operation*

$$F_* : C_0^\infty(\Omega_1) \to C_0^\infty(\Omega_2)$$

is continuous. $\square$

We obtain by duality a continuous map

$$(F_*)^\vee : C^{-\infty}(\Omega_2) \to C^{-\infty}(\Omega_1),$$

uniquely determined by

$$\langle (F_*)^\vee u, v\rangle = \langle u, (F_*v\rangle, \;\; \forall v \in C_0^\infty(\Omega_1.$$

From (1.3.5) we deduce that if $u \in C^\infty(\Omega_2) \subset C^{-\infty}(\Omega_2)$ then $(F_*)^\vee u \in C^\infty(\Omega_2)$, more precisely

$$(F_*)^\vee u = u \circ F = F^* u. \tag{1.3.6}$$

Because of this equality we will refer to the operation $(F_*)^\vee$ as the *pullback* of a generalized function via a diffeomorphism and we will denote it by F^*.

If $\Omega_1 \xrightarrow{F} \Omega_2 \xrightarrow{G} \Omega_3$ are diffeomorphisms then

$$(G \circ F)_* = G_* \circ F_* \text{ and } (G \circ F)^* = F^* \circ G^*.$$

Example 1.3.8. Let $\Omega = (0,\infty) \subset \mathbb{R}$ and $F : \Omega \to \Omega$ the diffeomorphism $f(x) = x^k$, $k \neq 0$. Fix $x_0, y_0 \in (0,\infty)$. We want to compute $F_*\delta_{x_0}$ and $F^*\delta_{y_0}$.

We have

$$\langle F_*\delta_{x_0}, \varphi\rangle = \langle \delta_{x_0}, F^*\varphi\rangle = \langle \delta_{x_0}, \varphi(x^k)\rangle = \varphi(x_0^k).$$

Hence

$$F_*\delta_{x_0} = \delta_{x_0^k} = \delta_{F(x_0)}.$$

To find the pullback of δ_{y_0} we need to describe $F_*\varphi$ for $\varphi \in C_0^\infty(\Omega)$. We let $y = F(x) = x^k$, so that $x = F^{-1}(y) = y^{1/k}$. Using (1.3.5) We have

$$(F_*\varphi)(y) = \varphi(x) \cdot \left|\frac{dx}{dy}\right| = \frac{1}{k} y^{1/k-1}\varphi(y^{1/k}).$$

Then

$$\langle F^*\delta_{y_0}, \varphi\rangle = \langle \delta_{y_0}, F_*\varphi\rangle = \frac{1}{k} y_0^{1/k-1}\varphi\big(\,y_0^{1/k}\,\big).$$

This shows that

$$F^*\delta_{y_0} = \frac{1}{k} y_0^{1/k-1}\delta_{F^{-1}(y_0)}.$$ □

Let $u \in C^{-\infty}(\Omega)$. We say that u is *smooth* at $x_0 \in \Omega$ if there exists an open neighborhood $\mathcal{O}$ of x_0 in Ω and a function $v \in C^\infty(\mathcal{O})$ such that $u|_{\mathcal{O}} = f$, i.e.

$$\langle u, \varphi\rangle = \int_{\mathcal{O}} v(x)\varphi(x)\,|dx|, \;\; \forall \varphi \in C_0^\infty(\mathcal{O}).$$

The *singular support* of u is the set of points x such that u is not smooth at x. The singular support is a closed subset of Ω denoted by $\operatorname{sing\,supp} u$.

We conclude this section with a fundamental result due to Laurent Schwartz. We need to introduce some notation. Given $u, v \in C^\infty(\Omega)$ we define $u \boxtimes v \in C^\infty(\Omega \times \Omega)$ by

$$(u \boxtimes v)(x,y) = u(x)v(y), \;\; \forall x, y \in \Omega.$$

Observe that any generalized function $K \in C^{-\infty}(\Omega \times \Omega)$ defines a linear operator

$$T_K : C_0^\infty(\Omega) \to C^{-\infty}(\Omega),$$

uniquely determined by

$$\langle T_K u, v\rangle = \langle K, v \boxtimes u\rangle, \;\; \forall u, v \in C_0^\infty(\Omega).$$

Observe that if K were a genuine smooth function $\Omega \times \Omega$, then the above equality would imply that

$$(T_K u)(x) = \int_\Omega K(x,y)u(y)\,|dy|, \;\; \forall u \in C_0^\infty(\Omega), \;\; x \in \Omega.$$

Theorem 1.3.9 (The Kernel Theorem)**.** *(a) For any generalized function $K \in C^{-\infty}(\Omega\times\Omega)$ the induced operator $T_K : C_0^\infty(\Omega) \to C^{-\infty}(\Omega)$ is continuous.*[3]

(b) If $T : C_0^\infty(\Omega) \to C^{-\infty}(\Omega)$ is a linear continuous[4] *operator, then there exists a unique generalized function $K \in C^{-\infty}(\Omega \times \Omega)$ such that $T_K = T$. The generalized function K is called the* Schwartz kernel *of T.* □

For a proof we refer to [**H1**, §5.2].

1.4. Generalized sections of a vector bundle

Often in geometry we need to work with vector valued functions. Suppose that E is complex Hermitian vector space of complex dimension r. We denote by $\boldsymbol{E}^\vee$ its complex dual,

$$E^\vee := \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C}).$$

We can define in a similar way the space $\mathcal{S}(\boldsymbol{V}, E)$ of smooth functions $f : \boldsymbol{V} \to E$ with temperate growth. The Fourier transform of such a function is then the function

$$\widehat{f}(\xi) := \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(\xi,x)} f(x)\, |dx|_*.$$

The dual $\mathcal{S}(\boldsymbol{V}, E)^\vee$ is defined in a similar fashion and we observe that we have an inclusion

$$\mathcal{S}(\boldsymbol{V}, E^\vee) \hookrightarrow \mathcal{S}(\boldsymbol{V}, E)^\vee, \;\; \mathcal{S}(\boldsymbol{V}, E^\vee) \ni \varphi \mapsto u_\varphi \in \mathcal{S}(\boldsymbol{V}, E)^\vee$$

$$\langle u_\varphi, f\rangle = \langle\!\langle \varphi, f\rangle\!\rangle := \int_{\boldsymbol{V}} \langle \varphi, f\rangle_E\, |dx|, \;\; \forall f \in \mathcal{S}(\boldsymbol{V}, E^\vee),$$

where $\langle -, -\rangle_E : E \times E^\vee \to \mathbb{C}$ denotes the natural bilinear pairing between a vector space and its dual.

If Ω is an open subset of $\boldsymbol{V}$ then we define $C_0^\infty(\Omega, E)$ and $C^\infty(\Omega, E)$ in an obvious fashion. Their duals $C_0^\infty(\Omega, E)^\vee$ and $C^\infty(\Omega, E)^\vee$ are defined as before. Similarly $C^\infty(\Omega, E)^\vee$ can be identified with the subspace of $C_0^\infty(\Omega, E)^\vee$ consisting of distributions with compact support. Observe that we have natural inclusions

$$C^\infty(\Omega, E^\vee) \hookrightarrow C_0^\infty(\Omega, E)^\vee, \;\; C_0^\infty(\Omega, E^\vee) \hookrightarrow C^\infty(\Omega, E)^\vee$$

and for this reason we introduce the notations

$$C^{-\infty}(\Omega, E) := C_0^\infty(\Omega, E^\vee)^\vee, \;\; C_0^{-\infty}(\Omega, E) := C^\infty(\Omega, E^\vee)^\vee$$

More generally, let M be a smooth m-dimensional manifold. We denote by $\underline{\mathbb{C}}_M$ the trivial complex line bundle over M. Fix a smooth complex vector bundle $\boldsymbol{E} \to M$, a Riemann metric g on M, a hermitian metric h on $\boldsymbol{E}$, and a connection ∇ on $\boldsymbol{E}$, compatible with h. Denote by ∇^g the Levi-Civita connection, and by $|dV_g|$ the volume density determined by g. Denote by $\boldsymbol{E}^\vee$ the dual bundle of $\boldsymbol{E}$. By coupling the connection $\boldsymbol{E}$ with the Levi-Civita connection we obtain connections ${}^k\nabla^\vee$ on each of the bundles $(T^*M)^{\otimes(k-1)} \otimes E^\vee$, and then an operator

$$\nabla^{\otimes\nu} : C^\infty(\boldsymbol{E}^\vee) \to C^\infty(T^*M^{\otimes\nu} \otimes \boldsymbol{E}^\vee)$$

obtained from the composition

$$\boldsymbol{E}^\vee \stackrel{{}^1\nabla^\vee}{\longrightarrow} T^*M \otimes \boldsymbol{E}^\vee \stackrel{{}^2\nabla^\vee}{\longrightarrow} T^*M^{\otimes 2} \otimes \boldsymbol{E}^\vee \stackrel{{}^3\nabla^\vee}{\longrightarrow} \cdots \longrightarrow T^*M^{\otimes(\nu-1)} \otimes \boldsymbol{E}^\vee \stackrel{{}^\nu\nabla^\vee}{\longrightarrow} T^*M^{\otimes\nu} \otimes \boldsymbol{E}^\vee. \quad (1.4.1)$$

[3]The continuity should be understood with respect to the natural topology on $C_0^\infty(\Omega)$ and the weak topology on $C^{-\infty}(\Omega)$.

[4]Ditto.

The metric g and h also define metrics on the bundles $T^*M^{\otimes\nu}\otimes \boldsymbol{E}^\vee$.

For every compact subset $K\subset M$, any integer $\nu\geq 0$ and any smooth section f of $\boldsymbol{E}^\vee$ we set

$$\boldsymbol{p}_{\nu,K}(f)=\sup_{x\in K,\;j\leq\nu}|\nabla^{\otimes j}f(x)|_{g,h}.$$

A *generalized section* of $\boldsymbol{E}$ is then linear map $u: C_0^\infty(E^\vee)\to\mathbb{C}$ such that, for any compact set $K\subset M$ there exists a nonnegative integer ν and a constant $C>0$ such that

$$|u(f)|\leq Cp_{\nu,K}(f),\;\;\forall f\in C^\infty(\boldsymbol{E}^\vee),\;\;\operatorname{supp} f\subset K.$$

Observe that if ψ is a smooth section of $\boldsymbol{E}$, then ψ determines a generalized section u_ψ described by

$$u_\psi(\phi)=\langle\!\langle\psi,\phi\rangle\!\rangle:=\int_M\langle\psi,\phi\rangle_{\boldsymbol{E}}\,|dV_g|,\;\;\forall\phi\in C_0^\infty(E^\vee),$$

where $\langle -,-\rangle_{\boldsymbol{E}}:\boldsymbol{E}\otimes\boldsymbol{E}^\vee\to\underline{\mathbb{C}}_M$ denotes the natural pairing between a bundle and its dual.

* * *

☞ ***A word of warning!*** Let us observe that the above correspondence

$$C^\infty\ni\psi\longmapsto u_\psi\in C^{-\infty}$$

depends on the choice of metric g! To see how this happens, for every $\psi\in C^\infty(\boldsymbol{E}^\vee)$ and any metric g on M we denote by $u_{\psi,g}\in C^{-\infty}(\boldsymbol{E})$ the associated generalized section. If g_0,g_1 are two metrics on M then

$$u_{\psi,g_1}=\frac{1}{\rho_{g_1,g_0}}\cdot u_{\psi,g_0},\tag{1.4.2}$$

where ρ_{g_1,g_0} is the smooth positive function on M uniquely determined by the equality

$$|dV_{g_1}(x)|=\rho_{g_1,g_0}(x)\,|dV_{g_0}(x)|.$$

To eliminate this pesky dependence on metric we would have to introduce the notion of half-density, and generalized half-density, but we will not follow this approach in these notes. A nice presentation of this point of view can be found in [GS, Chap.VII].

* * *

We denote by $C^{-\infty}(\boldsymbol{E})$ the space of generalized sections of E, and by $C_0^{-\infty}(\boldsymbol{E})$ the space of generalized sections with compact support,

$$C^{-\infty}(\boldsymbol{E}):=C_0^\infty(\boldsymbol{E}^\vee)^\vee,\;\;C_0^{-\infty}(E):=C^\infty(\boldsymbol{E}^\vee)^\vee$$

The proof of the following result is left to the reader.

Proposition 1.4.1. *The isomorphism classes of the topological vector spaces $C^{-\infty}(\boldsymbol{E})$ and $C_0^{-\infty}(\boldsymbol{E})$* are independent *of the choices of metrics g,h and connection ∇.* □

Let us observe that when M is an open subset of the Euclidean vector space $\boldsymbol{V}$ and $\boldsymbol{E}=\underline{\mathbb{C}}_M$ then

$$C^{-\infty}(\underline{\mathbb{C}}_M)=C^{-\infty}(M).$$

Remark 1.4.2. Suppose that $\Omega_1, \Omega_2 \subset \boldsymbol{V}$ and g_1, g_2 are Riemann metrics on Ω_1 and respectively Ω_2. If $F : \Omega_1 \to \Omega_2$ is a diffeomorphism, then the induced push-forward map depends on these metrics.

More precisely, if we describe the inverse of F as a collection of smooth functions

$$x_j = x_j(y_1, \dots, y_m), \;\; j = 1, \dots, m,$$

where (x_j) and (y_i) are Euclidean coordinates on Ω_1 and respectively Ω_2, then we can write

$$|dV_{g_1}(x)| = w_1(x)|dx|, \;\; |dV_{g_2}(y)| = w_2(y)\,|dy|.$$

If $u \in C_0^\infty(\Omega_1)$ and $v \in C_0^\infty(\Omega_2)$ we have

$$\langle F_* u, v\rangle_{\Omega_2} = \langle u, F^* v\rangle_{\Omega_1} = \int_{\Omega_1} u(x)v(y(x))w_1(x)\,|dx|$$

$$= \int_{\Omega_2} u(x(y))v(y)w_1(x(y)) \left|\frac{\partial x}{\partial y}\right| \,|dy| = \int_{\Omega_2} u(x(y))v(y)\frac{w_1(x(y))}{w_2(y)} \left|\frac{\partial x}{\partial y}\right| \,w_2(y)|dy|$$

$$= \int_{\Omega_2} \left(u(x(y))\frac{w_1(x(y))}{w_2(y)} \left|\frac{\partial x}{\partial y}\right| \right) \cdot v(y)\,|dV_{g_2}(y)|.$$

Hence

$$F_* u = u(x(y))\frac{w_1(x(y))}{w_2(y)} \left|\frac{\partial x}{\partial y}\right|.$$

If $y_0 \in \Omega_2$, then for any $u \in C_0^\infty(\Omega_1)$ we have

$$\langle F^*\delta_{y_0}, u\rangle_{\Omega_1} = \langle \delta_{y_0}, F_* u\rangle_{\Omega_2} = u(x(y_0))\frac{w_1(x(y_0))}{w_2(y_0)} \left|\frac{\partial x}{\partial y}\right|_{y=y_0}.$$

so that, if we set $x_0 = F^{-1}(y_0)$ we deduce

$$F^*\delta_{y_0} = \frac{w_1(x_0)}{w_2(y_0)} \left|\frac{\partial x}{\partial y}\right|_{y=y_0} \delta_{x_0}.$$

If $\Omega_1 = \Omega_2$, $F = \mathbb{1}$ and g_1 is the Euclidean metric, then $w_1 = 1$. We set $w = w_2$, and we deduce

$$(\mathbb{1}_* u)(x) = \frac{1}{w(x)}u(x). \qquad \square$$

Example 1.4.3. Suppose (M, g) is smooth Riemann manifold of dimension m. For every $0 \leq k \leq m$ we set

$$\Lambda^k_{\mathbb{C}} TM := \Lambda^k TM \otimes \mathbb{C}, \;\; \Lambda^k_{\mathbb{C}} T^*M := \Lambda^k T^*M \otimes \mathbb{C}.$$

Observe that $\Lambda^k_{\mathbb{C}} TM^{\vee}$ can be identified with $\Lambda_{\mathbb{C}} T^*M$ so that a generalized section of $\Lambda^k_{\mathbb{C}} TM^{\vee}$ can be identified with a continuous linear functional

$$u : \Omega^k_0(M) := C_0^\infty(\Lambda^k T^*M) \to \mathbb{C}.$$

These are known in geometry as *currents* of dimension k. The space of such currents is denoted by $\Omega_k(M)$, so that

$$C^{-\infty}(\Lambda^k_{\mathbb{C}} T^*M) := \Omega_k(M).$$

Observe that an orientation of M induces an inclusion

$$\Omega^{m-k}(M) \ni \eta \mapsto u_\eta \in \Omega_k(M), \;\; \langle u_\eta, \alpha\rangle = \int_M \eta \wedge \alpha, \;\; \forall \alpha \in \Omega^k_0(M),$$

where the orientation of M is needed to make sense of the above integral.

Any oriented, properly embedded, k-dimensional submanifold $S \hookrightarrow M$ defines a current $[S] \in \Omega_k(M)$,

$$\langle [S], \alpha\rangle := \int_S \alpha, \ \ \forall \alpha \in \Omega_0^k(M). \qquad \square$$

The kernel theorem extends to this more general context but its formulation requires more care.

For $i = 0, 1$ we denote by $\pi_i : M \times M \to M$ the projection $(x_0, x_1) \longmapsto x_i$. Given complex vector bundles $\boldsymbol{E}_i \to M$, $i = 0, 1$ we define the vector bundle $\boldsymbol{E}_0 \boxtimes \boldsymbol{E}_1 \to M \times M$ by

$$\boldsymbol{E}_0 \boxtimes \boldsymbol{E}_1 := \pi_0^* \boldsymbol{E}_0 \otimes \pi_1 \boldsymbol{E}_1.$$

Given sections $u_i \in C^\infty(\boldsymbol{E}_i)$ we define $u_0 \boxtimes u_1 \in C^\infty(E_0 E_1)$ to be the sections $\pi_0^* \otimes \pi_1^* u_1$.

A generalized section $K \in C^{-\infty}(\boldsymbol{E}_1{}^\vee \boxtimes \boldsymbol{E}_0)$ defines a linear operator

$$T_K : C_0^\infty(\boldsymbol{E}_0) \to C^{-\infty}(\boldsymbol{E}_1)$$

uniquely determined by the equality

$$\langle T_K u, v\rangle = \langle K, v \boxtimes u\rangle, \ \ \forall u \in C_0^\infty(\boldsymbol{E}_0), \ \ v \in C_0^\infty(\boldsymbol{E}_1{}^\vee),$$

where we used the natural identification

$$\big(\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee\big)^\vee \cong \boldsymbol{E}_1{}^\vee \boxtimes \boldsymbol{E}_0.$$

Observe that if $K \in C^\infty(\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee)$ and we identify the bundle $\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee$ with the bundle $\operatorname{Hom}(\pi_0^* \boldsymbol{E}_0, \pi_1^* \boldsymbol{E}_1)$, then we can alternatively define T_K via the equality

$$(T_K u)(x) = \int_M K(x, y) u(y)\, |dV_g(y)| \in \boldsymbol{E}_1(x), \ \ \forall x \in M, \ \ u \in C_0^\infty(\boldsymbol{E}_0).$$

The kernel theorem generalizes as follows.

Theorem 1.4.4. *(a) For any generalized section $K \in C^{-\infty}(\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee)$ the induced linear operator $T_K : C_0^\infty(\boldsymbol{E}_0) \to C^{-\infty}(\boldsymbol{E}_1)$ is continuous.*

(b) If $T : C_0^\infty(\boldsymbol{E}_0) \to C^{-\infty}(\boldsymbol{E}_1)$ is a linear continuous operator, then there exists a unique generalized section $K \in C^{-\infty}(\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee)$ such that $T_K = T$. The generalized section K is called the Schwartz kernel *of T.* $\square$

1.5. Sobolev spaces

For every $s \in \mathbb{R}$ we define $\widehat{\Lambda}_s : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ to be the continuous linear operator

$$\mathcal{S}(\boldsymbol{V}) \ni f(x) \mapsto \langle x\rangle^s f(x) = (1 + |x|^2)^{s/2} f(x) \in \mathcal{S}(\boldsymbol{V}).$$

This defines by duality a linear operator

$$\widehat{\Lambda}_s{}^\vee : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee,$$

whose restriction to $\mathcal{S}(\boldsymbol{V})$ coincides with $\widehat{\Lambda}_s$. For this reason we will continue to denote the operator $\widehat{\Lambda}_s{}^\vee$ by $\widehat{\Lambda}_s$. Note that it is bijective and its inverse is $\widehat{\Lambda}_{-s}$.

We define the *Sobolev space* $H^s(\boldsymbol{V})$ to be the complex subspace of $\mathcal{S}(\boldsymbol{V})^\vee$ consisting of distributions f such that $\widehat{\Lambda}_s \widehat{f} \in L^2(\boldsymbol{V}, |d\xi|)$. Equivalently, this means that

$$\widehat{f} \in L^2(V, \langle \xi\rangle^{2s} |d\xi|), \ \ \text{or} \ \ \widehat{f} \in \widehat{\Lambda}_{-s} L^2(\boldsymbol{V}, |d\xi|),$$

so we can define

$$H^s(\boldsymbol{V}) := \boldsymbol{\mathfrak{F}}^{-1}\Big(L^2\big(V,\, \langle\xi\rangle^{2s}|d\xi|\,\big)\,\Big) = \boldsymbol{\mathfrak{F}}^{-1}\Big(\widehat{\Lambda}_{-s} L^2(\boldsymbol{V}, |d\xi|)\,\Big).$$

We can equip $H^s(V)$ with the inner product

$$\langle f, g\rangle_s := \int_{\boldsymbol{V}} \widehat{f}(\xi)\,\cdot\,\overline{\widehat{g}(\xi)}\,\langle\xi\rangle^{2s}|d\xi| = \langle \widehat{f}, \widehat{g}\rangle_{L^2(V,\langle\xi\rangle^{2s}|d\xi|)},$$

and corresponding norm

$$\|f\|_s := \Big(\int_{\boldsymbol{V}} |\widehat{f}(\xi)|^2(1+|\xi|^2)^s|d\xi|\,\Big)^{1/2}.$$

This proves that the Fourier transform defines an *isometry*.

$$\boldsymbol{\mathfrak{F}} : H^s(\boldsymbol{V}) \to L^2(V, \langle\xi\rangle^{2s}|d\xi|).$$

From Plancherel's theorem we deduce that

$$H^0(\boldsymbol{V}) = L^2(\boldsymbol{V},\, |dx|).$$

The following result is an immediate consequence of the above definitions.

Proposition 1.5.1. *The space $H^s(\boldsymbol{V})$ equipped with the inner product $\langle -, -\rangle_s$ is a separable Hilbert space. Moreover $\mathcal{S}(\boldsymbol{V})$ is a dense subspace in $H^s(\boldsymbol{V})$.* □

Proof. We use the fact that $\mathcal{S}(\boldsymbol{V})$ is a dense subspace of the space $L^2(\boldsymbol{V}, |dx|)$. We have $\widehat{\Lambda}_s\widehat{f} \in L^2(\boldsymbol{V})$. We can then find a sequence of functions $g_\nu \in \mathcal{S}(\boldsymbol{V})$ such that $g_\nu \xrightarrow{L^2} \widehat{\Lambda}_s\widehat{f}$. We set $f_\nu := \boldsymbol{\mathfrak{F}}^{-1}\big(\widehat{\Lambda}_{-s}g_\nu\,\big)$ and we observe that $f_\nu \in \mathcal{S}(\boldsymbol{V})$ since $\widehat{\Lambda}_{-s}g_\nu \in \mathcal{S}(\boldsymbol{V})$. Then

$$\|f_\nu - f\|_s^2 = \|g_\nu - \widehat{\Lambda}_s\widehat{f}\|_{L^2}^2 \to 0.$$

□

Observe that the inclusion $\mathcal{S}(\boldsymbol{V}) \hookrightarrow H^s(\boldsymbol{V})$ is continuous with respect to the natural topology on $\mathcal{S}(\boldsymbol{V})$ and the above Hilbert space topology on $H^s(\boldsymbol{V})$. Since $C_0^\infty(\boldsymbol{V})$ is dense in $\mathcal{S}(\boldsymbol{V})$ (see Exercise 1.3) we deduce the following useful density result.

Corollary 1.5.2. *$C_0^\infty(\boldsymbol{V})$ is dense in $H^s(\boldsymbol{V})$, $\forall s \in \mathbb{R}$.* □

Observe that $H^0(\boldsymbol{V})$ is isometric to the space $L^2(\boldsymbol{V}, |dx|)$, while for $s_0 \leq s_1$ we have an inclusion

$$H^{s_1}(\boldsymbol{V}) \subset H^{s_0}(\boldsymbol{V}),\;\; \|u\|_{s_0} \leq \|u\|_{s_1},\;\; \forall u \in H^{s_1}(\boldsymbol{V}). \tag{1.5.1}$$

Proposition 1.5.3. *For any multi-index α, and any real number s the linear operator*

$$D^\alpha : \mathcal{S}(\boldsymbol{V})^{\mathsf{v}} \to \mathcal{S}(\boldsymbol{V})^{\mathsf{v}}$$

induces a continuous operator $D^\alpha : H^s(\boldsymbol{V}) \to H^{s-|\alpha|}(\boldsymbol{V})$.

Proof. We use the formula $\widehat{D^\alpha f}(\xi) = \xi^\alpha\widehat{f}(\xi)$ to deduce that

$$\|D^\alpha f\|_{s-|\alpha|}^2 = \int_{\boldsymbol{V}} |\xi^\alpha|^2|\widehat{f}(\xi)|^2\left(1+|\xi|^2\right)^{s-|\alpha|}|d\xi| \leq \int_{\boldsymbol{V}} |\widehat{f}(\xi)|^2\left(1+|\xi|^2\right)^{s}|d\xi| = \|f\|_s^2.$$

□

From Proposition 1.5.3 we obtain the following alternate characterization of the spaces $H^k(\boldsymbol{V})$, k nonnegative integer.

Proposition 1.5.4. *A temperate distribution $u \in \mathcal{S}(\boldsymbol{V})^{\vee}$ belongs to the Sobolev space $H^k(\boldsymbol{V})$, $k \in \mathbb{Z}_{\geq 0}$ if and only if $u \in L^2(\boldsymbol{V}, |dx|)$ and all the distributional derivatives $\partial_x^{\alpha} u$, $|\alpha| \leq k$, belong to $L^2(\boldsymbol{V}, |dx|)$. Moreover*

$$\|f\|_k^2 = \sum_{|\alpha| \leq k} \int_{\boldsymbol{V}} |D_x^{\alpha} f(x)|^2 \, |dx|. \qquad \square$$

We denote by $H^s(\boldsymbol{V})^{\vee}$ the topological dual of $H^s(\boldsymbol{V})$, and by

$$\langle -, - \rangle_s : H^s(\boldsymbol{V})^{\vee} \times H^s(\boldsymbol{V}) \to \mathbb{C}$$

the natural pairing, between a Banach space and its dual. For $u, v \in \mathcal{S}(\boldsymbol{V})$ we have

$$\langle\!\langle u, v \rangle\!\rangle = \int_{\boldsymbol{V}} u(x) v(x) \, |dx| \stackrel{(1.1.13)}{=} \int_{\boldsymbol{V}} \widehat{u}(\xi) \, \widehat{v}(-\xi) \, |d\xi|,$$

which implies that

$$|\langle\!\langle u, v \rangle\!\rangle| \leq \|u\|_{-s} \cdot \|v\|_s, \quad \forall u, v \in \mathcal{S}(\boldsymbol{V}), \quad \forall s \in \mathbb{R}. \tag{1.5.2}$$

Since $\mathcal{S}(\boldsymbol{V})$ is dense in $H^s(\boldsymbol{V})$, $\forall s \in \mathbb{R}$, the above inequality shows that the pairing

$$\langle\!\langle -, - \rangle\!\rangle : \mathcal{S}(\boldsymbol{V}) \times \mathcal{S}(\boldsymbol{V}) \to \mathbb{C}$$

extends to a continuous bilinear map

$$\langle\!\langle -, - \rangle\!\rangle : H^{-s}(\boldsymbol{V}) \times H^s(\boldsymbol{V}) \to \mathbb{C}.$$

We obtain a continuous linear map

$$\mathcal{L}_s : H^{-s}(\boldsymbol{V}) \to H^s(\boldsymbol{V})^{\vee}, \quad u \longmapsto \mathcal{L}_s(u) := \langle\!\langle u, - \rangle\!\rangle,$$

i.e.,

$$\langle \mathcal{L}_s(u), v \rangle_s = \langle\!\langle u, v \rangle\!\rangle, \quad \forall u \in H^{-s}(\boldsymbol{V}), \quad v \in H^s(\boldsymbol{V}).$$

Theorem 1.5.5 (Duality Principle)**.** *The linear map*

$$\mathcal{L}_s : H^{-s}(\boldsymbol{V}) \to H^s(\boldsymbol{V})^{\vee}, \quad u \longmapsto \mathcal{L}(u) := \langle\!\langle u, - \rangle\!\rangle$$

is isometry *of Banach spaces.*

Proof. We carry the proof in two steps.

Step 1. The case $s = 0$. The bijectivity of linear map $\mathcal{L}_0 : L^2(\boldsymbol{V}) \to L^2(\boldsymbol{V})^{\vee}$ is the classical Riesz representation theorem for Hilbert spaces. The fact that it is an isometry follows from the elementary fact

$$\sup_{\|v\|_{L^2}=1} |\langle\!\langle u, v \rangle\!\rangle| = \|u\|.$$

Step 2. The general case. For any $r \in \mathbb{R}$ we consider the operator

$$\Lambda_r : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V}), \quad u \mapsto \boldsymbol{\mathfrak{F}}^{-1}\big(\widehat{\Lambda}_r \widehat{u}\big).$$

Since $\langle \xi \rangle = \langle -\xi \rangle$ we deduce

$$\langle\!\langle \Lambda_r u, v \rangle\!\rangle = \langle\!\langle u, \Lambda_r v \rangle\!\rangle, \quad \forall u, v \in \mathcal{S}(\boldsymbol{V}). \tag{1.5.3}$$

By construction, the maps Λ_r induce isometries

$$H^s(\boldsymbol{V}) \to H^{s-r}(\boldsymbol{V}), \;\; \forall s, r \in \mathbb{R}.$$

In particular, it induces isometries

$$\Lambda_r{}^{\vee} : H^{s-r}(\boldsymbol{V})^{\vee} \to H^s(\boldsymbol{V})^{\vee}, \;\; \forall s, r \in \mathbb{R}.$$

The bijectivity of $\mathcal{L}_s$ is a consequence of the identity

$$\mathcal{L}_s = \Lambda_{-s}{}^{\vee} \circ \mathcal{L}_0 \circ \Lambda_s.$$

Indeed, for any $u, v \in \mathcal{S}(\boldsymbol{V})$ we have

$$\big\langle\, \Lambda_{-s}{}^{\vee} \circ \mathcal{L}_0 \circ \Lambda_s u,\; v \,\big\rangle_s = \big\langle\, \mathcal{L}_0 \circ \Lambda_s u,\; \Lambda_{-s} v \,\big\rangle_0 = \langle\!\langle\, \Lambda_s u, \Lambda_{-s} v \rangle\!\rangle \overset{(1.5.3)}{=} \langle\!\langle \Lambda_{-s}\Lambda_s u, v \rangle\!\rangle = \langle\!\langle u, v \rangle\!\rangle.$$

Since $\mathcal{S}(\boldsymbol{V})$ is dense in all the subspaces $H^t(\boldsymbol{V})$ we deduce that the above equality holds for all $u, v \in L^2(\boldsymbol{V})$. We see that $\mathcal{L}_s$ is an isometry since it is a composition of isometries. □

Proposition 1.5.6 (Interpolation inequality)**.** *For any real numbers $s_0 < s_1 < s_2$ and any $\varepsilon > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, s_0, s_1, s_1) > 0$ such that*

$$\|f\|_{s_1} \leq \varepsilon \|f\|_{s_2} + C(\varepsilon)\|f\|_{s_0}, \;\; \forall f \in \mathcal{S}(\boldsymbol{V}). \tag{1.5.4}$$

Proof. Fix $\varepsilon > 0$ and consider the function

$$g_\varepsilon : [1, \infty) \to \mathbb{R}, \;\; g_\varepsilon(r) = \frac{r^{2s_1} - \varepsilon^2 r^{2s_2}}{r^{2s_0}}.$$

Note that $\lim_{r\to\infty} g_\varepsilon(r) = -\infty$ so that

$$C(\varepsilon)^2 = \sup_{r \geq 1} g_\varepsilon(r) < \infty.$$

Thus, if we set $r = \langle \xi \rangle$ we deduce

$$\langle \xi \rangle^{2s_1} \leq \varepsilon^2 \langle \xi \rangle^{2s_2} + C(\varepsilon)^2 \langle \xi \rangle^{2s_0}$$

so that, for any $f \in \mathcal{S}(\boldsymbol{V})$ we have

$$\|f\|_{s_1}^2 \leq \varepsilon^2 \|f\|_{s_2}^2 + C(\varepsilon)^2 \|f\|_{s_0}^2 \leq \big(\varepsilon \|f\|_{s_2} + C(\varepsilon)\|f\|_{s_0} \big)^2.$$

□

Remark 1.5.7. Sometimes it is useful to have some idea on the dependence of $C(\varepsilon)$ on ε. To do this we use the classical inequality[5]

$$a^t b^{(1-t)} \leq ta + (1-t)b, \;\; \forall a, b > 0, \;\; t \in (0,1)$$

We take

$$t = \frac{s_1 - s_0}{s_2 - s_0}, \;\; a = \varepsilon^2 r^{2s_2}, \;\; b = \varepsilon^{-\frac{2t}{1-t}} r^{2s_0},$$

so that

$$s_1 = (1-t)s_0 + ts_2, \;\; a^t b^{(1-t)} = r^{2s_1},$$

and we deduce

$$r^{2s_1} \leq t\varepsilon^2 r^{2s_2} + (1-t)\varepsilon^{-\frac{2t}{(1-t)}} r^{2s_0} \leq \varepsilon^2 r^{2s_2} + \varepsilon^{-2\frac{s_1 - s_0}{s_2 - s_1}} r^{s_0}.$$

Thus we can take $C(\varepsilon) = \varepsilon^{-\frac{s_1 - s_0}{s_2 - s_1}}$. □

[5]Use Jensen's inequality for the concave function $x \mapsto \log x$.

The previous three results are often used in conjunction with the following trick.

Theorem 1.5.8 (Interpolation theorem)**.** *Suppose A is a linear operator*

$$A : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})^{\vee}$$

such that there exist real numbers $s_0 < s_1, t_0 < t_1$ and positive constants C_0, C_1 with the property

$$\|Af\|_{t_j} \leq C_j \|f\|_{s_j}, \forall j = 0, 1, \quad f \in \mathcal{S}(\boldsymbol{V}).$$

Then for every $u \in [0, 1]$ we have

$$\|Af\|_{t(u)} \leq C_0^{1-u} C_1^u \|f\|_{s(u)}, \quad \forall f \in \mathcal{S}(\boldsymbol{V}),$$

where $s(z) = (1 - z)s_0 + zs_1$, $t(z) = (1 - z)t_0 + zt_1$, $\forall z \in \mathbb{C}$.

Proof. We follow the approach in [**Se**, Thm.2.5] based on a classical result of complex analysis.

Phragmen-Lindelöf Theorem *If $F(z)$ is bounded and analytic for $\mathbf{Re}\, z \in [0, 1]$ and*

$$|F(\boldsymbol{i}y)| \leq C_0, \quad |F(1 + \boldsymbol{i}y)| \leq C_1, \quad \forall y \in \mathbb{R},$$

then

$$|F(x + \boldsymbol{i}y)| \leq C_0^{1-x} C_1^x \quad \forall x \in [0, 1], \quad y \in \mathbb{R}. \qquad \square$$

For a proof we refer to [**La**, Thm. XII.6.4].

For a complex number z we denote by Λ_z the linear operator

$$\Lambda_z : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V}), \quad \Lambda_z f = \boldsymbol{\mathfrak{F}}^{-1}\big(\langle \xi \rangle^z \widehat{f}(\xi) \big).$$

Then Λ_z is an isometry

$$\Lambda_z : H^{s+\mathbf{Re}(z)}(\boldsymbol{V}) \to H^s(\boldsymbol{V}),$$

i.e.,

$$\|\Lambda_z f\|_s = \|f\|_{s+\mathbf{Re}(z)}, \quad \forall f \in \mathcal{S}(\boldsymbol{V}).$$

Given $f, g \in \mathcal{S}(\boldsymbol{V})$ and $z \in \mathbb{C}$ we define

$$F(z) = \langle\!\langle A\Lambda_{-s(z)} f, \Lambda_{t(z)} g \rangle\!\rangle.$$

We obtain a holomorphic function $F(z) : \mathbb{C} \to \mathbb{C}$. Let us prove that it is bounded in the strip $\{0 \leq \mathbf{Re}\, z \leq 1\}$. For $z = x + \boldsymbol{i}y$ we have

$$|\langle\!\langle A\Lambda_{-s(z)} f, \Lambda_{t(z)} g \rangle\!\rangle| \overset{(1.5.2)}{\leq} \|A\Lambda_{-s(z)} f\|_{t_1} \|\Lambda_{t(z)} g\|_{-t_1} \leq C \|\Lambda_{-s(z)} f\|_{s_1} \|\Lambda_{t(x)} g\|_{t(x)-t_1}$$

$$= C \|f\|_{s_1 - s(x)} \|g\|_{t(x)-t_1} \overset{(1.5.1)}{\leq} C \|f\|_{s_1 - s_0} \|g\|_{t_0 - t_1}.$$

Now observe that

$$\sup_{\mathbf{Re}\, z = 0} |F(z)| = \sup_{y \in \mathbb{R}} \|A\Lambda_{-s(\boldsymbol{i}y)} f\|_{t_0} \|\Lambda_{t(\boldsymbol{i}y)} g\|_{-t_0} \leq C_0 \|\Lambda_{-s_0} f\|_{s_0} \|\Lambda_{t_0} g\|_{-t_0} = C_0 \|f\|_{L^2} \|g\|_{L^2},$$

and similarly

$$\sup_{\mathbf{Re}\, z = 1} |F(z)| \leq C_1 \|f\|_{L^2} \|g\|_{L^2}.$$

Invoking the Phragmen-Lindelöf theorem we deduce that for any $x \in [0, 1]$ we have

$$|\langle\!\langle A\Lambda_{-s(x+\boldsymbol{i}y)} f, \Lambda_{t(x+\boldsymbol{i}y)} g \rangle\!\rangle| \leq C_0^{1-x} C_1^x \|f\|_{L^2} \|g\|_{L^2}, \quad \forall f, g \in \mathcal{S}(\boldsymbol{V}).$$

Now choose f and g of the form

$$f = \Lambda_{s(x+\boldsymbol{i}y)} \tilde{f}, \quad g = \Lambda_{-t(x+\boldsymbol{i}y)} \tilde{g},$$

to conclude that for any $\tilde{f}, \tilde{g} \in \mathcal{S}(\boldsymbol{V})$ we have

$$|\langle\langle A\tilde{f}, \tilde{g}\rangle\rangle| \leq C_0^{1-x} C_1^x \|\tilde{f}\|_{s(x)} \|\tilde{g}\|_{-t(x)}$$

The duality principle implies

$$\|A\tilde{f}\|_{t(x)} \leq C_0^{1-x} C_1^x \|\tilde{f}\|_{s(x)}, \;\; \forall \tilde{f} \in \mathcal{S}(\boldsymbol{V}).$$

□

Corollary 1.5.9. *Let $\varphi \in C_0^\infty(\boldsymbol{V})$ then, for any $s \in \mathbb{R}$ there exists a constant $C = C(s, \varphi) > 0$ such that*

$$\|\varphi u\|_s \leq C\|u\|_s, \;\; \forall u \in \mathcal{S}(\boldsymbol{V}).$$

In particular, the operation of multiplication by φ induces a bounded linear operator $H^s(\boldsymbol{V}) \to H^s(\boldsymbol{V})$.

Proof. Let $s \geq 0$ and $k = \lfloor s \rfloor + 1$. Consider the linear operator

$$A : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V}) \subset \mathcal{S}(\boldsymbol{V})^\vee, \;\; u \mapsto \varphi u.$$

We have

$$\|Au\|_0^2 = \int_{\boldsymbol{V}} |\varphi u|^2 \, |dx| \leq \Big(\sup_{x \in \boldsymbol{V}} |\varphi(x)| \Big)^2 \cdot \|u\|_0^2,$$

and

$$\|Au\|_k^2 = \|\varphi u\|_k^2 = \sum_{|\alpha| \leq k} \int_{\boldsymbol{V}} |D^\alpha(\varphi u)|^2 \, |dx| \leq C(k, \varphi) \sum_{|\alpha| \leq k} \int_{\boldsymbol{V}} |D^\alpha u|^2 \, |dx| = C(k, \varphi) \|u\|_k^2.$$

Using the interpolation theorem we deduce that for any $s \in [0, k]$ there exists a constant $C = C(s, \varphi) > 0$ such that

$$\|Au\|_s \leq C\|u\|_s, \;\; \forall u \in \mathcal{S}(\boldsymbol{V}).$$

This proves the claim for $s \geq 0$. Now observe that for any $u, v \in \mathcal{S}(\boldsymbol{V})$ and $s \geq 0$ we have

$$|\langle\langle Au, v\rangle\rangle| = |\langle\langle u, Av\rangle\rangle| \leq \|u\|_{-s} \|Av\|_s \leq C_s \|u\|_{-s} \|v\|_s.$$

Invoking the duality principle we conclude

$$\|Au\|_{-s} = \|\mathcal{L}_s(Au)\|_{H^2(\boldsymbol{V})^\vee} \leq C_s \|u\|_{-s}, \;\; \forall u \in \mathcal{S}(\boldsymbol{V})$$

which proves the claim for negative exponents $-s$. □

Theorem 1.5.10 (Morrey)**.** *Let $s > m/2 = \dim \boldsymbol{V}/2$. Then for any $\alpha \in (0,1)$ such that $s \geq \alpha + m/2$ and any $f \in H^s(\boldsymbol{V})$ there exists a Hölder continuous function $\widetilde{f} \in C^\alpha(\boldsymbol{V})$ such that $f = \widetilde{f}$ a.e. on $\boldsymbol{V}$, i.e.,*

$$\langle f, g\rangle = \int_{\boldsymbol{V}} \widetilde{f}(x) \, g(x) \, |dx|, \;\; \forall g \in \mathcal{S}(\boldsymbol{V}).$$

Moreover, there exists a constant $C > 0$ that depends only on s, α and m such that, for any $v \in \boldsymbol{V}$, $|v| \leq 1$ we have

$$|f(u)| \leq C\|f\|_s, \;\; |f(u+v) - f(u)| \leq C\|f\|_s \cdot |v|^\alpha, \;\; \forall u \in \boldsymbol{V}. \tag{1.5.5}$$

Proof. Let us observe that for any $v \in \boldsymbol{V}$ we have (see (1.2.5))

$$\widehat{\delta_v}(\xi) = \frac{1}{(2\pi)^{m/2}} e^{-\boldsymbol{i}(v,\xi)} \in L^\infty(\boldsymbol{V})$$

and we deduce

$$\langle \xi \rangle^{-s} \widehat{\delta_v}(\xi) \in L^2(\boldsymbol{V}), \quad \forall s > m/2.$$

Using the pairing $\langle\!\langle -,- \rangle\!\rangle : H^{-s}(\boldsymbol{V}) \times H^s(\boldsymbol{V}) \to \mathbb{C}$ we deduce that for any $f \in \mathcal{S}(\boldsymbol{V})$ we have

$$\langle\!\langle \delta_u, f \rangle\!\rangle = \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(u,\xi)} \widehat{f}(-\xi)\, |d\xi|_* = f(u).$$

Using (1.5.2) we deduce

$$|f(u)| = |\langle\!\langle \delta_u f \rangle\!\rangle| \leq \|\delta_u\|_{-s} \|f\|_s,$$
$$|f(u+v) - f(u)| = |\langle \delta_{u+v} - \delta_u, f \rangle| \leq \|\delta_{u+v} - \delta_v\|_{-s} \cdot \|f\|_s.$$

Thus, we need to estimate $\|\delta_u\|_{-s}^2$ and $\|\delta_{u+v} - \delta_u\|_{-s}^2$, for $u, v \in \boldsymbol{V}$, $|v| \leq 1$. We have

$$\|\delta_u\|_{-s}^2 = \int_{\boldsymbol{V}} (1+|\xi|^2)^{-s}\, |d\xi| \stackrel{(1.1.2)}{=} \frac{\boldsymbol{\sigma}_{m-2}}{2} \frac{\Gamma(p)\Gamma(s-p)}{\Gamma(s)}, \;\; p = \frac{(m-2)}{2}.$$

Next we have,

$$\|\delta_{u+v} - \delta_u\|_{-s}^2 = \int_{\boldsymbol{V}} (1+|\xi|^2)^{-s} |e^{-\boldsymbol{i}(u+v,\xi)} - e^{-\boldsymbol{i}(u,\xi)}|^2\, |d\xi|$$
$$= \int_{\boldsymbol{V}} (1+|\xi|^2)^{-s} |e^{-\boldsymbol{i}(v,\xi)} - 1|^2$$
$$= \int_{|\xi| \leq 1/|v|} (1+|\xi|^2)^{-s} |e^{-\boldsymbol{i}(v,\xi)} - 1|^2\, |d\xi| + \int_{|\xi| \geq 1/|v|} (1+|\xi|^2)^{-s} |e^{-\boldsymbol{i}(v,\xi)} - 1|^2\, |d\xi|$$
$$\leq |v|^2 \underbrace{\int_{|\xi| \leq 1/|v|} |\xi|^2 (1+|\xi|^2)^{-s} |e^{-\boldsymbol{i}(v,\xi)} - 1|^2\, |d\xi|}_{I_1} + 4 \underbrace{\int_{|\xi| \geq 1/|v|} (1+|\xi|^2)^{-s}\, |d\xi|}_{I_2}.$$

Now observe that

$$I_1 \leq \boldsymbol{\sigma}_{m-1} |v|^2 \int_0^{1/|v|} \frac{r^{m+1}}{(1+r^2)^s} dr.$$

Now choose $\alpha \in (0,1)$ such that $s \geq \alpha + m/2$ so that

$$(1+r^2)^s \geq (1+r^2)^{\alpha + m/2} \geq r^{2\alpha + m}.$$

We conclude that[6]

$$I_1 \leq \boldsymbol{\sigma}_{m-1} |v|^2 \int_0^{1/|v|} r^{1-2\alpha} dr = \frac{\boldsymbol{\sigma}_{m-1}}{2-2\alpha} |v|^{2\alpha}.$$

Since $|v| \leq 1$ and $2s - m > 2\alpha$ we deduce

$$I_2 = \boldsymbol{\sigma}_{m-1} \int_{1/|v|}^\infty \frac{r^{m-1}}{(1+r^2)^s} dr \leq \boldsymbol{\sigma}_{m-1} \int_{1/|v|}^\infty r^{m-1-2s}\, dr = \frac{\boldsymbol{\sigma}_{m-1}}{2s-m} |v|^{2s-m} \leq \frac{\boldsymbol{\sigma}_{m-1}}{2s-m} |v|^{2\alpha}.$$

This proves the inequality (1.5.5) for any $f \in \mathcal{S}_V$. To prove it for any $f \in H^s(\boldsymbol{V})$ it suffices to choose a sequence (f_ν) in $\mathcal{S}(\boldsymbol{V})$ that converges to f in the norm of $H^s(\boldsymbol{V})$. Then $f_\nu(x) \to f(x)$ for almost all $x \in \boldsymbol{V}$. We can now let $\nu \to \infty$ in the inequalities

$$|f_\nu(u)| \leq C \|f_\nu\|_s, \;\; |f_\nu(u+v) - f_\nu(u)| \leq C \|f_\nu\|_s \cdot |v|^\alpha.$$

□

[6] Here we use the assumption $\alpha < 1$.

Remark 1.5.11. The above theorem can be a bit strengthened. Namely one can prove that if $f \in H^s(\boldsymbol{V})$, then besides being Hölder continuous, the function f decays to 0 at infinity.

To prove this let us first observe that $\widehat{f} \in L^1(\boldsymbol{V}, |d\xi|)$. Indeed,

$$\int_{\boldsymbol{V}} |\widehat{f}(\xi)|\,|d\xi| = \int_{\boldsymbol{V}} |\widehat{f}(\xi)|\langle\xi\rangle^s\langle\xi\rangle^{-s}\,|d\xi|$$

$$\leq \left(\int_{\boldsymbol{V}} |\widehat{f}(\xi)|\langle\xi\rangle^{2s}\,|d\xi|\right)^{1/2}\left(\int_{\boldsymbol{V}} \langle\xi\rangle^{-2s}\,|d\xi|\right)^{1/2} = C(s,m)\|f\|_s.$$

From the Fourier inversion formula we deduce that f is the Fourier transform of the L^1-function $\xi \mapsto \widehat{f}(-\xi)$. We can now invoke the *Riemann-Lebesgue lemma* to conclude that $\lim_{|x|\to\infty} f(x) = 0$. Here is fast proof of this fact courtesy of [**ReSi**, Thm. IX.7].

Observe first that if $f \in \mathcal{S}(\boldsymbol{V})$, then $\widehat{f} \in \mathcal{S}(\boldsymbol{V})$ and thus decays to zero at ∞. Moreover,

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

The space $\mathcal{S}(\boldsymbol{V})$ is dense in both $L^1(\boldsymbol{V})$ and the Banach space $C^0(\boldsymbol{V},\infty)$ of continuous functions vanishing at ∞ equipped with the sup-norm. Thus the Fourier transform extends to a continuous map $\mathcal{F} : L^1(\boldsymbol{V}) \to C^\infty(\boldsymbol{V},\infty)$. □

Theorem 1.5.10 coupled with Proposition 1.5.3 imply immediately the following result.

Corollary 1.5.12. *Let k be a nonnegative integer, $\mu \in (0,1)$, and $s \geq \mu + k + m/2$. Then any function $f \in H^s(\boldsymbol{V})$ belongs to the Hölder space $C^{k,\mu}(\boldsymbol{V})$ and there exists a positive constant C that depends only on s, μ and m such that,*

$$|D^\alpha f(u)| \leq C\|f\|_s, \quad |D^\alpha f(u+v) - D^\alpha f(u)| \leq C\|f\|_s \cdot |v|^\mu,$$

$\forall u, v \in \boldsymbol{V}, \quad \alpha \in \mathbb{Z}^m_{\geq 0}, \quad |v| \leq 1, \; \|\alpha| \leq k.$ □

Theorem 1.5.13 (Rellich-Kondrachov)**.** *Fix real numbers $t > s$ and a compact subset $K \subset \boldsymbol{V}$. If $(u_\nu) \subset H^t(\boldsymbol{V})$ is a bounded sequence such that*

$$\operatorname{supp} u_\nu \subset K, \quad \forall \nu,$$

then a subsequence of (u_ν) converges in the norm of $H^s(\boldsymbol{V})$.

Proof. First, we replace the sequence (u_ν) with a sequence (f_ν) of *smooth* compactly supoorted functions such that

$$\lim_{\nu\to\infty} \|u_\nu - f_\nu\|_t = 0.$$

Choose a compactly supported smooth function $\varphi : \boldsymbol{V} \to \mathbb{R}$ such that $\varphi \equiv 1$ on K. Next, choose a sequence of functions (g_ν) in $\mathcal{S}(\boldsymbol{V})$ such that

$$\|g_\nu - u_\nu\|_s \leq \|g_\nu - u_\nu\|_t \leq \frac{1}{\nu}.$$

Set $f_\nu = \varphi g_\nu$. Observe that $u_\nu = \varphi u_\nu$ so that $f_\nu - u_\nu = \varphi(g_\nu - u_\nu)$. Corollary 1.5.9 implies that there exists a constant $C > 0$, independent of ν such that

$$\|f_\nu - u_\nu\|_s + \|f_\nu - u_\nu\|_t \leq \frac{C}{\nu}, \quad \forall \nu > 0. \tag{1.5.6}$$

We will show that f_ν admits a subsequence convergent in H^s. The inequalities (1.5.6) will then imply that the same is true for the original sequence (u_ν).

Using (1.2.8) and the equality $f_\nu = \varphi g_\nu$ we deduce

$$(2\pi)^{m/2}\widehat{f_\nu}(-\xi) = \widehat{\varphi} * \widehat{g_\nu}(\xi) = \int_{\boldsymbol{V}} \widehat{\varphi}(\xi-\eta)\widehat{g_\nu}(\eta)\,|d\eta|, \;\; \forall \nu, \xi.$$

We deduce that $\widehat{f_\nu}(\xi)$ is differentiable and

$$\partial_{\xi_j}\widehat{f_\nu}(-\xi) = (2\pi)^{-m/2}\int_{\boldsymbol{V}} \partial_{\xi_j}\widehat{\varphi}(\xi-\eta)\widehat{g_\nu}(\eta)\,|d\eta|.$$

Hence

$$|\partial_{\xi_j}\widehat{f_\nu}(-\xi)| \leq (2\pi)^{-m/2}\int_{\boldsymbol{V}} |\partial_{\xi_j}\widehat{\varphi}(\xi-\eta)|\langle\eta\rangle^{-t/2}|\widehat{g_\nu}(\eta)|\langle\eta\rangle^{t/2}\,|d\eta|$$

$$\leq (2\pi)^{-m/2}\|g_\nu\|_t\left(\int_{\boldsymbol{V}} |\partial_{\xi_j}\widehat{\varphi}(\xi-\eta)|^2\langle\eta\rangle^{-t}\,|d\eta|\right)^{1/2}.$$

Since $\widehat{\varphi} \in \mathcal{S}(\boldsymbol{V})$ we deduce that for some constant $C > 0$ we have

$$|\partial_{\xi_j}\widehat{\varphi}(\xi-\eta)|^2 \leq C\langle\xi-\eta\rangle^{-m-1-2|t|} \overset{(1.1.15)}{\leq} C\langle\xi\rangle^{m+1+2|t|}\langle\eta\rangle^{-1-m-2|t|}$$

and we deduce that, for some constant $C > 0$ independent of ν we have

$$|\partial_{\xi_j}\widehat{f_\nu}(-\xi)| \leq Ch(\xi)\|g_\nu\|_t, \;\; h(\xi) = \langle\xi\rangle^{(m+1+2|t|)/2}, \;\; \forall \nu, \xi.$$

A completely analogous argument yields a similar estimate for $|\widehat{f_\nu}(\xi)|$.

From the Arzela-Ascoli theorem we deduce that a subsequence of $\widehat{f_\nu}$ converges uniformly on the compacts of $\boldsymbol{V}$. For simplicity we continue denote this subsequence with $(\widehat{f_\nu})$. We want to prove that $\widehat{f_\nu}$ is a Cauchy sequence in the norm of $L^2(\boldsymbol{V}, \langle\xi\rangle^{2s}|d\xi|)$.

Fix $\varepsilon > 0$. We have

$$\|f_\nu - f_\mu\|_s^s = \int_{\boldsymbol{V}} |\widehat{f_\nu}(\xi) - \widehat{f_\mu}(\xi)|^2\langle\xi\rangle^{2s}|d\xi|$$

$$= \underbrace{\int_{|\xi|\leq r} |\widehat{f_\nu}(\xi) - \widehat{f_\mu}(\xi)|^2\langle\xi\rangle^{2s}|d\xi|}_{I_{<r}} + \underbrace{\int_{|\xi|\geq r} |\widehat{f_\nu}(\xi) - \widehat{f_\mu}(\xi)|^2\langle\xi\rangle^{2s}|d\xi|}_{I_{>r}}$$

Now observe that

$$I_{>r} = \int_{|\xi|>r} \langle\xi\rangle^{2s-2t}|\widehat{f_\nu}(\xi) - \widehat{f_\mu}(\xi)|^2\langle\xi\rangle^{2t}|d\xi| \leq (1+r^2)^{-2(t-s)}\|f_\nu - f_\mu\|_t^2.$$

Now *fix* $r > 0$ such that

$$(1+r^2)^{-2(t-s)}\|f_\nu - f_\mu\|_t^2 < \frac{\varepsilon^2}{2}, \;\; \forall \nu, \mu.$$

With $r > 0$ fixed as above, we deduce from the uniform convergence of $\widehat{f_\nu}(\xi)$ on the compact $\{|\xi| \leq r\}$ we deduce that there exists $N \geq 0$ such that for $\nu, \mu > N$, and any $|\xi| \leq r$ we have

$$|\widehat{f_\nu}(\xi) - \widehat{f_\mu}(\xi)|^2\langle\xi\rangle^{2s} \leq \frac{\varepsilon^2}{2\mathrm{vol}\,\{|\xi| \leq r\}} = \frac{\varepsilon^2}{2\boldsymbol{\omega}_m r^m}$$

We deduce that

$$\|f_\nu - f_\mu\|_s < \varepsilon, \;\; \forall \mu, \nu \geq N.$$

□

The proof of Theorem 1.5.13 also yields the following useful corollary.

Corollary 1.5.14. *Let $\varphi : \boldsymbol{V} \to \mathbb{C}$ be a smooth, compactly supported function. Then for any $t > s$ the linear map*

$$H^t(\boldsymbol{V}) \ni f \mapsto \varphi f \in H^s(\boldsymbol{V})$$

is continuous and compact. □

Let Ω be an open subset of $\boldsymbol{V}$. For $s \in \mathbb{R}$, and $K \subset \Omega$ a compact set we define

$$H^s_{\rm loc}(\Omega) := \big\{ u \in C^{-\infty}(\Omega);\;\; \varphi u \in H^s(\boldsymbol{V}),\;\; \forall \varphi \in \mathcal{D}(\Omega) \big\},$$

$$H^s_K(\Omega) = \big\{ u \in H^s(\boldsymbol{V});\;\; \operatorname{supp} u \subset K \big\}.$$

The space $H^s_K(\Omega)$ is a Hilbert space. In fact, it is a closed subspace of $H^s(\boldsymbol{V})$. We then define

$$H^s_{\rm comp}(\Omega) = \bigcup_K H^s_K(\Omega).$$

We equip $H^s_{\rm comp}(\Omega)$ with the finest locally convex topology such that all the inclusion maps

$$H^s_K(\Omega) \hookrightarrow H^s_{\rm comp}(\Omega)$$

are continuous.

We can put a locally convex topology on $H^s_{\rm loc}(\Omega)$ as follows.

- Choose an exhausting sequence of open precompact sets

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega_n \Subset \Omega_{n+1} \Subset \cdots \Subset \Omega,\;\; \Omega = \bigcup_{n \geq 1} \Omega_n.$$

- For any $n \in \mathbb{Z}_{>0}$ choose smooth function $\varphi_n \in \mathcal{D}(\Omega_{n+1})$, $\varphi_n \equiv 1$ on Ω_n.
- Define

$$p_n = p_{s,n} : H^s_{\rm loc}(\Omega) \to \mathbb{R},\;\; p_{s,n}(f) = \|\varphi_n f\|_s,\;\; \forall f \in H^s_{\rm loc}(\Omega).$$

- The locally convex topology of $H^s_{\rm loc}(\Omega)$ is the topology defined by the collection of seminorms $\{p_{s,n}\}_{n \geq 1}$.

Proposition 1.5.15. *The inclusion of $C^\infty_0(\Omega)$ in $H^s_{\rm comp}(\Omega)$ is continuous and has dense image.*

Proof. We follow the approach in [**Pet**, Lemma 4.5.2]. Let

$$C^\infty_K(\Omega) = \big\{ u \in C^\infty_0(\Omega);\;\; \operatorname{supp} u \subset K \big\}.$$

The inclusion $C^\infty_K(\Omega) \to H^s_K(\Omega)$ is continuous and thus the inclusion of $C^\infty_K(\Omega) \to H^s_{\rm comp}(\Omega)$ is continuous for any compact $K \subset \Omega$. This is equivalent to the fact that the inclusion $C^\infty_0(\Omega) \hookrightarrow H^s_{\rm comp}(\Omega)$ is continuous.

If $u \in H^s_{\rm comp}(\Omega)$ we can find $\varphi \in C^\infty_0(\Omega)$ such that $\varphi u = u$. Now choose $u_n \in \mathcal{S}(\boldsymbol{V})$ such that $u_n \to u$ in $H^s(\boldsymbol{V})$. From Corollary 1.5.9 we deduce that there exists a constant $C > 0$ depending only on φ and s such that

$$\|\varphi(u - u_n)\|_s \leq C\|u - u_n\|_s,\;\; \forall n.$$

Thus,

$$\varphi u_n \subset C^\infty_0(\Omega) \;\text{ and }\; \varphi u_n \to \varphi u = u \;\text{ in }\; H^s_{\operatorname{supp}\varphi}(\Omega).$$

□

Proposition 1.5.16. *The space $C^\infty_0(\Omega)$ is dense in $H^s_{\rm loc}(\Omega)$, for any $s \in \mathbb{R}$.*

Proof. We need to prove that for any $u \in H^s_{\text{loc}}(\Omega)$, and any $\varphi \in C_0^\infty(\Omega)$ there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi(u - u_n)\|_s = 0.$$

Choose a function $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on $\operatorname{supp}\varphi$. Then $\psi u \in H^s_{\text{comp}}(\Omega)$ and there exists $u_n \in C_0^\infty(\Omega)$ such that $\|u_n - \psi u\|_s \to 0$. We deduce

$$\|\varphi u_n - \varphi u\|_s = \|\varphi u_n - \varphi\psi u\|_s \leq C\|u_n - \psi u\|_s \to 0.$$

□

Another simple application of the Interpolation Theorem 1.5.8 is the following useful result.

Proposition 1.5.17. *Let $F : \Omega_1 \to \Omega_2$ be a diffeomorphism, and $\varphi \in C_0^\infty(\Omega_1)$, $\eta \in C_0^\infty(\Omega)$. Then for any $s \in \mathbb{R}$ there exists a constant $C > 0$ such that for any $u \in H^s_{\text{loc}}(\Omega_1)$ and any $v \in H^s_{\text{loc}}(\Omega_2)$ we have*

$$\frac{1}{C}\|\varphi u\|_s \leq \|F_*(\varphi u)\|_s \leq C\|\varphi u\|_s, \;\; \frac{1}{C}\|\varphi u\|_s \leq \|F^*(\eta v)\|_s \leq C\|\eta v\|_s.$$ □

Remark 1.5.18. The Sobolev spaces have an obvious vectorial counterpart. If E is a complex Hermitian vector space of dimension r, then

$$H^2(\boldsymbol{V}, E) = \Big\{ u \in \mathcal{S}(\boldsymbol{V}, E)^\vee; \;\; \int_{\boldsymbol{V}} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, |d\xi| < \infty \Big\}.$$

The Duality Principle (Theorem 1.5.5) continues to hold for vector valued Sobolev distribution and has the following form. We have a natural pairing

$$\langle\!\langle -, - \rangle\!\rangle : \mathcal{S}(E^\vee) \times \mathcal{S}(E) \to \mathbb{C}, \;\; \langle\!\langle u, v \rangle\!\rangle = \int_{\boldsymbol{V}} \langle u(x), v(x) \rangle_E \, |dx|,$$

where $\langle -, - \rangle_E : E^\vee \times E \to \mathbb{C}$ is the natural pairing between a vector space and its dual. This pairing satisfies the inequalities

$$|\langle\!\langle u, v \rangle\!\rangle| \leq \|u\|_{-s} \cdot \|v\|_s,$$

and in this fashion we obtain a continuous linear map

$$\mathcal{L}_E : H^{-s}(E^\vee) \to H^s(E)^\vee \tag{1.5.7}$$

and as in the scalar case we deduce that this is a bijection. The spaces H^s_{comp} and H^s_{loc} are defined in a similar fashion. □

1.6. Exercises

Exercise 1.1. (a) Prove that function

$$d : \mathcal{S}(\boldsymbol{V}) \times \mathcal{S}(\boldsymbol{V}) \to [0, \infty), \;\; d(f, g) = \sum_{n\geq 0} \frac{1}{2^n} \min(\boldsymbol{p}_n(f - g), 1)$$

is a complete, translation invariant metric on $\mathcal{S}(\boldsymbol{V})$, and the topology defined by this metric coincides with the natural topology[7] of $\mathcal{S}(\boldsymbol{V})$, i.e.,

$$\lim_{n\to\infty} d(f_\nu, f) = 0 \Longleftrightarrow f_\nu \to f \;\text{ in the natural topology of } \mathcal{S}(\boldsymbol{V}).$$

(b)* Suppose $\mathcal{N} \subset \mathcal{S}(\boldsymbol{V})$ is a *barrel* i.e., it satisfies the following conditions.

[7]In modern parlance, the space $\mathcal{S}(\boldsymbol{V})$ with its natural topology is a *Frèchet space*.

(b0) It is *closed*.

(b1) It is *absorbing*, i.e., for every $f \in \mathcal{S}(\boldsymbol{V})$ there exists $\varepsilon_f > 0$ such that $tf \in \mathcal{N}$, $\forall t \in \mathbb{C}$, $|t| < \varepsilon_f$.

(b2) It is *convex*.

(b3) It is *balanced*, i.e., $\lambda\mathcal{N} \subset \mathcal{N}$, $\forall \lambda \in \mathbb{C}$, $|\lambda| \leq 1$.

Prove that $\mathcal{N}$ is a neighborhood of 0. **Hint.** Use Baire's theorem stating that a complete metric space cannot be written as a countable union of closed sets with empty interiors. □

Exercise 1.2. Prove Proposition 1.1.2. □

Exercise 1.3. Prove that for any $f \in \mathcal{S}(\boldsymbol{V})$ there exists a sequence of smooth, *compactly supported* functions $f_n : E \to \mathbb{C}$ such that $f_n \to f$ in the topology of $\mathcal{S}(\boldsymbol{V})$ as $n \to \infty$.
Hint: Choose a compactly supported function $\varphi : E \to \mathbb{C}$ such that $|\varphi(x)| = 1$, $\forall |x| \leq 1$, define

$$\varphi_n(x) = \varphi(x/n), \quad \forall x \in \boldsymbol{V}, \quad n \in \mathbb{Z}_{>0},$$

and then show that $\varphi_n f \to f$ in $\mathcal{S}(\boldsymbol{V})$. □

Exercise 1.4. Prove that the Fourier transform $\boldsymbol{\mathcal{F}} : \mathcal{S}(\boldsymbol{V})^{\vee} \to \mathcal{S}(\boldsymbol{V})^{\vee}$ is continuous with respect to the natural topology on $\mathcal{S}(\boldsymbol{V})^{\vee}$. □

Exercise 1.5. Let $p \in (1, \infty)$. Prove that the natural inclusion

$$L^p(\boldsymbol{V}, |dx|) \to \mathcal{S}(\boldsymbol{V})^{\vee}, \quad L^p(\boldsymbol{V}, |dx|) \ni f \mapsto u_f \in \mathcal{S}(\boldsymbol{V})^{\vee}$$

is continuous, with respect to the natural topology on $L^p(\boldsymbol{V}, |dx|)$ and the natural topology on $\mathcal{S}(\boldsymbol{V})^{\vee}$. □

Exercise 1.6. A subset $\mathcal{A} \subset \mathcal{S}(\boldsymbol{V})$ is called *bounded* if for every $p, s \geq 0$ we have

$$\sup_{f \in \mathcal{A}} \sup_{x \in \boldsymbol{V},\, |\alpha| \leq s} |x|^s |D^\alpha f(x)| < \infty.$$

(a) Prove that if $\mathcal{A} \subset \mathcal{S}(\boldsymbol{V})$ is a bounded subset in $\mathcal{S}(\boldsymbol{V})$ then its closure is also bounded.

(b) Prove that $\mathcal{A}$ is bounded if and only if for any neighborhood $\mathcal{N}$ of $0 \in \mathcal{S}(\boldsymbol{V})$ there exists $\varepsilon_0 > 0$ such that

$$tf \in \mathcal{N}, \quad \forall t \in \mathbb{C}, |t| \leq \varepsilon_0, \quad \forall f \in \mathcal{A}.$$

(c) Prove that if $\mathcal{A}$ is a closed and bounded subset of $\mathcal{S}(\boldsymbol{V})$, then any sequence in $\mathcal{A}$ admits a subsequence that is convergent in the natural topology of $\mathcal{S}(\boldsymbol{V})$.

(d) If $u_n \in \mathcal{S}(\boldsymbol{V})'$ is a sequence of temperate distributions converging weakly to $u \in \mathcal{S}(\boldsymbol{V})'$ then for any $\varepsilon > 0$ the set

$$\big\{ f \in \mathcal{S}(\boldsymbol{V}); \quad |\langle f, u_n \rangle| \leq \varepsilon, \quad \forall n \geq 1 \big\}$$

is a barrel (see Exercise 1.1(b)).

(e)* If $u_n \in \mathcal{S}(\boldsymbol{V})^{\vee}$ is a sequence of temperate distributions converging weakly to $u \in \mathcal{S}(\boldsymbol{V})^{\vee}$, and $\mathcal{A} \subset \mathcal{S}(\boldsymbol{V})$ is a bounded subset, then the resulting linear functions $u_n : \mathcal{A} \to \mathbb{C}$ converge uniformly (on $\mathcal{A}$) to the function $u : \mathcal{A} \to \mathbb{C}$. □

Exercise 1.7. Prove that if $f \in L^1(\boldsymbol{V})$, then the Fourier transform of the temperate distribution defined by f is the distribution defined by the bounded function

$$\xi \mapsto \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\xi)} f(x)\, |dx|_*. \qquad \square$$

Exercise 1.8. Consider the function

$$\varphi : \boldsymbol{V} \setminus \{0\} \to \mathbb{C}, \;\; \varphi(x) = |x|^{-\lambda}, \;\; 0 < \lambda < m = \dim \boldsymbol{V}.$$

As explained in Example 1.2.1 this function is locally integrable and has polynomial growth and thus it defines a temperate distribution u_φ. Show that its Fourier transform is the temperate distribution represented by the locally integrable function with polynomial growth $C|\xi|^{\lambda-m}$, where the constant C is determined from the equality

$$C \int_{\boldsymbol{V}} |\xi|^{\lambda-m} e^{-|\xi|^2/2}\, |d\xi| = \int_{\boldsymbol{V}} |x|^{-\lambda} e^{-|x|^2|/2}\, |dx|. \qquad \square$$

Exercise 1.9. Let $u \in C^{-\infty}(\boldsymbol{V})$ Prove that the following statements are equivalent.

(a) The support of u is the origin $\{0\} \subset \boldsymbol{V}$.

(b) The distribution u is a finite linear combination of the Dirac distribution δ_0 and some of its derivatives.

$\square$

Exercise 1.10. Consider the diffeomorphism $F : \mathbb{R} \to \mathbb{R}$, $F(x) = cx$, $c > 0$. Let δ_0 be the Dirac distribution concentrated at 0 and denote by δ_0' its derivative. Express the distributions $F_*\delta_0$, $F^*\delta_0$, $F_*\delta_0'$ and $F^*\delta_0'$ as linear combinations of δ_0 and δ_0'. $\square$

Exercise 1.11. Let $s > \frac{1}{2}\dim \boldsymbol{V}$.

(a) Prove that the map

$$\boldsymbol{V} \ni v \mapsto \delta_v \in H^{-s}(\boldsymbol{V})$$

is continuous with respect to the natural topologies on $\boldsymbol{V}$ and $H^{-s}(\boldsymbol{V})$.

(b) Suppose $A : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ is a linear map such that, for any $s > 0$ there exists $C_s > 0$ such that

$$\|Au\|_s \leq C_s \|u\|_{-s}, \;\; \forall u \in \mathcal{S}(\boldsymbol{V}).$$

Prove that $A : \mathcal{S}(\boldsymbol{V}) \to \mathcal{S}(\boldsymbol{V})$ is continuous and the dual map $A^\vee : \mathcal{S}(\boldsymbol{V})^\vee \to \mathcal{S}(\boldsymbol{V})^\vee$ induces continuous linear maps $A^\vee : H^{-s}(\boldsymbol{V}) \to H^s(\boldsymbol{V})$ for all $s > 0$.

(c) Let A as in part (b) . For any $x, y \in \boldsymbol{V}$ and $s > \dim V/2$ we set

$$K_A(x,y) := \langle\!\langle \delta_y, A^\vee \delta_x \rangle\!\rangle_s.$$

where

$$\langle\!\langle -, - \rangle\!\rangle_s : H^{-s}(\boldsymbol{V}) \times H^s(\boldsymbol{V}) \to \mathbb{C}$$

is the pairing in Theorem 1.5.5. Prove that $K_A(x,y)$ depends smoothly on $x, y \in \boldsymbol{V}$, for every $\mathcal{S}(\boldsymbol{V})$ the function $y \to K_A(x,y)f(y)$ is integrable and

$$(Af)(x) = \int_{\boldsymbol{V}} K_A(x,y) f(y) dy. \qquad \square$$

Exercise 1.12. Prove Proposition 1.5.17. **Hint:** Mimic the proof of Corollary 1.5.9. $\square$

Exercise 1.13. Let $f \in H^1(\boldsymbol{V})$. Fix an orthonormal basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of $\boldsymbol{V}$. Let $h = \sum_{i=1}^m h_i \boldsymbol{e}_i \in \boldsymbol{V}$, and set

$$f_t(x) = \frac{1}{t}\big(f(x+th) - f(x)\big).$$

Prove that as $t \to 0$ the functions f_t converge in the L^2-norm to the function $\sum_{i=1}^m \partial_i f(x) h_i$, where $\partial_i f \in L^2(\boldsymbol{V})$ are the distributional derivatives of f. □

Chapter 2

Pseudo-differential operators on $\mathbb{R}^n$.

In this chapter, we will continue to denote by $\boldsymbol{V}$ a fixed, real Euclidean space of dimension m, and by Ω an open subset in $\boldsymbol{V}$. We will define the pseudo-differential operators following the approach in [H3, Shu] based on oscillatory integrals.

2.1. Oscillatory Integrals

Let Ω be an open subset of $\boldsymbol{V}$. We consider a scalar differential operator

$$A = C^\infty(\Omega) \to C^\infty(\Omega), \quad Au = \sum_{|\alpha|\leq k} a_\alpha(x)\partial_x^\alpha u = \sum_{|\alpha|\leq k} \boldsymbol{i}^{|\alpha|} a_\alpha(x) D_x^\alpha u.$$

Define the *total symbol* of A to be the function

$$\sigma_A(x,\xi) : \Omega \times \boldsymbol{V} \to \mathbb{C}, \quad \sigma_A(x,\xi) = \sum_{|\alpha|\leq k} \boldsymbol{i}^{|\alpha|} a_\alpha(x)\xi^\alpha.$$

For any $u \in \mathcal{D}(\Omega)$ we have $u \in \mathcal{S}(\boldsymbol{V})$ and we can write

$$Lu = \sum_{|\alpha|\leq k} \boldsymbol{i}^{|\alpha|} a_\alpha(x) \boldsymbol{\mathcal{F}}^{-1}\widehat{D^\alpha u} = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(\xi,x)} \underbrace{\Big(\sum_{|\alpha|\leq k} \boldsymbol{i}^{|\alpha|} a_\alpha(x)\xi^\alpha\Big)}_{\sigma_A(x,\xi)} \widehat{u}(\xi)\,|d\xi|_*$$

$$\int_{\boldsymbol{V}} e^{\boldsymbol{i}(\xi,x)}\sigma_A(x,\xi)\left(\int_\Omega e^{-\boldsymbol{i}(\xi,y)}u(y)\,|dy|_*\right)|d\xi|_* = \int_{\boldsymbol{V}}\left(\int_\Omega e^{\boldsymbol{i}(x-y,\xi)}\sigma_A(x,\xi)u(y)\,|dy|_*\right)|d\xi|_*.$$

If we close our eyes, and we pretend that we do not have any integrability concerns, we can define a "function" on $\Omega\times\Omega$

$$K(x,y) = (2\pi)^{-m/2}\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)}\sigma_A(x,\xi)\,|d\xi|_* \tag{2.1.1}$$

and then we can define the action of the differential operator A as the action of an integral operator

$$Au(x) = \int_\Omega K(x,y)u(y)|dy|. \tag{2.1.2}$$

The integral in (2.1.1) is a special example of *oscillatory integral*. It is not convergent in any meaningful sense but in this section we will explain to explain how to correctly interpret $K(x, y)$ as a generalized function (or distribution) on $\Omega \times \Omega$, namely the Schwartz kernel of A. We will achieve this by relying on the concept of oscillatory integral.

We fix another real Euclidean space $\boldsymbol{U}$ of dimension N, an open set $\mathcal{O} \subset \boldsymbol{U}$, a smooth complex valued function

$$a : \mathcal{O} \times \boldsymbol{V} \to \mathbb{C}, \ \ a = a(z, \xi),$$

called *amplitude* and a smooth *real valued* function

$$\Phi : \Omega \times (\boldsymbol{V} \setminus \{0\}) \to \mathbb{R}, \ \ \Phi = \Phi(z, \xi)$$

called *phase*. We want to give a meaning to integrals of the form

$$\int_{\mathcal{O}\times\boldsymbol{V}} e^{\boldsymbol{i}\Phi(z,\xi)} a(z, \xi)u(z)\,|dz\,d\xi|, \ \ u \in \mathcal{D}(\mathcal{O}).$$

Definition 2.1.1. (a) Fix a real number k. An *amplitude* of order $\leq k$ on $\mathcal{O} \times \boldsymbol{V}$ is a smooth function $a : \mathcal{O} \times \boldsymbol{V} \to \mathbb{V}$ such that for any multi-indices $\alpha \in \mathbb{Z}^N_{\geq 0}$ and $\beta \in \mathbb{Z}^m_{\geq 0}$, and any compact set $K \subset \mathcal{O}$, there exists a constant $C = C_{\alpha,\beta,K}(a) > 0$ such that

$$\sup_{z\in K,\xi\in\boldsymbol{V}} |D^\alpha_z D^\beta_\xi a(z, \xi)| \leq C\langle\xi\rangle^{k-|\beta|}. \tag{2.1.3}$$

We denote by $\boldsymbol{\mathcal{A}}^k(\mathcal{O} \times \boldsymbol{V})$ the space of amplitudes of order $\leq k$, and we set

$$\boldsymbol{\mathcal{A}}(\mathcal{O} \times \boldsymbol{V}) := \bigcup_{k\in\mathbb{R}} \boldsymbol{\mathcal{A}}^k(\mathcal{O} \times \boldsymbol{V}), \ \ \boldsymbol{\mathcal{A}}^{-\infty}(\mathcal{O} \times \boldsymbol{V}) := \bigcap_{k\in\mathbb{R}} \boldsymbol{\mathcal{A}}^k(\mathcal{O} \times \boldsymbol{V}).$$

(b) An *admissible phase* function on $\mathcal{O} \times \boldsymbol{V}$ is a smooth function $\Phi : \mathcal{O} \times (\boldsymbol{V} \setminus \{0\}) \to \mathbb{R}$ satisfying the following conditions.

(b1) The function Φ is positively homogeneous in ξ, i.e., for any $t > 0$ and any $(z, \xi) \in \mathcal{O} \times (\boldsymbol{V} \setminus \{0\})$ we have

$$\Phi(z, t\xi) = t\Phi(z, \xi).$$

(b2) The function Φ does not have critical points, i.e., for any $(z, \xi) \in \mathcal{O} \times (\boldsymbol{V} \setminus \{0\})$ we have

$$|d_z\Phi(z, \xi)| + |d_\xi\Phi(z, \xi)| \neq 0.$$

We denote by $\Theta(\mathcal{O} \times \boldsymbol{V})$ the space of admissible phases. □

Note that $\boldsymbol{\mathcal{A}}^k(\mathcal{O}\times\boldsymbol{V})$ is a Frèchet space with respect to the seminorms defined by the best constants $C_{\alpha,\beta,K}(a)$ in (2.1.3). We topologize $\boldsymbol{\mathcal{A}}(\mathcal{O} \times \boldsymbol{V})$ as an inductive limit of Frèchet spaces. In other words, the topology of $\boldsymbol{\mathcal{A}}$ is the largest locally convex topology such that all the inclusion maps $\boldsymbol{\mathcal{A}}^k \hookrightarrow \boldsymbol{\mathcal{A}}$ are continuous. We will need the following fact [**Tr**, Chap. 13, 14]

Theorem 2.1.2. *(a) If X is a locally convex topological vector space, then a linear map $L : \boldsymbol{\mathcal{A}} \to X$ is continuous if and only if its restriction to any $\boldsymbol{\mathcal{A}}^k$ is continuous.*

(b) A sequence $a_n \in \boldsymbol{\mathcal{A}}(\mathcal{O} \times \boldsymbol{V})$ converges to $a \in \boldsymbol{\mathcal{A}}(\mathcal{O} \times \boldsymbol{V})$ in the above inductive topology of $\boldsymbol{\mathcal{A}}$ if and only there exists $k \in \mathbb{R}$ such that

$$a, a_n \in \boldsymbol{\mathcal{A}}^k, \ \ \forall n \ \ \textit{and} \ \ a_n \to a \in \boldsymbol{\mathcal{A}}^k.$$ □

We denote by $\mathcal{A}_0(\mathcal{O}\times\boldsymbol{V})$ the subspace of $\mathcal{A}(\mathcal{O}\times\boldsymbol{V})$ consisting of amplitudes $a(z,\xi)$ such that

$$\exists R>0:\ \ a(z,\xi)=0\ \ \forall z\in\mathcal{O},\ \ |\xi|>R.$$

Proposition 2.1.3. *The space $\mathcal{A}_0(\mathcal{O}\times\boldsymbol{V})$ is dense in $\mathcal{A}(\mathcal{O}\times\boldsymbol{V})$.*

Proof. We follow the presentation in [Me, Chap.2]. Let

$$a\in\mathcal{A}^k(\mathcal{O}\times\boldsymbol{V})\subset\mathcal{A}^{k+1}(\mathcal{O}\times\boldsymbol{V})\subset\mathcal{A}(\mathcal{O}\times\boldsymbol{V}).$$

We will construct a sequence $a_n\in\mathcal{A}_0(\mathcal{O}\times\boldsymbol{V})$ such that

$$a_n\to a\ \ \text{in}\ \ \mathcal{A}^{k+1}.$$

To prove this we consider a smooth, even cutoff function

$$\varphi:\boldsymbol{V}\to[0,1],\ \ \varphi(\xi)=\begin{cases}1, & |\xi|\le 1,\\ 0, & |\xi|\ge 2.\end{cases} \tag{2.1.4}$$

For any positive integer νwe set $\varphi_\nu(\xi)=\varphi(\xi/\nu)$ and for $a\in\mathcal{A}^k(\mathcal{O}\times\boldsymbol{V})$ we define

$$a_\nu(z,\xi)=\varphi_\nu(\xi)a(z,\xi),\ \ \nu\in\mathbb{Z}_{>0}.$$

Then $a_\nu\in\mathcal{A}_0(\mathcal{O}\times\boldsymbol{V})$.

For any $b\in\mathcal{A}^{k+1}(\mathcal{O}\times\boldsymbol{V})$, $\alpha,\beta\in\mathbb{Z}^m_{\ge0}$ and any compact $K\subset\mathcal{O}$ we set

$$p_{\alpha,\beta,K}(b)=\sup_{x\in K,\xi\in\boldsymbol{V}}\langle\xi\rangle^{|\beta|-k-1}|D^\alpha_xD^\beta_\xi b(x,\xi)|. \tag{2.1.5}$$

We need to prove that

$$\lim_{\nu\to\infty}p_{\alpha,\beta,K}(a_\nu-a)=0.$$

Observe that

$$a_\nu(x,\xi)-a(x,\xi)=0,\ \ \forall|\xi|\le\nu$$

so we only need to investigate the difference $a_\nu(x,\xi)-a(x,\xi)$ for $|\xi|>\nu$. In this region we have $\langle\xi\rangle\ge(1+\nu^2)^{1/2}$ and thus

$$\langle\xi\rangle^{-k-1}|a_\nu(x,\xi)-a(x,\xi)|\le(1+\nu^2)^{-1/2}\sup_{x,\xi}\langle\xi\rangle^{-k}|a(x,\xi)|\to0\ \ \text{as}\ \nu\to\infty.$$

Next, consider the ξ derivatives of $a_\nu-a$. At this point we want to invoke the following elementary result whose proof is left to the reader as an exercise.

Lemma 2.1.4 (Leibniz formula)**.** *For any multi-index $\gamma\in\mathbb{Z}^m_{\ge0}$, any $x=(x_1,\dots,x_m)\in\boldsymbol{V}$, $y=(y_1,\dots,y_m)\in\boldsymbol{V}$ and any $f,g\in C^\infty(\Omega)$ we have*

$$\partial^\gamma_x\big(f(x)g(x)\big)=\sum_{\kappa+\lambda=\gamma}\frac{\gamma!}{\kappa!\lambda!}\partial^\kappa_xf(x)\partial^\lambda_xg(x), \tag{2.1.6}$$

where $\alpha!=(\alpha_1!)\cdots(\alpha_m!)$, $\forall(\alpha_1,\dots,\alpha_m)\in\mathbb{Z}^m_{\ge0}$. □

We have

$$D^\beta_\xi(a-a_\nu)=\sum_{\kappa+\lambda=\beta}\frac{\beta!}{\kappa!\lambda!}D^\kappa_\xi\big(1-\varphi(\xi/\nu)\big)D^\lambda_\xi a(x,\xi)$$

$$=\big(1-\varphi(\xi/\nu)\big)D^\beta_\xi a(x,\xi)-\sum_{\substack{\kappa+\lambda=\beta\\ \kappa\ne0}}\nu^{-|\kappa|}D^\kappa\varphi(x/\nu)D^\lambda_\xi a(x,\xi)$$

Since $D_\xi^\beta a \in \boldsymbol{\mathcal{A}}^{k-|\beta|}$ we deduce as above that

$$\lim_{\nu\to\infty} \sup_{x\in K,\,\xi\in \boldsymbol{V}} \langle\xi\rangle^{-k-1+|\beta|} \bigl|\,\bigl(1-\varphi(\xi/\nu)\bigr) D_\xi^\beta a(x,\xi)\,\bigr| = 0.$$

All the other terms have compact supports in ξ. This proves (2.1.5) for $\alpha = 0$. For general α observe that

$$D^\alpha a_\nu(x,\xi) = D_x^\alpha\bigl(\,\varphi_\nu(\xi) a(x,\xi)\bigr)\,\bigr) = \varphi_\nu(\xi) D_x^\alpha a(x,\xi).$$

The equality (2.1.5) for a general α follws from the equality (2.1.5) for $\alpha = 0$ involving the amplitude $D_x^\alpha a \in \boldsymbol{\mathcal{A}}^k$. □

Lemma 2.1.5. *For any $s,t\in\mathbb{R}$, any $1\le \ell\le N$ and any $1\le j\le m$ we have*

$$\boldsymbol{\mathcal{A}}^s(\mathcal{O}\times \boldsymbol{V})\cdot \boldsymbol{\mathcal{A}}^t(\mathcal{O}\times \boldsymbol{V}) \subset \boldsymbol{\mathcal{A}}^{s+t}(\mathcal{O}\times \boldsymbol{V}),$$

$$\partial_{z_\ell}\boldsymbol{\mathcal{A}}^t(\mathcal{O}\times \boldsymbol{V}) \subset \boldsymbol{\mathcal{A}}^t(\mathcal{O}\times \boldsymbol{V}),\;\; \partial_{\xi_j}\boldsymbol{\mathcal{A}}^t(\mathcal{O}\times \boldsymbol{V}) \subset \boldsymbol{\mathcal{A}}^{t-1}(\mathcal{O}\times \boldsymbol{V}).$$

Proof. The inclusion $\boldsymbol{\mathcal{A}}^s\cdot\boldsymbol{\mathcal{A}}^t \subset \boldsymbol{\mathcal{A}}^{s+t}$ follows easily using Leibniz' formula (2.1.6) while the remaining two follow directly from the definition of the spaces $\boldsymbol{\mathcal{A}}^t$. □

Observe that any phase function Φ defines a linear map

$$I_\Phi : \boldsymbol{\mathcal{A}}_0(\mathcal{O}\times \boldsymbol{V})\times \mathcal{D}(\mathcal{O}) \to \mathbb{C}$$

$$(a,u) \longmapsto I_\Phi(au) := \int_{\mathcal{O}\times\boldsymbol{V}} e^{\boldsymbol{i}\Phi(z,\xi)} a(z,\xi)u(z)\,|dz\,d\xi| \in \mathbb{C}. \tag{2.1.7}$$

We want to show that for appropriate choices of phase function we can extend this linear operator to very general choices of amplitudes.

Theorem 2.1.6. *Suppose Φ is an admissible phase function and $k\in\mathbb{R}$. Then there exists a unique linear map*

$$I_\Phi^\sim : \boldsymbol{\mathcal{A}}(\mathcal{O}\times \boldsymbol{V})\times \mathcal{D}(\mathcal{O}) \to \mathbb{C}, \tag{2.1.8}$$

separately continuous in the variables a and u, whose restriction to $\boldsymbol{\mathcal{A}}_0(\mathcal{O}\times \boldsymbol{V})\subset \boldsymbol{\mathcal{A}}(\mathcal{O}\times \boldsymbol{V})$ coincides with the oscillatory integral $I_\Phi(au)$ defined in (2.1.7).

Proof. The theorem contains three separate statements: existence, continuity and uniqueness. We will deal with them one by one.

Existence. We explain how to extend the linear operator I_Φ to $\boldsymbol{\mathcal{A}}^k(\mathcal{O}\times \boldsymbol{V})\times \mathcal{D}(\mathcal{O}\times \boldsymbol{V})$. The proof is based on the following elementary fact.

Lemma 2.1.7. *There exists a first order differential operator on $\mathcal{O}\times \boldsymbol{V}$*

$$L = L_\Phi = \sum_{j=1}^m a_j(z,\xi)\partial_{\xi_j} + \sum_{\ell=1}^N b_\ell(z,\xi)\partial_{z_\ell} + c(z,\xi),$$

such that

$$a_j \in \boldsymbol{\mathcal{A}}^0(\mathcal{O}\times \boldsymbol{V}),\;\; b_\ell,\;\; c\in \boldsymbol{\mathcal{A}}^{-1}(\mathcal{O}\times \boldsymbol{V}), \forall 1\le j\le m,\;\; 1\le \ell\le N, \tag{2.1.9}$$

and

$$L^\dagger e^{\boldsymbol{i}\Phi} = e^{\boldsymbol{i}\Phi}, \tag{2.1.10}$$

where $L^{\vee}$ is the formal transpose of L defined by

$$L^{\vee}u = -\sum_{j=1}^{m}\partial_{\xi_j}(a_j u) - \sum_{\ell=1}^{N}\partial_{z_\ell}(b_\ell u) + cu, \;\; \forall u \in C^{\infty}(\mathcal{O}\times \boldsymbol{V}). \qquad \square$$

Before we present a proof of this lemma, let us explain how it implies the existence of a linear extension to $\boldsymbol{\mathcal{A}}(\mathcal{O}\times\boldsymbol{V})\times\mathcal{D}(\mathcal{O})$ of the map $I_\Phi : \boldsymbol{\mathcal{A}}_0(\mathcal{O}\times\boldsymbol{V})\times\mathcal{D}(\mathcal{O})\to\mathbb{C}$.

Observe that if $a\in\boldsymbol{\mathcal{A}}_0(\mathcal{O}\times\boldsymbol{V})$, $u\in\mathcal{D}(\mathcal{O})$ and L is a first order differential operator $\mathcal{O}\times\boldsymbol{V}$ as in the above lemma, then for any positive integer n we have.

$$I_\Phi(au) = \int_{\mathcal{O}\times\boldsymbol{V}} (L^{\vee})^n(e^{\boldsymbol{i}\Phi(z,\xi)})a(z,\xi)u(z)\,|dz\,|d\xi| =$$

$$= \int_{\mathcal{O}\times\boldsymbol{V}} e^{\boldsymbol{i}\Phi(z,\xi)}L^n\big(\,a(z,\xi)u(z)\,\big)\,|dz\,d\xi|.$$

We will show that if $a\in\boldsymbol{\mathcal{A}}^k(\mathcal{O}\times\boldsymbol{V})$ then the above integral is convergent if n is sufficiently large. The properties of symbols show

$$L^n\big(\,a(z,\xi)u(z)\,\big)\in\boldsymbol{\mathcal{A}}^{k-n}(\mathcal{O}\times\boldsymbol{V}).$$

Indeed, observe that $ua\in\boldsymbol{\mathcal{A}}^k$, while Lemma 2.1.5 implies that $L\boldsymbol{\mathcal{A}}^k\subset\boldsymbol{\mathcal{A}}^{k-1}$. We take $n>k+m$ and define

$$I_\Phi^{\sim}(au) := \int_{\mathcal{O}\times\boldsymbol{V}} e^{\boldsymbol{i}\Phi(z,\xi)}L^n\big(\,a(z,\xi)u(z)\,\big)\,|dz\,|d\xi|. \tag{2.1.11}$$

Continuity. It suffices to prove that for any $k\in\mathbb{R}$ and any compact set $K\subset\mathcal{O}$ there exist a constant $C>0$ and an integer $\nu>0$ such that for any $u\in\mathcal{D}(\mathcal{O})$, $\operatorname{supp} u\subset\mathcal{O}$ and any $a\in\boldsymbol{\mathcal{A}}^k(\mathcal{O}\times\boldsymbol{V})$ we have

$$\big|\,I_\Phi^{\sim}(au)\,\big| \le C\sup_{z\in K,\xi\in\boldsymbol{V}\ |\alpha|,|\beta|\le\nu}\langle\xi\rangle^{|\beta|-k}|D_z^\alpha D_\xi^\beta a(x,\xi)|\cdot\sup_{z\in K,|\alpha|\le\nu|}|D_z^\alpha u(z)|.$$

This follows by observing that (2.1.9) implies that there exists a constant $C>0$ such that

$$\sup_{z\in K,\xi\in\boldsymbol{V}}\langle\xi\rangle^{n-k}|L^n(a(z,\xi)u(z)|\le C\sup_{z\in K,\ |\alpha|,|\beta|\le n}\langle\xi\rangle^{|\beta|-k}|D_z^\alpha D_\xi^\beta a(x,\xi)|\cdot\sup_{z\in K,|\alpha|\le n|}|D_z^\alpha u(z)|.$$

Uniqueness. This follows from the continuity of $a\mapsto I_\Phi^{\sim}(au)$ for fixed u and the density of in $\boldsymbol{\mathcal{A}}(\mathcal{O}\times\boldsymbol{V})$. This proves Theorem 2.1.6.

$\square$

Proof of Lemma 2.1.7. We have

$$\partial_{\xi_j}e^{\boldsymbol{i}\Phi} = \boldsymbol{i}\Phi'_{\xi_j}e^{\boldsymbol{i}\Phi},\;\; \partial_{z_\ell}e^{\boldsymbol{i}\Phi} = \Phi'_{z_\ell}e^{\boldsymbol{i}\Phi}.$$

so that

$$-\boldsymbol{i}\Big(|\xi|^2\sum_{j=1}^{m}\Phi'_{\xi_j}\partial_{\xi_j} + \sum_{\ell=1}^{N}\Phi'_{z_\ell}\partial_{z_\ell}\Big)e^{\boldsymbol{i}\Phi} = \Big(|\xi|^2\sum_{j=1}^{m}|\Phi'_{\xi_j}|^2 + \sum_{\ell=1}^{N}|\Phi'_{z_\ell}|^2\Big)e^{\boldsymbol{i}\Phi} = \frac{1}{\psi}e^{\boldsymbol{i}\Phi},$$

where $\psi\in C^{\infty}(\mathcal{O}\times\boldsymbol{V}\setminus\{0\})$ is homogeneous of degree -2 in ξ, i.e.,

$$\psi(z,t\xi) = t^{-2}\psi(z,\xi),\;\; \forall t>0,\;\; \xi\in\boldsymbol{V}\setminus 0.$$

Now choose a smooth cutoff function $\varphi(\xi)$ as in (2.1.4) and define the linear operator

$$M = -\boldsymbol{i}(1-\varphi)\psi\Big(|\xi|^2\sum_{j=1}^m \Phi'_{\xi_j}\partial_{\xi_j} + \sum_{\ell=1}^N \Phi'_{z_\ell}\partial_{z_\ell}\Big) + \varphi, \;\; L = M^\vee.$$

One can check immediately that the coefficients of L satisfy the decay conditions (2.1.9). □

Given $a \in \boldsymbol{\mathcal{A}}^k(\mathcal{O} \times \boldsymbol{V})$ we thus obtain a continuous linear map

$$\mathcal{D}(\mathcal{O}) \ni u \mapsto I_\Phi^\sim(au) \in \mathbb{C}.$$

It thus defines a distribution $I_\Phi(a) \in C^{-\infty}(\mathcal{O})$.

Definition 2.1.8. The distribution $I_\Phi^\sim(a) \in C^{-\infty}(\mathcal{O})$, $a \in \boldsymbol{\mathcal{A}}(\mathcal{O} \otimes \mathbb{V})$ is called the *oscillatory integral* with amplitude a and phase Φ and we will denote it

$$I_\Phi^\sim(a) = \int_{\boldsymbol{V}}^\sim e^{\boldsymbol{i}\Phi(z,\xi)} a(z,\xi)\,|d\xi|.$$

□

Definition 2.1.9. A first order differential operator satisfying the conditions (2.1.9) and (2.1.10) in Lemma 2.1.7 is said to be mollifying (with respect to the phase Φ.) □

Example 2.1.10. Let us illustrate the above general theory on a simple example. Namely, we want to compute the oscillatory integral

$$\mathcal{I}(x) = \int_{\mathbb{R}}^\sim e^{\boldsymbol{i}x\xi}|d\xi| \in C^{-\infty}(\mathbb{R}).$$

In this case $\Phi = x\xi$, $a = 1 \in \boldsymbol{\mathcal{A}}^0(\mathbb{R} \times \mathbb{R})$. Choose a smooth function $\varphi(\xi)$ as in (2.1.4) and set

$$\varphi_n(\xi) = \varphi(\xi/n).$$

Then $\varphi_n \to a$ in $\boldsymbol{\mathcal{A}}$ and we set

$$\mathcal{I}_n(x) = \int_{\mathbb{R}} e^{\boldsymbol{i}x\xi}\varphi_n(\xi)|d\xi| = (2\pi)^{1/2}\boldsymbol{\mathcal{F}}^{-1}[\varphi_n] \in \boldsymbol{\mathcal{S}}(\mathbb{R}).$$

Using the substitution $\xi = n\tau$ and the fact that φ is even we deduce

$$\mathcal{I}_n(x) = n\int_{\mathbb{R}} e^{-\boldsymbol{i}nx\tau}\varphi(\tau)\,|d\tau| = (2\pi)^{1/2} n\psi(nx),$$

where $\psi = \widehat{\varphi}$. We claim that $\mathcal{I}_n \to (2\pi)^{1/2}\delta_0$ in $C^{-\infty}(\mathbb{R})$ as $n \to \infty$. Indeed, given $u = u(x) \in C_0^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \mathcal{I}_n(x)u(x)\,|dx| = (2\pi)^{1/2}\int_{\mathbb{R}} \widehat{u}(\xi)\varphi_n(-\xi)|d\xi| \to (2\pi)^{1/2}\int_{\mathbb{R}} \widehat{u}(\xi)\,|d\xi| = (2\pi)^{1/2}u(0). \quad \square$$

Remark 2.1.11. The construction of the oscillatory integral $I_\Phi^\sim(a)$ used a mollifying operator L but the uniqueness of this integral shows that it is in fact independent of the choice of such an operator. In fact, by choosing this mollifying operator carefully we can obtain various interesting properties of the oscillatory integral. The next result illustrates this principle. □

Proposition 2.1.12. *Let* $a \in \boldsymbol{\mathcal{A}}^k(\mathcal{O} \otimes \mathbb{V})$ *and* $\Phi \in \Theta(\mathcal{O} \times \boldsymbol{V})$. *Define*

$$C_\Phi := \{ z \in \mathcal{O};\ \ \exists \xi \in \boldsymbol{V} \setminus 0;\ \ \partial_{\xi_j} \Phi(x, \xi) = 0,\ \ \forall j = 1, \dots, m \}.$$

Then

$$\operatorname{sing\,supp} I^{\sim}_{\Phi}(a) \subset C_\Phi.$$

Proof. Set $R_\Phi := \mathcal{O} \setminus C_\Phi$. The inclusion $\operatorname{sing\,supp} I^{\sim}_{\Phi}(a) \subset C_\Phi$ is equivalent to the existence of a smooth function $A \in C^\infty(R_\Phi)$ such that, for any $u \in C_0^\infty(R_\Phi)$ we have

$$\langle I^{\sim}_{\Phi}(a), u \rangle = \int_{R_\Phi} A(z) u(z)\, |dz|. \tag{2.1.12}$$

For each $z \in R_\Phi$ we define $a_z \in \boldsymbol{\mathcal{A}}(\boldsymbol{V})$ and $\Phi_z \in \Theta(\boldsymbol{V})$,

$$a_z(\xi) = a(z, \xi),\ \ \Phi_z(\xi) = \Phi(z, \xi).$$

Observe that $z \in R_\Phi \Longleftrightarrow \Phi_z \in \Theta(\boldsymbol{V})$. Now define

$$A(z) = I^{\sim}_{\Phi_z}(a_z) = \int_{\boldsymbol{V}} e^{\boldsymbol{i}\Phi_z(\xi)} L^n\big(a_z(\xi) \big) |d\xi| \in \mathbb{C},\ \ n > m + k,$$

where the mollifying operator L is defined by

$$L^{\mathsf{V}} = -\boldsymbol{i} \frac{1 - \varphi(\xi)}{|\nabla_\xi \Phi|^2} \sum_{j=1}^m \frac{\partial \Phi_z}{\partial \xi_j} \partial_{\xi_j} + \varphi(\xi),$$

where φ is as in (2.1.4). The proof of Theorem 2.1.6 shows that $A(z)$ depends smoothly on z. To prove (2.1.12) we regard L_z as a differential operator on $R_\Phi \times \boldsymbol{V}$ and we observe that for any $u \in C_0^\infty(R_\Phi)$ we have

$$L_z\big(a(z, \xi) u(z) \big) = L_z\big(a(z, \xi) \big)\, u(z).$$

□

2.2. Pseudo-differential operators

Let Ω be an open subset of $\boldsymbol{V}$. For any amplitude $a \in \boldsymbol{\mathcal{A}}^k(\Omega \times \Omega \times \boldsymbol{V})$ and any admissible phase Φ on $\Omega \times \Omega \times \boldsymbol{V}$ we obtain a distribution

$$K_{\Phi,a} = (2\pi)^{-m/2} \int_V^{\sim} e^{\boldsymbol{i}\Phi(x,y))} a(x, y, \xi) |d\xi|_* \in C^{-\infty}(\Omega \times \Omega).$$

Using (2.1.2) as inspiration we define a continuous linear map

$$\mathbf{Op}_\Phi(a) : C_0^\infty(\Omega) \to C^{-\infty}(\Omega),\ \ \langle \mathbf{Op}_\Phi(a) u, v \rangle := \langle K_{\Phi,a}, v \boxtimes u \rangle,\ \ \forall u, v \in C_0^\infty(\Omega), \tag{2.2.1}$$

where $v \boxtimes u \in C_0^\infty(\Omega \times \Omega)$ is the function

$$\Omega \times \Omega \ni (x, y) \mapsto (v \boxtimes u)(x, y) := v(x) u(y) \in \mathbb{C}.$$

Loosely speaking,

$$\mathbf{Op}(a) u(x) = \int_\Omega \int_V^{\sim} e^{\boldsymbol{i}\Phi(x,y))} a(x, y, \xi) u(y) |d\xi|_* |dy|_*.$$

Equivalently, this means that $K_{\Phi,a}$ is the Schwartz kernel of $\mathbf{Op}_\Phi(a)$.

Definition 2.2.1. A *pseudo-differential operator* (ψdo) of order $\leq k$ on Ω is an operator of the form $\mathbf{Op}_\Phi(a) : C_0^\infty(\Omega) \to C^{-\infty}(\Omega)$ with phase $\Phi(x,y,\xi) = (x-y,\xi)$, and amplitude $a \in \boldsymbol{\mathcal{A}}^k(\Omega \times \Omega)$. We denote by $\boldsymbol{\Psi}^k(\Omega)$ the space of pseudo-differential operators of order $\leq k$, and we set

$$\boldsymbol{\Psi}(\Omega) := \bigcup_{k\in\mathbb{R}} \boldsymbol{\Psi}^k(\Omega), \quad \boldsymbol{\Psi}^{-\infty}(\Omega) = \bigcap_{k\in\mathbb{R}} \boldsymbol{\Psi}^k(\Omega).$$

Then operators in $\boldsymbol{\Psi}^{-\infty}(\Omega)$ are called *smoothing operators*. □

The uniqueness statement in Proposition 2.1.6 implies the following useful result.

Proposition 2.2.2 (Universality trick)**.** *Suppose*

$$L : \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V}) \times \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{C}$$

is a linear map separately continuous in each of its variables such that

$$L(a,u,v) = \langle \mathbf{Op}(a)u, v\rangle, \ \ \forall (a,u,v) \in \boldsymbol{\mathcal{A}}_0(\Omega \times \Omega \times \boldsymbol{V}) \times \mathcal{D}(\Omega) \times \mathcal{D}(\Omega).$$

Then the above equality holds for any $(a,u,v) \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V}) \times \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$. □

A pseudo-differential operator is uniquely determined by its amplitude $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$. We will denote such an operator by $\mathbf{Op}(a)$. Its Schwartz kernel K_a is given by the oscillatory integral

$$K_a = (2\pi)^{-m/2} \int_V^\sim e^{\boldsymbol{i}(x-y,\xi)} a(x,y,\xi) |d\xi|_* \in C^{-\infty}(\Omega \times \Omega).$$

Proposition 2.1.12 implies that

$$\operatorname{sing\,supp} K_a \subset \Delta_\Omega := \big\{ (x,y) \in \Omega \times \Omega; \ \ x = y \big\}. \tag{2.2.2}$$

We have a linear map

$$\boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V}) \ni a \mapsto \mathbf{Op}(a) \in \boldsymbol{\Psi}(\Omega).$$

Proposition 2.2.3. *If* $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$ *then* $\mathbf{Op}(a) C_0^\infty(\Omega) \subset C^\infty(\Omega)$.

Proof. If $u, v \in C_0^\infty(\Omega)$ then $\mathbf{Op}(a)u$ is defined by the oscillatory integral

$$\mathbf{Op}(a)u = \int_\Omega \int_{\boldsymbol{V}}^\sim e^{\boldsymbol{i}(x-y,\xi)} a(x,y,\xi) u(y) |d\xi|_* |dy|_* \in C^{-\infty}(\Omega),$$

i.e.,

$$\langle \mathbf{Op}(a)u, v\rangle = \int_\Omega \left(\int_\Omega \Big(\int_{\boldsymbol{V}}^\sim e^{\boldsymbol{i}(x-y,\xi)} a(x,y,\xi) u(y) v(x) |d\xi|_* \Big) |dy|_* \right) |dx|.$$

To compute this oscillatory integral we can use any of the mollifying operators $L_x = {M_x}^\vee$ or $L_y = {M_y}^\vee$, where

$$M_x = -\boldsymbol{i} \frac{1-\varphi(\xi)}{|\xi|^2(1+|x-y|^2)} \Big(|\xi|^2 \sum_{j=1}^m (x_j - y_j)\partial_{\xi_j} + \sum_{j=1}^m \xi_j \partial_{x_j} \Big) + \varphi(\xi),$$

$$M_y = -\boldsymbol{i} \frac{1-\varphi(\xi)}{|\xi|^2(1+|x-y|^2)} \Big(|\xi|^2 \sum_{j=1}^m (x_j - y_j)\partial_{\xi_j} - \sum_{j=1}^m \xi_j \partial_{y_j} \Big) + \varphi(\xi),$$

and $\varphi(\xi)$ is a cutoff function as in (2.1.4). Observe that

$$\langle \mathbf{Op}(a)u, v\rangle = \int_{\Omega}\left(\int_{\Omega\times\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)} L_y^N\big(a(x,y,\xi)u(y)\big)v(x)|dy|_* d\xi|_*\right)|dx|$$

$$= \int_{\Omega}\underbrace{\left(\int_{\Omega\times\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)} L_y^N\big(a(x,y,\xi)u(y)\big)|dy|_* d\xi|_*\right)}_{U(x)} v(x)|dx|$$

The integrand $U(x)$ is a smooth function on Ω that can be identified with the distribution $\mathbf{Op}(a)u$. □

Thus, for any $a \in \boldsymbol{\mathcal{A}}(\Omega\times\Omega\times\boldsymbol{V})$ we get a linear operator

$$\mathbf{Op}(a) : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega).$$

The arguments in the proof of Theorem 2.1.6 yield the following more precise result.

Theorem 2.2.4. *For any amplitude $a \in \boldsymbol{\mathcal{A}}(\Omega\times\Omega\times\boldsymbol{V})$ the operator $\mathbf{Op}(a)$ induces a continuous linear operator*

$$\mathbf{Op}(a) : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega).$$

□

Observe that we have a *transposition map*

$$\boldsymbol{\mathcal{A}}(\Omega\times\Omega\times\boldsymbol{V}) \ni a \mapsto a^{\top} \in \boldsymbol{\mathcal{A}}(\Omega\times\Omega\times\boldsymbol{V}),\ \ a^{\top}(x,y,\xi) := a(y,x,-\xi).$$

The universality trick implies that for any $a \in \mathcal{A}_0(\Omega\times\Omega\times\boldsymbol{V})$ we have

$$\langle\!\langle \mathbf{Op}(a^{\top})u, v\rangle\!\rangle = \langle\!\langle u, \mathbf{Op}(a)v\rangle\!\rangle,\ \ u,v\in C_0^{\infty}(\Omega). \tag{2.2.3}$$

We say that $\mathbf{Op}(a^{\top})$ is the *formal dual* of $A = \mathbf{Op}(a) : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega)$.

We can allow $\mathbf{Op}(a)$ to act on rather singular functions. More precisely, we can give a rigorous meaning to $\mathbf{Op}(a)u$, when $u \in C_0^{-\infty}(\Omega)$.

The continuous linear operator

$$\mathbf{Op}(a^{\top}) : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega)$$

induces by duality, a continuous linear operator

$$\mathbf{Op}(a^{\top})^{\vee} : C_0^{-\infty}(\Omega) \to C^{-\infty}(\Omega),$$

defined by

$$\langle \mathbf{Op}(a^{\top})^{\vee}u, v\rangle = \langle u, \mathbf{Op}(a^{\top})v\rangle,\ \ \forall u\in C_0^{-\infty}(\Omega),\ \ v\in C_0^{\infty}(\Omega).$$

From (2.2.3) we deduce the following result.

Theorem 2.2.5. *The continuous linear operator $\mathbf{Op}(a^{\top})^{\vee} : C_0^{-\infty}(\Omega) \to C^{-\infty}(\Omega)$ is an extension of the continuous linear operator $\mathbf{Op}(a) : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega)$.* □

Thus, for any $u \in C_0^{-\infty}(\Omega)$ we define $\mathbf{Op}(a)u \in C^{-\infty}(\Omega)$ via the rule

$$\langle \mathbf{Op}(a)u, v\rangle := \langle u, \mathbf{Op}(a^{\top})v\rangle,\ \ \forall v\in C_0^{\infty}(\Omega).$$

For this reason, when no confusion is possible, we will write $\mathbf{Op}(a)^{\vee}$ instead of $\mathbf{Op}(a^{\top})$.

Proposition 2.2.6. *Suppose $A : C_0^{\infty}(\Omega) \to C^{-\infty}(\Omega)$ be a continuous linear operator. Then the following statements are equivalent.*

(a) $A \in \mathbf{\Psi}^{-\infty}(\Omega)$.

(b) *There exists a smooth function* $K \in C^\infty(\Omega \times \Omega)$ *such that*

$$(Au)(x) = (T_K u)(x) := \int_\Omega K(x,y)u(y)\,|dy|, \ \ \forall u \in C_0^\infty(\Omega).$$

Proof. (a) $\Rightarrow$ (b) Let $A = \mathbf{Op}(a)$, $a \in \boldsymbol{\mathcal{A}}^{-\infty}(\Omega \times \Omega \times \boldsymbol{V})$. Then the integral

$$K_a(x,y) := (2\pi)^{-m/2} \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)} a(x,y,\xi)\,|d\xi|_*$$

is absolutely convergent since a decays very fast as $|\xi| \to \infty$. The functions $K_a(x,y)$ depends smoothly on x, y and, by definition $T_{K_a} = \mathbf{Op}(a)$.

(b) $\Rightarrow$ (a) Choose a function $\varphi \in C_0^\infty(\boldsymbol{V})$ such that

$$\int_{\boldsymbol{V}} \varphi(\xi)\,|d\xi|_* = (2\pi)^{m/2},$$

and set

$$\widetilde{a}(x,y,\xi) := e^{-\boldsymbol{i}(x-y,\xi)} K(x,y)\varphi(\xi).$$

Clearly $\widetilde{a} \in \boldsymbol{\mathcal{A}}^{-\infty}(\Omega \times \Omega \times \boldsymbol{V})$ and

$$K_{\widetilde{a}}(x,y) = (2\pi)^{-m/2} \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)} a(x,y,\xi)\,|d\xi|_* = K(x,y).$$

Hence $T_K = \mathbf{Op}(\widetilde{a})$.

□

The next result perhaps explains why the operators in $\mathbf{\Psi}^{-\infty}$ are called smoothing.

Proposition 2.2.7. *If* $A \in \mathbf{\Psi}^{-\infty}(\Omega)$ *then* $A\big(C_0^{-\infty}(\Omega)\big) \subset C^\infty(\Omega)$. □

The proof is left to the reader as an exercise.

Example 2.2.8 (Quantization)**.** Consider an amplitude

$$a(x,y,\xi) \in \boldsymbol{\mathcal{A}}^k(\Omega \times \Omega \times \boldsymbol{V}),$$

that is *independent* of y $a = a(x,\xi)$. We want to show that for any $u \in C_0^\infty(\Omega)$ we have

$$\mathbf{Op}(a)u(x) = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} a(x,\xi)\widehat{u}(\xi)\,|d\xi|_*. \tag{2.2.4}$$

This is clearly true for $a \in \boldsymbol{\mathcal{A}}_0(\Omega \times \Omega \times \boldsymbol{V})$ because in this case we can write

$$\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} a(x,\xi)\widehat{u}(\xi)\,|d\xi|_* = (2\pi)^{-m/2} \int_{\boldsymbol{V}} \left(\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x-y)} a(x,\xi)\,|d\xi|_* \right) u(y)|dy|$$
$$= \int_{\boldsymbol{V}} K_a(x,y)u(y)\,|dy|,$$

where we recall that

$$K_a(x,y) = (2\pi)^{-m/2} \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x-y)} a(x,\xi)\,|d\xi|_*.$$

The general case follows by invoking the universality trick. When a is independent of both x and y that we say that the operator $\mathbf{Op}(a)$ is a *Fourier multiplier*.

The equality (2.2.4) shows that if

$$A = \sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha$$

is a differential operator on Ω and

$$\sigma_A = \sigma_A(x,\xi) = \sum_{|\alpha| \leq k} a_\alpha(x)\xi^\alpha,$$

is its symbol, then $\mathbf{Op}(\sigma_A) = A$.

The correspondence $\boldsymbol{\mathcal{A}}(\Omega \times \boldsymbol{V}) \ni a(x,\xi) \mapsto \mathbf{Op}(a) \in \boldsymbol{\Psi}(\Omega)$ is called *quantization*. Observe that $\Omega \times \boldsymbol{V}$ can be identified with the total space of the cotangent bundle $T^*\Omega$ which is the classical phase space. An amplitude a is a function on the phase space, i.e., a classical physical quantity and the operation of quantization associates to this function a linear operator $\mathbf{Op}(a)$ which is a quantum physical quantity. □

Theorem 2.2.9. *The pseudo-differential operators are* pseudo-local*, i.e., for* $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$ *and* $u \in C_0^{-\infty}(\Omega)$ *we have*

$$\text{sing supp}\, \mathbf{Op}(a)u \subset \text{sing supp}\, u.$$

Proof. We imitate the proof of Proposition 2.1.12. Let $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$, $u \in C_0^{-\infty}(\Omega)$ and set

$$R_u := \Omega \setminus \text{sing supp}\, u.$$

We need to show that there exists a function $A_u \in C^\infty(R_u)$ such that

$$\langle \mathbf{Op}(a)u, v\rangle = \int_{R_u} A_u(x)v(x)\,|dx|, \;\; \forall v \in C_0^\infty(R_u).$$

Denote by $\tilde{u} \in C^\infty(R_u)$ the smooth function $\tilde{u} := u|_{R_u}$. Let L_y denote the first order partial differential operator defined in the proof of Proposition 2.2.3. For $v \in C_0^\infty(R_u)$ we have

$$\langle \mathbf{Op}(a)u, v\rangle = \int_\Omega \left(\int_{\Omega\times\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)} L_y^N\big(a(x,y,\xi)u(y)\big)|dy|_*|d\xi|_* \right) v(x)\,|dx|.$$

We see that the smooth function

$$A_u(x) = \left(\int_{\Omega\times\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)} L_y^N\big(a(x,y,\xi)u(y)\big)|dy|_*|d\xi|_* \right)$$

will do the trick. □

2.3. Properly supported ψdo's

We say that a distribution $K \in C^{-\infty}(\Omega \times \Omega)$ is *properly supported* if the restrictions to $\text{supp}\, K$ of the natural projections

$$\ell, r : \Omega \times \Omega \to \Omega, \;\; \ell(x,y) = x, \;\; r(x,y) = y$$

are proper maps. For example, a distribution on $\Omega \times \Omega$ whose support is the diagonal

$$\Delta_\Omega := \big\{ (x,y) \in \Omega \times \Omega; \;\; x = y \big\}$$

is properly supported. A pseudo-differential operator $\mathbf{Op}(a)$, $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$ is called *properly-supported* if its Schwartz kernel $K_a \in C^{-\infty}(\Omega \times \Omega)$ given by the oscillatory integral

$$K_a(x, y) = (2\pi)^{-m/2} \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)} a(x, y, \xi) \, |d\xi|_*$$

is properly supported.

Proposition 2.3.1. *Suppose $\mathbf{Op}(a)$ is a properly supported pseudo-differential operator on Ω, $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$. Then $\mathbf{Op}(a)$ induces continuous linear operators*

$$C_0^\infty(\Omega) \to C_0^\infty(\Omega), \;\; C^{-\infty}(\Omega) \to C^{-\infty}(\Omega)$$

such that

$$\mathbf{Op}(a)\big(C^\infty(\Omega) \big) \subset C^\infty(\Omega) \;\; \textit{and} \;\; \mathbf{Op}(a)\big(C_0^{-\infty}(\Omega) \big) \subset C_0^{-\infty}(\Omega).$$

Proof. Observe that for any $u \in C_0^\infty(\Omega)$ we have

$$\operatorname{supp} \mathbf{Op}(a) u \subset \operatorname{supp} K_a \circ \operatorname{supp} u := \big\{ x \in \Omega; \;\; \exists y \in \operatorname{supp} u : (x, y) \in \operatorname{supp} K_a \,\big\}.$$

Indeed, if $v \in C_0^\infty(\Omega)$ and $\operatorname{supp} v \cap \operatorname{supp} K_a \circ \operatorname{supp} u = \emptyset$ then $\operatorname{supp} K_a \cap \operatorname{supp} u(y)v(x) = \emptyset$. This proves that $\operatorname{supp} \mathbf{Op}(a)u$ is compact since K_a is properly supported. This proves that $\mathbf{Op}(a)$ induces a continuous linear map $C_0^\infty(\Omega) \to C_0^\infty(\Omega)$.

Let us now observe that $K_{a^\top}$, the Schwartz kernel of $A^\vee$ is also properly supported since

$$\operatorname{supp} K_{a^\top} = R(\operatorname{supp} K_a),$$

where $R : \Omega \times \Omega \to \Omega \times \Omega$ is the reflection $(x, y) \mapsto (y, x)$. Thus we have a continuous map

$$\mathbf{Op}(a^\top) : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$$

and by duality, a continuous linear map $\mathbf{Op}(a^\top)^\vee = \mathbf{Op}(a) : C^{-\infty}(\Omega) \to C^{-\infty}(\Omega)$. The pseudo-locality of ψdo-s implies that $\mathbf{Op}(a)$ maps $C^\infty(\Omega)$ to $C^\infty(\Omega)$.

Finally, using the continuous map $\mathbf{Op}(a^\top) : C^\infty(\Omega) \to C^\infty(\Omega)$ we deduce that the dual $\mathbf{Op}(a^\top)^\vee = \mathbf{Op}(a)$ maps $C_0^{-\infty}(\Omega)$ to itself. □

We have the following characterization of properly supported operators whose proof is left as an exercise.

Proposition 2.3.2. *Let $A \in \boldsymbol{\Psi}(\Omega)$. Then A is properly supported if and only if for any compact subset $K \subset \Omega$ there exists a compact set $K' \subset \Omega$ such that*

$$u \in C^{-\infty}(\Omega), \;\; \operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Au, \;\; \operatorname{supp} A^\vee u \subset K'.$$ □

We have the following proper counterpart of Proposition 2.2.7 whose proof is left to the reader as an exercise.

Proposition 2.3.3. *If $A \in \boldsymbol{\Psi}_0^{-\infty}(\Omega)$ then $A\big(C^{-\infty}(\Omega) \big) \subset C^\infty(\Omega)$.* □

Proposition 2.3.4. *If $A \in \mathbf{\Psi}_0(\Omega)$ and $S \in \mathbf{\Psi}^{-\infty}(\Omega)$ then the operators*

$$AS : C_0^\infty(\Omega) \xrightarrow{S} C^\infty(\Omega) \xrightarrow{A} C^\infty(\Omega)$$

and

$$SA : C_0^\infty(\Omega) \xrightarrow{A} C_0^\infty(\Omega) \xrightarrow{S} C^\infty(\Omega)$$

are smoothing.

Proof. We will describe only the main steps in the proof leaving some technical details (marked with ?s) to the reader. Let $K_A \in C^{-\infty}(\Omega \times \Omega)$ denote the Schwartz kernel of A and $K_S \in C^\infty(\Omega \times \Omega)$ denote the Schwartz kernel of S. For every $z \in \Omega$ we define $\rho_z : \Omega \to \Omega \times \Omega$ to be the inclusion

$$y \mapsto \rho_z(y) = (y, z).$$

Thus

$$\rho_z^* K_S(y) = K_S(y, z), \quad \forall y, z \in \Omega.$$

Then for any $u, v \in C_0^\infty(\Omega)$ we have

$$\langle ASu, v\rangle = \langle K_A, v \boxtimes Su\rangle = \langle K_A, v(x)Su(y)\rangle$$

$$= \Big\langle K_A\,,\, v(x)\int_\Omega K_S(y,z)u(z)\,|dz|\Big\rangle \overset{???}{=} \int_\Omega \Big\langle K_A, v(x)(\rho_z^*K_S)(y)\Big\rangle\, u(z)\,|dz|$$

$$= \int_\Omega \Big\langle K_A, v \boxtimes (\rho_z^* K_S)\Big\rangle\, u(z)\,|dz| = \int_\Omega \big\langle A(\rho_z^*K_S), v\big\rangle\, u(z)\,|dz|.$$

Observe that for any z we have $A(\rho_z^* K_S) \in C^\infty(\Omega)$, and in fact the resulting function

$$(x, z) \mapsto W(z, x) := A(\rho_z^* K_S)(x)$$

is smooth (???). We deduce

$$\langle ASu, v\rangle = \int_\Omega \left(\int_\Omega W(z,x)v(x)\,|dx| \right) u(z)\,|dz|$$

so that the Schwartz kernel of AS is the smooth function W. This proves that AS is smoothing.

To prove that SA is smoothing we will use the fact that the dual $R^\vee$ of a smoothing operator $C_0^\infty(\Omega) \to C^\infty(\Omega)$ is a smoothing operator. Then $SA = (A^\vee S^\vee)^\vee$. Using the result that we have just proved we deduce that $A^\vee S^\vee$ is smoothing since $S^\vee$ is smoothing, $A^\vee$ is properly supported. □

Definition 2.3.5. (a) A relatively closed subset $C \subset \Omega \times \Omega$ is called *proper* if the restriction to C of the projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are proper maps.

(b) For a function $a : \Omega \times \Omega \times \boldsymbol{V} \to \mathbb{C}$ we denote by $\operatorname{supp}_{x,y} a$ the closure of the projection of the support of a onto the component $\Omega \times \Omega$.

(c) The function $a : \Omega \times \Omega \times \boldsymbol{V} \to \mathbb{C}$ is said to be *properly supported* if $\operatorname{supp}_{x,y} a$ is a proper subset of $\Omega \times \Omega$. □

The following result is left to the reader as an exercise (Exercise 2.5).

Lemma 2.3.6. *If $C \subset \Omega \times \Omega$ is a proper subset, then there exists a smooth function $\chi : \Omega \times \Omega \to [0, \infty)$ such that $\chi|_C \equiv 1$ and $\operatorname{supp} \chi$ is a proper subset of $\Omega \times \Omega$.* □

Proposition 2.3.7. *Any* ψdo *on* Ω *can be decomposed as a sum between a properly supported* ψdo *and a smoothing operator.*

Proof. Let $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$. Choose a smooth, properly supported function

$$\chi : \Omega \times \Omega \to [0, \infty)$$

such that $\chi \equiv 1$ in a neighborhood of the diagonal Δ_Ω. Define

$$a_0(x, y, \xi) = \chi(x, y)a(x, y, \xi), \quad a_1 = a - a_0.$$

Then $\mathbf{Op}(a) = \mathbf{Op}(a_0) + \mathbf{Op}(a_1)$ and $\mathbf{Op}(a_0)$ is properly supported. To show that $\mathbf{Op}(a_1)$ is smoothing we denote by K_a the Schwartz kernel of $\mathbf{Op}(a)$ any by K_{a_0} the Schwartz kernel of $\mathbf{Op}(a_0)$. Then

$$K_{a_0} = \chi(x, y)K_a$$

and we deduce that the Schwartz kernel of $\mathbf{Op}(a_1)$ is

$$K_{a_1} = (1 - \chi)K_a.$$

Note that K_{a_1} is identically zero in a neighborhood of the diagonal, and since its singular support is contained in the diagonal, we deduce that K_{a_1} has trivial singular support. In other words, K_{a_1} is smooth. □

Definition 2.3.8. We say that two ψdo's $A, B \in \boldsymbol{\Psi}(\Omega)$ are *smoothly equivalent* (or s-equivalent), and we denote this by $A \sim B$ if they differ by a smoothing operator, i.e., $A - B \in \boldsymbol{\Psi}^{-\infty}(\Omega)$. □

We can rephrase the above result as saying that any ψdo is s-equivalent to a proper one.

Proposition 2.3.9. *Suppose* $A \in \boldsymbol{\Psi}(\Omega)$ *is a properly supported* ψdo*. Then there exists a properly supported amplitude* $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$ *such that* $A = \mathbf{Op}(a)$.

Proof. Let $\tilde{a} \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$ such that $A = \mathbf{Op}(\tilde{a})$. Consider the kernel of A, i.e., the distribution $K \in C_0^{-\infty}(\Omega \times \Omega)$ given by the oscillatory integral

$$K = (2\pi)^{-m/2} \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)} a(x, y, \xi) |d\xi|_*$$

Now choose a smooth function $\chi : \Omega \times \Omega \to [0, \infty)$ with proper support such that $\chi|_{\operatorname{supp} K} = 1$, and set

$$a(x, y, \xi) := \chi(x, y)\tilde{a}(x, y, \xi).$$

Then a is a properly supported amplitude. Then $\chi K = K$ and the universality trick shows that we have an equality of distributions

$$\chi \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)} a(x, y, \xi) |d\xi|_* = \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)} \chi(x, y) a(x, y, \xi) |d\xi|_*$$

so that $A = \mathbf{Op}(a)$, a properly supported. □

Definition 2.3.10. We will denote by $\boldsymbol{\Psi}_0^k(\Omega)$ the space of properly supported ψdo's of order $\leq k$ and we set

$$\boldsymbol{\Psi}_0(\Omega) = \bigcup_{k \in \mathbb{R}} \boldsymbol{\Psi}_0^k(\Omega).$$ □

2.4. Symbols and asymptotic expansions

For any $\xi \in \boldsymbol{V}$ we define $e_\xi \in C^\infty(\boldsymbol{V})$

$$e_\xi(x) = e^{\boldsymbol{i}(\xi,x)}, \quad \forall x \in \boldsymbol{V}.$$

Observe that for any $u \in C_0^\infty(\boldsymbol{V})$ we have

$$\langle e_\xi, u\rangle = \langle\!\langle e_\xi, u\rangle\!\rangle = \int_{\boldsymbol{V}} e_\xi(x)u(x)\,|dx| = (2\pi)^{m/2}\widehat{u}(-\xi). \tag{2.4.1}$$

Suppose $A = \mathbf{Op}(a)$ is a *properly supported* ψdo on Ω. Then its *symbol* is the function

$$\sigma_A(x,\xi) := e_{-\xi} A e_\xi. \tag{2.4.2}$$

Proposition 2.4.1. *If A is a properly supported ψdo on Ω, then for any $u, v \in \mathcal{D}(\Omega)$, and we have*

$$v(x)\sigma_A(x,\xi)\widehat{u}(\xi) \in C^\infty(\Omega\times\boldsymbol{V}) \cap L^1(\Omega\times\boldsymbol{V})$$

and

$$Au(x) = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)}\sigma_A(x,\xi)\widehat{u}(\xi)\,|d\xi|_*. \tag{2.4.3}$$

Proof. Suppose $A = \mathbf{Op}(a)$, $a \in \boldsymbol{\mathcal{A}}^k(\Omega\times\Omega\times\boldsymbol{V})$. Set $K = \operatorname{supp} v$. Since the operator $A : C^\infty(\Omega) \to C^\infty(\Omega)$ is continuous we deduce that there exists a compact $K_1 \subset \Omega$, and integer $n > 0$ and a constant $C > 0$ such that for any ξ we have

$$\sup_{x\in K}|\sigma_A(x,\xi)| = \sup_{x\in K}|Ae_\xi(x)| \leq C \sup_{x\in K_1, |\alpha|\leq n}|D_\alpha e_\xi(x)| = C\max_{|\alpha|\leq n}|\xi^\alpha|.$$

This proves the integrability statement since $\widehat{u}(\xi) \in \mathcal{S}(\boldsymbol{V})$.

A similar argument shows that for every $x \in \Omega$ and every multi-index α the map

$$\xi \mapsto D_x^\alpha \sigma_A(x,\xi)\widehat{u}(\xi)$$

is integrable and thus we get a continuous linear map

$$C_0^\infty(\Omega) \ni u(x) \mapsto A_0u(x) := \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)}\sigma_A(x,\xi)\widehat{u}(\xi)\,|d\xi|_* \in C^\infty(\Omega).$$

We have to prove that $A_0u = Au$, $\forall u \in C_0^\infty(\Omega)$. If $v \in C_0^\infty(\Omega)$ we have

$$\langle\!\langle A_0u, v\rangle\!\rangle = \int_\Omega\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)}v(x)\sigma_A(x,\xi)\widehat{u}(\xi)\,|d\xi|_*|dx|$$

$$= \int_{\boldsymbol{V}}\Big(\int_\Omega v(x)e_\xi\sigma_A(x,\xi)|dx|\Big)\widehat{u}(\xi)|d\xi|_* = \int_{\boldsymbol{V}}\widehat{u}(\xi)\langle Ae_\xi, v\rangle|d\xi|_*$$

$$= \int_{\boldsymbol{V}}\widehat{u}(\xi)\,\langle \mathbf{Op}(a^\top)^\vee e_\xi, v\rangle\,|d\xi|_* = \int_{\boldsymbol{V}}\widehat{u}(\xi)\,\langle e_\xi, \mathbf{Op}(a^\top)v\rangle\,|d\xi|_*$$

$$\stackrel{(2.4.1)}{=} (2\pi)^{m/2}\int_{\boldsymbol{V}}\widehat{u}(\xi)\,\boldsymbol{\mathcal{F}}[\mathbf{Op}(a^\top)v\,](-\xi)|d\xi|_* = \int_{\boldsymbol{V}}\widehat{u}(\xi)\,\boldsymbol{\mathcal{F}}[\,\mathbf{Op}(a^\top)v\,](-\xi)|d\xi|$$

$$\stackrel{(1.1.13)}{=} \int_{\boldsymbol{V}} u(x)\,\mathbf{Op}(a^\top)v(x)\,|dx| = \langle\!\langle u, \mathbf{Op}(a^\top)v\rangle\!\rangle = \langle\!\langle \mathbf{Op}(a)u, v\rangle\!\rangle.$$

This proves (2.4.3). □

Remark 2.4.2. The equality (2.4.3) implies that the Schwartz kernel of A can be expressed in terms of the symbol $\sigma_A(x,\xi)$ as the oscillatory integral

$$K_A = (2\pi)^{-m/2}\int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)}\sigma_A(x,\xi)\,|d\xi|_*. \tag{2.4.4}$$

□

The above equality tacitly assumes that $\sigma_A \in \boldsymbol{\mathcal{A}}(\Omega\times V)$. This is what we intend to show next. We will achieve in several steps of independent interest.

Definition 2.4.3. Let $a\in C^\infty(\Omega\times\boldsymbol{V})$ and suppose $a_j\in\boldsymbol{\mathcal{A}}^{k_j}(\Omega\times\boldsymbol{V})$, $j=0,1,2,\dots$, where $(k_j)_{j\geq 0}$ is a strictly decreasing, unbounded sequence of real numbers. We write

$$a\sim\sum_{j=0}^\infty a_j \tag{2.4.5}$$

if for any integer $r\geq 0$ we have

$$a-\sum_{j=0}^{r-1}a_j\in\boldsymbol{\mathcal{A}}^{k_r}(\Omega\times\boldsymbol{V}). \tag{2.4.6}$$

We will refer to a relation such as (2.4.5) as an *asymptotic* expansion of a. Observe that in this case $a\in\boldsymbol{\mathcal{A}}^{k_0}(\Omega\times\boldsymbol{V})$. □

Proposition 2.4.4 (Completeness)**.** *For any sequence $a_j\in\boldsymbol{\mathcal{A}}^{k_j}(\Omega\times\boldsymbol{V})$ such that $k_j\searrow-\infty$ there exists a function $a\in\boldsymbol{\mathcal{A}}(\Omega\times\boldsymbol{V})$ such that*

$$a\sim\sum_{j=0}^\infty a_j. \tag{2.4.7}$$

Moreover if $a'\in\boldsymbol{\mathcal{A}}^{k_0}(\Omega\times\boldsymbol{V})$ satisfies the same asymptotic expansion as a, then

$$a-a'\in\boldsymbol{\mathcal{A}}^{-\infty}(\Omega\times\boldsymbol{V}).$$

Proof. The proof is based on an old trick of E. Borel. We begin by choosing an exhaustion of Ω by open precompact sets

$$\Omega_0\Subset\Omega_1\Subset\cdots\Omega,\ \ \Omega=\bigcup_{\ell\geq 0}\Omega_\ell,$$

and smooth cutoff function

$$\chi:\boldsymbol{V}\to[0,1],\ \ \chi(\xi)=\begin{cases}0, & |\xi|\leq 1\\ 1, & |\xi|>2.\end{cases}$$

Observe that for any multi-index α there exists a constant C_α such that

$$|\partial_\xi^\alpha\chi(\xi/t)|\leq C_\alpha\langle\xi\rangle^{-\alpha},\ \ \forall t\geq 1.$$

We want to emphasize that the above constant C_α is *independent of* t.

Since $a_j\in\boldsymbol{\mathcal{A}}^{k_j}(\Omega\times\boldsymbol{V})$ there we deduce that there exists a constant $C_j>0$ such that

$$\big|\,\partial_x^\beta\partial_\xi^\alpha\big(\,\chi(\xi/t)a_j(x,\xi)\,\big)\,\big|\leq C_j\langle\xi\rangle^{k_j-|\alpha|},\ \ \forall x\in\Omega_j,\ \ t\geq 1,\ \ |\alpha|+|\beta|\leq j.$$

Observe that

$$\chi(\xi/t)a_j(x,\xi) = 0, \quad \forall |\xi| \leq t.$$

Fix $j_0 > 0$ such that $k_j < -3$, $\forall j \geq j_0$. Next, for $j \geq j_0$ choose $t_j > 0$ such that

$$C_j \langle \xi \rangle^{k_j - |\alpha|} \leq 2^{-j} \langle \xi \rangle^{k_{j-1} - |\alpha|}, \quad \forall |\xi| \geq t_j, \quad |\alpha| \leq j.$$

Equivalently, this means that

$$\left(1 + t_j^2\right)^{\frac{k_{j-1} - k_j}{2}} \geq C_j 2^j.$$

We deduce that for any $j > j_0$ we have

$$\sup_{x \in \Omega_j,\, |\alpha| + |\beta| \leq j} |\partial_\xi^\alpha \partial_x^\beta (\chi(\xi/t) a_j(x,\xi))| \leq 2^{-j} \langle \xi \rangle^{k_{j-1} - |\alpha|} \leq 2^{-j} \langle \xi \rangle^{-2}.$$

If K is a compact subset of Ω, then there exists $j(K) > j_0$ such that

$$\Omega_j \Supset K, \quad \forall j \geq j(K).$$

We deduce that for any positive integer N we have and any $j \geq \max(j(K), N)$ we have

$$\sup_{x \in K,\, |\alpha| + |\beta| \leq N} |\partial_\xi^\alpha \partial_x^\beta (\chi(\xi/t) a_j(x,\xi))| \leq 2^{-j} \langle \xi \rangle^{k_{j-1} - |\alpha|}, \quad \forall j \geq \max(j(K), N). \tag{2.4.8}$$

This proves that the series

$$\sum_{j=0}^{\infty} \widetilde{a}_j(x,\xi), \quad \widetilde{a}_j(x,\xi) := \chi(\xi/t_j) a_j(x,\xi),$$

and the corresponding series of partial derivatives converge uniformly on the compacts of $\Omega \times \boldsymbol{V}$. Thus, there exists a function $a(x,\xi) \in C^\infty(\Omega \times \boldsymbol{V})$ such that

$$a(x,\xi) = \sum_{j=0}^{\infty} \chi(\xi/t_j) a_j(x,\xi),$$

and the partial derivatives of a are described by the corresponding series of partial derivatives.

Let us show that for any $r \geq 0$ we have

$$a - \sum_{i=0}^{r-1} a_i \in \boldsymbol{\mathcal{A}}^{k_r}(\Omega \times \boldsymbol{V}).$$

Fix multi-indices α, β and the compact set $K \subset \Omega$. We need to show that there exists a constant $C > 0$ such that

$$\sup_{x \in K} \left| \partial_\xi^\alpha \partial_x^\beta \Big(a - \sum_{i=0}^{r-1} a_i \Big) \right| \leq C \langle \xi \rangle^{k_r - |\alpha|}.$$

Let $N := |\alpha| + |\beta|$, and fix

$$j_1 > \max(j(K), N, r).$$

Then

$$a - \sum_{i=0}^{r-1} a_i = \underbrace{\sum_{i=0}^{r-1} (\widetilde{a}_i - a_i)}_{T_1} + \underbrace{\sum_{r \leq j \leq j_1} \widetilde{a}_j}_{T_2} + \underbrace{\sum_{j > j_1} \widetilde{a}_j}_{T_3}.$$

Clearly $T_2 \in \boldsymbol{\mathcal{A}}^{k_r}$. Next, observe that

$$T_1(x,\xi) = 0, \quad \forall |\xi| \geq 2t_r.$$

so that $T_1 \in \boldsymbol{\mathcal{A}}^{-\infty}$. Finally, using (2.4.8) we deduce that

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta T_3(x,\xi)| \leq 2^{-j_1}\langle\xi\rangle^{k_{j_1}-|\alpha|} < \langle\xi\rangle^{k_r-|\alpha|}.$$

□

The conditions in the definition of an asymptotic expansion are cumbersome in many concrete situations since they amount to checking growth conditions for infinitely many partial derivative. The next result, describes one instance when we can relax some of these requirements.

Proposition 2.4.5. *Let $a_j \in \boldsymbol{\mathcal{A}}^{k_j}(\Omega\times\boldsymbol{V})$, $j = 0,1,\dots$, $k_j \searrow -\infty$, and $a \in C^\infty(\Omega\times\boldsymbol{V})$ such that for any multi-indices α,β and any compact set K there exists a real number $\mu = \mu(\alpha,\beta,K)$ and a constant $C = C(\alpha,\beta,K) > 0$ such that*

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C\langle\xi\rangle^{\mu}, \;\; \forall \xi\in\boldsymbol{V}. \tag{2.4.9}$$

Then the following statements are equivalent.

(a) $a \sim \sum_{j\geq 0} a_j$.

(b) *For any compact set $K\subset\Omega$ there exists a sequence of real numbers $\mu_r \searrow -\infty$ and constants $C_r > 0$, $r = 1,2,\dots$,such that*

$$\sup_{x\in K}\Bigl|\, a(x,\xi) - \sum_{j=0}^{r-1} a_j(x,\xi)\,\Bigr| \leq C_r\langle\xi\rangle^{\mu_r}, \;\; \forall r\geq 1, \;\; \xi\in\boldsymbol{V}. \tag{2.4.10}$$

Proof. The implication (a) ⇒ (b) is obvious so we only need to prove that (b) ⇒ (a). We follow the very elegant presentation in [**H3**, Prop. 18.14].

Choose $b \in \boldsymbol{\mathcal{A}}^{k_0}(\Omega\times\boldsymbol{V})$ such that

$$b \sim \sum_{j\geq 0} a_j$$

We need to prove that $c = a-b \in \boldsymbol{\mathcal{A}}^{-\infty}(\Omega\times\boldsymbol{V})$. The hypothesis (2.4.10) implies that $c(x,\xi)$ is rapidly decreasing as $|\xi|\to\infty$ and we need to show that the same is true for all its partial derivatives. It suffices to do this for first order derivatives and then iterate. We will achieve this via a simple application of Taylor's formula.

Fix a compact set $K\subset\Omega$ and set $\delta_0 = \operatorname{dist}(K,\partial\Omega)$. Then for every $x\in K$, $v\in\boldsymbol{V}$, $|v| = 1$ and $0 < \varepsilon < \frac{\delta_0}{2}$ we have

$$c(x+\varepsilon v,\xi) = c(x,\xi) + \varepsilon d_x c(x,\xi)v + \frac{1}{2}\int_0^\varepsilon \frac{d^2}{dt^2} c(x+tv,\xi)dt.$$

so that

$$\varepsilon d_x c(x,\xi)v = c(x+\varepsilon v,\xi) - c(x,\xi) - \frac{1}{2}\int_0^\varepsilon \frac{d^2}{dt^2} c(x+tv,\xi)dt$$

so that

$$\varepsilon|d_x c(x,\xi)v| \leq |c(x+\varepsilon v,\xi)| + |c(x,\xi)| + C\varepsilon^2 \sup_{x\in K_\varepsilon} |d_x^2 c(x,\xi)|,$$

where

$$K_\varepsilon = \bigl\{\, x\in\Omega;\;\; \operatorname{dist}(x,K)\leq\varepsilon \,\bigr\}.$$

Now choose $N \gg 0$ and $\varepsilon = \frac{\delta_0}{4}\langle\xi\rangle^{-N}$. We deduce

$$|d_x c(x,\xi)v| \leq \frac{4}{\delta_0}\langle\xi\rangle^N\big(\,|c(x+\varepsilon v,\xi)| + |c(x,\xi)|\,\big) + \frac{C\delta_0}{4}\langle\xi\rangle^{-N}\sup_{x\in K_\varepsilon}|d_x^2 c(x,\xi)|.$$

The quantity $\sup_{x\in K_\varepsilon}|d_x^2 c(x,\xi)|$ grows at most polynomially in ξ, while the quantity

$$\frac{4}{\delta_0}\langle\xi\rangle^N\big(\,|c(x+\varepsilon v,\xi)| + |c(x,\xi)|\,\big)$$

is rapidly decreasing as $\xi \to \infty$ uniformly in $x \in K$. This proves that $d_x c$ is rapidly decreasing as $|\xi| \to \infty$.

Similarly

$$c(x,\xi+\varepsilon v) = c(x,\xi) + \varepsilon d_\xi c(x,\xi)v + \frac{1}{2}\int_0^\varepsilon \frac{d^2}{dt^2}c(x,\xi+tv)dt,$$

$$\varepsilon|d_\xi c(x,\xi)v| \leq |c(x,\xi+\varepsilon v)| + |c(x,\xi)| + C\varepsilon^2 \sup_{x\in K_\varepsilon, |t|\leq\varepsilon}|d_\xi^2 c(x,\xi+tv)|,$$

and we deduce in a similar fashion that $d_\xi c$ is rapidly decreasing as $|\xi| \to \infty$. □

We have the following important result referred to as the Workhorse Theorem in [**LM**, III.3].

Theorem 2.4.6. *Suppose* $A \in \mathbf{\Psi}_0^k(\Omega)$ *is a properly supported* ψdo*,*

$$A = \mathbf{Op}(a), \;\; a \in \boldsymbol{A}^k(\Omega\times\Omega\times\boldsymbol{V}).$$

Then its symbol $\sigma_A(x,\xi) = e_{-\xi}Ae_\xi$ *admits the asymptotic expansion*

$$\sigma_A(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}D_y^\alpha\partial_\xi^\alpha a(x,y,\xi)|_{x=y}, \tag{2.4.11}$$

where $(\alpha_1,\dots,\alpha_m)! = \alpha_1!\cdots\alpha_m!$.

Proof. We follow the approach in [**Shu**, Thm. 3.1]. We plan to use Proposition 2.4.5 which requires an a priori rough estimates of the type (2.4.9). We set

$$a^{(\alpha)}(x,y,\xi) := \partial_\xi^\alpha a(x,y,\xi).$$

First note that Proposition 2.3.9 implies that we can assume that the amplitude $a(x,y,\xi)$ is properly supported. We can then rewrite the equality $\sigma_A(x,\xi) = e_{-\xi}(x)(Ae_\xi)(x)$ as

$$\sigma_A(x,\xi) = \int_{\boldsymbol{V}}^{\sim}\left(\int_{\boldsymbol{V}} a(x,y,\xi)e^{\boldsymbol{i}(x-y,\theta)}e^{\boldsymbol{i}(y-x,\xi)}|dy|_*\right)|d\theta|_*.$$

Above, for every x the support of the function $y \mapsto a(x,y,\xi)$ is compact since a is properly supported. Making the change in variables $z = y-x$, $\eta = \theta-\xi$ and invoking the universality trick we deduce

$$\sigma_A(x,\xi) = \int_{\boldsymbol{V}}^{\sim}\left(\int_{\boldsymbol{V}} a(x,x+z,\xi+\eta)e^{-\boldsymbol{i}(z,\eta)}|dz|_*\right)|d\eta|_*. \tag{2.4.12}$$

Let L_z denote the partial differential operator

$$L_z = 1 + \sum_{j=1}^m D_{z_j}^2.$$

Observe that $L_z e^{-\boldsymbol{i}(z,\eta)} = \langle\eta\rangle^2 e^{-\boldsymbol{i}(z,\eta)}$. Integrating by parts in (2.4.12) we deduce

$$\sigma_A(x,\xi) = \int_{\boldsymbol{V}}^{\sim} \left(\int_{\boldsymbol{V}} L_z^\nu a(x, x+z, \xi+\eta) \langle\eta\rangle^{-2\nu} e^{-\boldsymbol{i}(z,\eta)} |dz|_* \right) |d\eta|_*, \tag{2.4.13}$$

where ν is an arbitrary positive integer. Using the condition $a \in \boldsymbol{\mathcal{A}}^k(\Omega\times\Omega\times\boldsymbol{V})$ we deduce that for any multi-indices α, β and any compacts $K, K' \subset \Omega$ there exists a positive constant $C = C(\alpha,\beta,K,K')$ such that

$$\sup_{x\in K, x+z\in K'} \bigl|\, \partial_x^\beta L_z^\nu a^{(\alpha)}(x, x+z, \xi+\eta) \,\bigr| \langle\eta\rangle^{-2\nu} \leq C\langle\xi+\eta\rangle^{k-|\alpha|} \langle\eta\rangle^{-2\nu}.$$

Peetre's inequality now implies

$$\langle\xi+\eta\rangle^{k-|\alpha|} \leq 2^{p/2} \langle\xi\rangle^{k-|\alpha|} \langle\eta\rangle^p, \;\; p = \bigl|\, k - |\alpha| \,\bigr|.$$

Using these inequalities in (2.4.13) we deduce

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \leq C\langle\xi\rangle^{k-|\alpha|} \int_{\boldsymbol{V}} \langle\eta\rangle^{p-2\nu} |d\nu|_*.$$

This proves the rough estimates of the type (2.4.9). We need to prove the estimates of the type (2.4.10). Fix a compact set $K \subset \Omega$.

Expanding $\eta \mapsto a(x, x+z, \xi+\eta)$ near $\eta = 0$ using Taylor formula we get

$$a(x, x+z, \xi+\eta) = \sum_{|\alpha|\leq N-1} a^{(\alpha)}(x, x+z, \xi) \frac{\eta^\alpha}{\alpha!} + r_N(x, x+z, \xi, \eta),$$

where

$$r_N(x, x+z, \xi, \eta) = \sum_{|\alpha|=N} \frac{N\eta^\alpha}{\alpha!} \int_0^1 (1-t)^{N-1} a^{(\alpha)}(x, x+z, \xi+t\eta) dt.$$

Now observe that

$$\begin{aligned}
&\int_{\boldsymbol{V}}^{\sim} \left(\int_{\boldsymbol{V}} a^{(\alpha)}(x, x+z, \xi) \eta^\alpha e^{-\boldsymbol{i}(z,\eta)} |dz|_* \right) |d\eta|_* \\
&= (-1)^{|\alpha|} \int_{\boldsymbol{V}}^{\sim} \left(\int_{\boldsymbol{V}} a^{(\alpha)}(x, x+z, \xi) D_z^\alpha e^{-\boldsymbol{i}(z,\eta)} |dz|_* \right) |d\eta|_* \\
&= \int_{\boldsymbol{V}}^{\sim} \int_{\boldsymbol{V}} \underbrace{D_z^\alpha a^{(\alpha)}(x, x+z, \xi)}_{f(z)} \, e^{-\boldsymbol{i}(z,\eta)} |dz|_* \, |d\eta|_* \\
&= \int_{\boldsymbol{V}} \widehat{f}(\eta) |d\eta|_* = f(0) = D_z^\alpha a^{(\alpha)}(x, x+z, \xi)|_{z=0},
\end{aligned}$$

where at the last step we used Fourier inversion formula. Using these facts in (2.4.12) we deduce

$$R_N(x,\xi) := \sigma_A(x,\xi) - \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!} D_y^\alpha a^{(\alpha)}(x, y, \xi)|_{x=y}$$

$$= \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 \int_{\boldsymbol{V}}^{\sim} \int_{\boldsymbol{V}} (1-t)^{N-1} a^{(\alpha)}(x, x+z, \xi+t\eta) \eta^\alpha e^{-\boldsymbol{i}(z,\eta)} |dz|_* |d\eta|_* dt$$

$$= \sum_{|\alpha|=N} \frac{N\boldsymbol{i}^N}{\alpha!} \int_0^1 \int_{\boldsymbol{V}}^{\sim} \int_{\boldsymbol{V}} (1-t)^{N-1} a^{(\alpha)}(x, x+z, \xi+t\eta) \partial_z^\alpha e^{-\boldsymbol{i}(z,\eta)} |dz|_* |d\eta|_* dt$$

$$= \sum_{|\alpha|=N} \frac{N(-\boldsymbol{i})^N}{\alpha!} \int_0^1 \int_{\boldsymbol{V}}^{\sim} \int_{\boldsymbol{V}} (1-t)^{N-1} \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta) e^{-\boldsymbol{i}(z,\eta)} |dz|_* |d\eta|_* dt.$$

For N sufficiently large these integrals are absolutely convergent, uniformly in $x \in K$, $|\xi| < R$. We need to produce estimates for the integrals

$$R_{\alpha,t} = \int_{\boldsymbol{V}} \int_{\boldsymbol{V}} \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta) e^{-\boldsymbol{i}(z,\eta)} \, |dz|_* |d\eta|_*, \quad |\alpha| = N,$$

uniform in $x \in K$ and $t \in [0,1]$. Assume $|\xi| > 1$. We split these integrals into two parts

$$R'_{\alpha,t} = \int_{|\eta| \le |\xi|/2} \int_{\boldsymbol{V}} \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta) e^{-\boldsymbol{i}(z,\eta)} \, |dz|_* |d\eta|_*,$$

$$R''_{\alpha,t} = \int_{|\eta| \ge |\xi|/2} \int_{\boldsymbol{V}} \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta) e^{-\boldsymbol{i}(z,\eta)} |dz|_* |d\eta|_*.$$

Note that

$$\mathrm{vol}\left\{ \eta; \quad |\eta| \le |\xi|/2 \right\} \sim \langle \xi \rangle^m,$$

and if $|\eta| \le |\xi|/2$, then we have

$$\sup_{x \in K, t \in [0,1]} |\partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta)| \le C \langle \xi \rangle^{k-N},$$

which proves that

$$\sup_{x \in K} |R'_{\alpha,t}(x,\xi)| \le C \langle \xi \rangle^{k+m-N}. \tag{2.4.14}$$

Consider the Laplacian

$$\Delta_z = \sum_{j=1}^m D_{z_j}^2.$$

Observe that

$$|\eta|^{-2} \Delta_z e^{-\boldsymbol{i}(z,\eta)} = e^{-\boldsymbol{i}(z,\eta)}$$

Then

$$R''_{\alpha,t} = \int_{|\eta| \ge |\xi|/2} \int_{\boldsymbol{V}} \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta) |\eta|^{-2\nu} \Delta_z^\nu e^{-\boldsymbol{i}(z,\xi)} |dz|_* |d\eta|_*$$

(integrate by parts in the z-integral)

$$= \int_{|\eta| \ge |\xi|/2} \int_{\boldsymbol{V}} \Delta_z^\nu \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta) |\eta|^{-2\nu} e^{-\boldsymbol{i}(z,\xi)} |dz|_* |d\eta|_*.$$

Now observe that

$$\sup_{x, \in K} |\Delta_z^\nu \partial_z^\alpha a^{(\alpha)}(x, x+z, \xi+t\eta)| \le C_\nu \langle \xi+t\eta \rangle^{k-N} \le C_\nu \langle \xi \rangle^{k-N} \langle t\eta \rangle^{N-k},$$

where at the second step we used Peetre's inequality, and C_ν stands for a positive constant that depends only on ν. Since $\langle t\eta \rangle \le \langle \eta \rangle$ we deduce

$$\sup_{x \in K} |R''_{\alpha,t}(x,\xi)| \le C_\nu \mathrm{vol}\,(K) \langle \xi \rangle^{k-N} \int_{|\eta| \ge |\xi|/2} \langle \eta \rangle^{N-k-2\nu} |d\nu|_*. \tag{2.4.15}$$

By choosing ν sufficiently large, $2\nu > m + N - k$, we deduce from (2.4.14) and (2.4.15) that for every compact subset $K \subset \Omega$ and any positive integer N there exists a positive constant $C = C(N,K)$ such that

$$\sup_{x \in K} \Big| a(x,\xi) - \sum_{|\alpha| \le N-1} \frac{1}{\alpha!} D_y^\alpha a^{(\alpha)}(x,y,\xi)|_{x=y} \Big| \le C \langle \xi \rangle^{k+m-N}.$$

This proves the estimate (2.4.10) and concludes the proof of the theorem. □

Remark 2.4.7. The result in Theorem 2.4.6 can be concisely formulated as follows. We introduce the second order partial differential operators

$$(\partial_x, \partial_\xi) := \sum_{j=1}^m \partial_{x_j}\partial_{\xi_j} = \sum_{j=1}^m \frac{\partial^2}{\partial x_j \partial \xi_j}.$$

Then the asymptotic expansion (2.4.7) can rewritten as

$$\sigma_A(x,\sigma) \sim \Big(e^{-\boldsymbol{i}(\partial_y,\partial_\xi)} a(x,y,\xi) \Big)|_{x=y}. \tag{2.4.16}$$

□

Corollary 2.4.8. *Suppose that $k \in \mathbb{R}$ and $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$ is a continuous linear operator such that for any $\eta, \varphi \in C_0^\infty(\Omega)$ we have $\varphi A \eta \in \boldsymbol{\Psi}^k(\Omega)$. Then $A \in \boldsymbol{\Psi}^k(\Omega)$.*

Proof. Choose a partition of unity of $(\varphi_i)_{i\in I}$ on Ω, $\varphi_i \in C_0^\infty(\Omega)$. Set $A_{ij} = \varphi_i A \varphi_j$. Then $A_{ij} \in \boldsymbol{\Psi}_0^k(\Omega)$ and we set $a_{ij}(x,\xi) = \sigma_{A_{ij}}$. Define

$$a'(x,\xi) = {\sum_{i,j}}' a_{ij}(x,\xi),$$

where $\sum'$ indicates that the summation is over pairs i,j such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \neq \emptyset$. The sum is locally finite and thus a' is well defined and $a' \in \mathbf{S}^k(\Omega)$. Set $A' = \mathbf{Op}(a')$. If K is the Schwartz kernel of A then the Schwartz kernel of $A - A'$ is

$${\sum_{i,j}}'' \varphi_i(x)\varphi_j(y) K$$

where $\sum''$ indicates that the summation is over pairs i,j such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$. Since the singular support of K is contained in the diagonal of $\Omega \times \Omega$ we deduce that the Schwartz kernel of $A - A'$ is smooth, so that $A - A' \in \boldsymbol{\Psi}^{-\infty}(\Omega)$, $A' \in \boldsymbol{\Psi}^k(\Omega)$. □

Let us summarize the facts we have uncovered so far. We denote by $\mathcal{S}^k(\Omega)$ the space $\boldsymbol{\mathcal{A}}(\Omega \times \boldsymbol{V})$ and we set

$$\mathcal{S}(\Omega) := \bigcup_{k\in\mathbb{R}} \mathcal{S}^k(\Omega), \quad \mathcal{S}^{-\infty}(\Omega) := \bigcap_{k\in\mathbb{R}} \mathcal{S}^k(\Omega).$$

We will refer to the functions in $\mathcal{S}(\Omega)$ as *symbols*.

Every symbol $\sigma \in \mathcal{S}^k(\Omega)$ can be viewed as an amplitude $\sigma \in \boldsymbol{\mathcal{A}}^k(\Omega \times \Omega \times \boldsymbol{V})$ and thus determine a ψdo $\mathbf{Op}(\sigma) : C_0^\infty(\Omega) \to C^\infty(\Omega)$ that can be alternatively defined by

$$\mathbf{Op}(\sigma)u(x) = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} \sigma(x,\xi)\widehat{u}(\xi)|d\xi|_*.$$

Conversely, to any properly supported ψdo $A \in \boldsymbol{\Psi}_0^k(\Omega)$ we can associate a symbol

$$\sigma_A(x,\xi) := e^{-\boldsymbol{i}(\xi,x)} A e^{\boldsymbol{i}(\xi,x)},$$

and $A = \mathbf{Op}(\sigma_A)$. Moreover, if $A \sim B$, $B \in \boldsymbol{\Psi}_0^k(\Omega)$, then $\sigma_A - \sigma_B \in \mathcal{S}^{-\infty}(\Omega)$. Since any ψdo is smoothly equivalent to a properly supported one we deduce that we have a natural linear bijection

$$\sigma : \boldsymbol{\Psi}(\Omega)/\boldsymbol{\Psi}^{-\infty}(\Omega) \to \mathcal{S}(\Omega)/\mathcal{S}^{-\infty}(\Omega), \tag{2.4.17}$$

that associates to a pseudo-differential operator A the symbol of a properly supported ψdo A' smoothly equivalent to A. The inverse of this map is called the *quantization* map.

2.5. Symbolic calculus

We want to prove that the composition of two properly supported ψdo's is a ψdo. This shows that space $\boldsymbol{\Psi}(\Omega)/\boldsymbol{\Psi}^{-\infty}(\Omega)$ is and algebra equipped with various other natural operations. Using the symbol map (2.4.17) we can transport these to operations on $\mathcal{S}(\Omega)/\mathcal{S}^{-\infty}(\Omega)$, and we will provide explicit descriptions of these operations on the space of symbols.

Suppose A is a properly supported ψdo. It defines a continuous linear operator $A : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$. Its *transpose* or *form dual* is the linear operator $A^\vee : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ uniquely determined by

$$\langle Au, v\rangle = \langle u, A^\vee v\rangle, \ \ \forall u, v \in C_0^\infty(\Omega).$$

The operator $A^\vee$ is also a ψdo. More precisely, if $A = \mathbf{Op}(a)$, $a \in \boldsymbol{\mathcal{A}}(\Omega \times \Omega \times \boldsymbol{V})$

$$Au(x) = \int_{\boldsymbol{V}}^{\sim} \int_\Omega e^{\boldsymbol{i}(x-y,\xi)} a(x, y, \xi) u(y) |dy||d\xi|_*,$$

then $A^\vee = \mathbf{Op}(a^\top)$,

$$A^\vee v(x) = \int_{\boldsymbol{V}}^{\sim} \int_\Omega e^{\boldsymbol{i}(x-y,\xi)} a(y, x, -\xi) v(y) |dy||d\xi|_*. \tag{2.5.1}$$

Theorem 2.5.1. *Suppose $A \in \boldsymbol{\Psi}_0^k(\Omega)$ is a properly supported ψdo with symbol $\sigma_A(\xi)$. Then $A^\vee \in \boldsymbol{\Psi}_0^k(\Omega)$ and*

$$\sigma_{A^\vee}(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \sigma_A(x, -\xi) = e^{-\boldsymbol{i}(\partial_x, \partial_\xi)} \sigma_A(x, -\xi). \tag{2.5.2}$$

Proof. We write $A = \mathbf{Op}(a)$ where $a \in \boldsymbol{\mathcal{A}}^k(\Omega \times \Omega \times \boldsymbol{V})$ is properly supported. We set $\sigma_A(x, y; \xi) := \sigma_A(x, \xi)$ so that

$$\sigma_A^\top(x, y, \xi) = \sigma_A(y, x, -\xi) = \sigma_A(y, -\xi).$$

From the equality $A = \mathbf{Op}(\sigma_A)$ we deduce $A^\vee = \mathbf{Op}(\sigma_A^\top)$ and therefore

$$\sigma_{A^\vee}(x, \xi) \sim e^{-\boldsymbol{i}(\partial_y, \partial_\xi)} \sigma_A^\top(x, y, \xi)_{y=x} = e^{-\boldsymbol{i}(\partial_y, \partial_\xi)} \sigma_A(y, -\xi)|_{y=x} = e^{-\boldsymbol{i}(\partial_x, \partial_\xi)} \sigma_A(x, -\xi).$$

□

If $A \in \boldsymbol{\Psi}_0^k(\Omega)$ is a properly supported ψdo we define its *formal adjoint* A^* to be the conjugate of its dual, i.e., for any $u \in C^\infty(\Omega)$ we have

$$A^* u = \overline{A^\vee \bar{u}}, \tag{2.5.3}$$

where for any smooth function $v : \Omega \to \mathbb{C}$ we denoted by $\bar{v}$ its conjugate. Recall that the L^2-inner product of two smooth, compactly supported functions $u, v : \Omega \to \mathbb{C}$ is

$$(u, v)_{L^2} = \langle u, \bar{v}\rangle = \int_\Omega u(x)\overline{v(x)}\,|dx|.$$

We deduce that A^* satisfies the equality

$$(u, A^* v)_{L^2} = (Au, v)_{L^2}, \ \ \forall u, v \in C_0^\infty(\Omega). \tag{2.5.4}$$

The equality (2.5.4) determines A^* uniquely,

$$\overline{A^* v} = A^{\vee}\overline{v} \Longleftrightarrow A^* v = \overline{A^{\vee}\overline{v}}.$$

From the definition we see

$$\sigma_{A^*}(x,\xi) = \overline{\sigma_{A^{\vee}}(x,-\xi)} \sim \sum_{\alpha} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{\sigma_A(x,\xi)}. \tag{2.5.5}$$

Theorem 2.5.2. *If $A \in \boldsymbol{\Psi}_0^k(\Omega)$ and $B \in \boldsymbol{\Psi}_0^\ell(\Omega)$ are properly supported ψdo's on Ω then the induced linear operator $A\circ B : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ is also a ψdo $A\circ B \in \boldsymbol{\Psi}_0^{k+\ell}(\Omega)$ and*

$$\sigma_{A\circ B}(x,\xi) \sim (\sigma_A \circledast \sigma_B)(x,\xi),$$

where

$$(\sigma_A \circledast \sigma_B)(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_A(x,\xi) D_x^\alpha \sigma_B(x,\xi). \tag{2.5.6}$$

Proof. The equality $B = (B^{\vee})^{\vee}$ shows that $B = \mathbf{Op}(\sigma_{B^{\vee}}^{\top})$ Using (2.5.1) we deduce that

$$Bu(x) = \int_{\boldsymbol{V}}\int_{\Omega} e^{\boldsymbol{i}(x-y,\xi)}\sigma_{B^{\vee}}(y,-\xi)u(y)|dy|_*|d\xi|_*, \quad \forall u \in C_0^\infty(\Omega).$$

Using the Fourier inversion formula we deduce

$$\widehat{Bu}(\xi) = \int_{\Omega} e^{-\boldsymbol{i}(y,\xi)}\sigma_{B^{\vee}}(y,-\xi)u(y)|dy|_*.$$

We deduce

$$ABu(x) = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)}\sigma_A(x,\xi)\widehat{Bu}(\xi)|d\xi|_* = \int_{\boldsymbol{V}}^{\sim}\int_{\Omega} e^{\boldsymbol{i}(x-y,\xi)}\sigma_A(x,\xi)\sigma_{B^{\vee}}(y,-\xi)u(y)\,|dy|_*\,|d\xi|_*.$$

Using Theorem 2.4.6 we deduce

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha}\frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha\big(\,\sigma_A(x,\xi)\sigma_{B^{\vee}}(y,-\xi)\,\big)_{y=x}$$

$$= \sum_{\alpha}\frac{1}{\alpha!}\partial_\xi^\alpha\big(\,\sigma_A(x,\xi)D_x^\alpha\sigma_{B^{\vee}}(x,-\xi)\,\big) \overset{Thm. 2.5.1}{\sim} \sum_{\alpha,\beta}\frac{1}{\alpha!\beta!}\partial_\xi^\alpha\big(\,\sigma_A(x,\xi)(-\partial_\xi)^\beta D_x^{\alpha+\beta}\sigma_B(x,\xi)\,\big).$$

At this point we want to invoke the following elementary result whose proof is left to the reader as an exercise.

Lemma 2.5.3 (Newton multinomial formula)**.** *For any multi-index $\gamma \in \mathbb{Z}_{\geq 0}^m$ and any $x = (x_1,\dots,x_m) \in \boldsymbol{V}$, $y = (y_1,\dots,y_m) \in \boldsymbol{V}$*

$$(x+y)^\gamma = \sum_{\kappa+\lambda=\gamma}\frac{\gamma!}{\kappa!\lambda!}x^\kappa y^\lambda. \tag{2.5.7}$$

□

Using Leibniz' formula (2.1.6) we deduce

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha,\beta,\kappa+\lambda=\alpha}\frac{(-1)^{|\beta|}}{\beta!\kappa!\lambda!}\partial_\xi^\kappa\sigma_A(x,\xi)\partial_\xi^{\lambda+\beta}D_x^{\alpha+\beta}\sigma_B(x,\xi)$$

$$= \sum_{\beta,\kappa,\lambda}\frac{(-1)^{|\beta|}}{\beta!\kappa!\lambda!}\partial_\xi^\kappa\sigma_A(x,\xi)\partial_\xi^{\lambda+\beta}D_x^{\kappa+\lambda+\beta}\sigma_B(x,\xi)$$

$$= \sum_{\kappa} \frac{1}{\kappa!} \sum_{\gamma} \left(\sum_{\beta+\lambda=\gamma} \frac{(-1)^{|\beta|}}{\beta!\lambda!} \right) \partial_{\xi}^{\kappa} \sigma_A(x,\xi) \partial_{\xi}^{\gamma} D_x^{\kappa+\gamma} \sigma_B(x,\xi)$$

Using (2.5.7) we deduce that

$$\sum_{\beta+\lambda=\gamma} \frac{(-1)^{|\beta|}}{\beta!\lambda!} = \begin{cases} 1, & \gamma = (0,\dots,0) \\ 0, & \text{otherwise.} \end{cases}$$

This shows that

$$\sigma_{AB}(x,\xi) \sim \sum_{\kappa} \frac{1}{\kappa!} \partial_{\xi}^{\kappa} \sigma_A(x,\xi) D_x^{\kappa} \sigma_B(x,\xi).$$

□

Remark 2.5.4. Note that we can reformulate (2.5.6) as

$$\sigma_{A\circ B}(x,\xi) \sim e^{-\boldsymbol{i}(\partial_y,\partial_\eta)} \sigma_A(x,\eta)\sigma_B(y,\xi)|_{\eta=\xi, y=x}.$$ □

We now want to introduce a special class of symbols, namely the *polyhomogeneous* or *classical* symbols.

Definition 2.5.5. (a) A symbol $a \in \mathcal{S}^k(\Omega)$ is called *polyhomogeneous* of degree k, if there exist smooth functions $a_j(x,\xi)$, $j = 0, 1, \dots$ that are positively homogeneous of degree $k - j$ in the variable ξ such that

$$a(x,\xi) \sim \sum_{j\geq 0} \varphi(\xi) a_j(x,\xi)$$

where $\varphi \in C^\infty(\boldsymbol{V})$,

$$\varphi(\xi) = \begin{cases} 0, & |\xi| \leq 1 \\ 1, & |\xi| \geq 2. \end{cases}.$$

We denote by $\mathcal{S}^k_{\text{phg}}(\Omega)$ the vector space of polyhomogeneous symbols of degree k and we set

$$\mathcal{S}_{\text{phg}}(\Omega) := \bigcup_{k\in\mathbb{R}} \mathcal{S}^k_{\text{phg}}(\Omega).$$

(b) A *classical* ψdo is a ψdo smoothly equivalent to a properly supported ψdo whose symbol is polyhomogeneous. We denote by ${\boldsymbol{\Psi}_{\text{phg}}}^k(\Omega)$ the set of classical ψdo's A such that $\sigma_A \in \mathcal{S}^k_{\text{phg}}(\Omega)/\mathcal{S}^{-\infty}_{\text{phg}}(\Omega)$ and we set

$$\boldsymbol{\Psi}_{\text{phg}}(\Omega) := \bigcup_{k\in\mathbb{R}} \boldsymbol{\Psi}^k_{\text{phg}}(\Omega).$$ □

We have the following immediate consequence of Theorem 2.5.1 and 2.5.2.

Corollary 2.5.6. *The transpose of a classical ψdo is a classical ψdo, and the composition of two properly supported classical ψdos is a classical ψdo.* □

2.6. Change of variables

In this section we want to investigate the effect of smooth changes in variables on ψdo's. Suppose $\Omega, \mathcal{O}$ are two open subsets in $\boldsymbol{V}$ and $F : \mathcal{O} \to \Omega$ is a diffeomorphism. Given a properly supported ψdo $A \in \boldsymbol{\Psi}_0^k(\Omega)$ we define $F^*A : C_0^\infty(\mathcal{O}) \to C_0^\infty(\mathcal{O})$ to be the linear operator defined by the commutative diagram

$$\begin{array}{ccc} C_0^\infty(\Omega) & \xrightarrow{\;A\;} & C_0^\infty(\Omega) \\ {\scriptstyle F^*}\big\downarrow & & \big\downarrow{\scriptstyle F^*} \\ C_0^\infty(\mathcal{O}) & \underset{F^*A}{\dashrightarrow} & C_0^\infty(\mathcal{O}) \end{array}$$

where $F^* : C_0^\infty(\Omega) \to C_0^\infty(\mathcal{O})$ is the pullback by F. We will refer to F^*A as the *pullback* of A via the diffeomorphism F. We denote by G the inverse of F, $G = F^{-1}$. For every $x \in \mathcal{O}$, we let $\dot{G}_x$ denote the differential of G at $F(x)$,

$$\dot{G}_x : T_{F(x)}\Omega \to T_x\mathcal{O},$$

and by $\dot{G}_x^\vee$ its transpose

$$\dot{G}_x^\vee : T_x^*\mathcal{O} \to T_{F(x)}^*\Omega.$$

Using the metric on $\boldsymbol{V}$ we can identify $T_{F(x)}^*\Omega \cong T_{F(x)}\Omega$ and $T_x^*\mathcal{O} \cong T_x\mathcal{O}$ so we can view ${}_x\dot{G}^\vee$ as a linear map

$$\dot{G}_x^\vee : T_x\mathcal{O} \to T_{F(x)}\Omega.$$

Theorem 2.6.1. *If $\mathcal{O}$, Ω, F, G and A are as above, then F^*A is a properly supported ψdo on $\mathcal{O}$, $\boldsymbol{\Psi}_0^k(\mathcal{O})$. Moreover,*

$$\sigma_{F^*A}\big(x, \eta\big) \sim \sum_\beta p_\beta(x,\eta)\sigma_A^{(\beta)}\big(F(x), \dot{G}_x^\vee \eta\big), \tag{2.6.1}$$

where

$$\sigma_A^{(\beta)}(x,\xi) := \partial_\xi^\beta \sigma_A(x,\xi),$$

$$p_\beta(x,\eta) \;\text{ is a polynomial in } \eta \text{ of degree} \leq |\beta|/2,$$

and $p_0(x,\xi) \equiv 1$. In particular, if A is classical, then so is F^*A.

Proof. Our approach is a compilation of the approaches in [**Tay**, II§5] and [**Shu**, §4]. We need an auxiliary result whose proof we defer to the end of the proof of Theorem 2.6.1.

Lemma 2.6.2. *There exists a neighborhood $\mathcal{N}$ of the diagonal $\Delta_\mathcal{O} \subset \mathcal{O} \times \mathcal{O}$ and a smooth map*

$$T : \mathcal{N} \to \mathrm{GL}(\boldsymbol{V})$$

such that

$$(F(x) - F(y), \eta) = (x - y, T(x,y)\eta), \;\; \forall (x,y) \in \mathcal{N}, \;\; \eta \in \boldsymbol{V}$$

and

$$\det T(x,x) = \dot{F}_x^\vee, \forall x \in \mathcal{O}. \qquad \square$$

We now want to present the proof of Theorem 2.6.1 assuming Lemma 2.6.2. Suppose $A \in \boldsymbol{\Psi}_0^k(\Omega)$. We set $\mathcal{A} = F^*A$. Then

$$\mathcal{A}u(x) = \int_{\boldsymbol{V}}^{\sim} \int_{\mathcal{O}} e^{\boldsymbol{i}(F(x)-F(y)\,,\,\xi)}\,\sigma_A\big(\,F(x),\xi\,\big)u(y)|\det \dot{F}_y||dy|_*|d\xi|_*.$$

Equivalently, this means that

$$\langle \mathcal{A}u, v\rangle = \langle K_{\mathcal{A}}, v\otimes u\rangle, \;\; \forall u, v \in C_0^\infty(\mathcal{O}),$$

where the kernel $K_{\mathcal{A}}$ is the distribution on $\mathcal{O}\times\mathcal{O}$ defined by the oscillatory integral.

$$K_{\mathcal{A}}(x,y) = (2\pi)^{-m/2}\int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(F(x)-F(y)\,,\,\xi)}\,\sigma_A\big(\,F(x),\xi\,\big)|\det \dot{F}_y||d\xi|_*$$

The phase $\Phi(x,y,\xi) = (F(x)-F(y)\,,\,\xi)$ satisfies all the assumptions in Lemma 2.6.2.

Choose a neighborhood $\mathcal{N}$ of the diagonal $\Delta_{\mathcal{O}}$ in $\mathcal{O}\times\mathcal{O}$ and a map $T:\mathcal{N}\to \mathrm{GL}(\boldsymbol{V})$ as in Lemma 2.6.2. Next choose another closed neighborhood $\mathcal{N}_1$ such that $\mathcal{N}_1 \subset \boldsymbol{int}\,\mathcal{N}$. Finally, choose a smooth function $\varphi:\mathcal{O}\times\mathcal{O}\to[0,\infty)$ such that $\varphi|_{\mathcal{N}_1}\equiv 1$ and $\operatorname{supp}\varphi\subset\mathcal{N}$. Then

$$K_{\mathcal{A}} = \varphi K_{\mathcal{A}} + (1-\varphi)K_{\mathcal{A}}.$$

From (2.2.2) we deduce that $\operatorname{sing\,supp} K_{\mathcal{A}}\subset\Delta_{\mathcal{O}}$ so that $(1-\varphi)K_{\mathcal{A}}\in C^\infty(\mathcal{O}\times\mathcal{O})$. Denote by $\mathcal{A}_\varphi$ the operator defined by the kernel $\varphi K_{\mathcal{A}}$. We deduce that $\mathcal{A}-\mathcal{A}_\varphi$ is the operator defined by the smooth kernel $(1-\varphi)K_{\mathcal{A}}$. Proposition 2.2.6 then implies that $\mathcal{A}-\mathcal{A}_\varphi$ is a smoothing operator. Thus, it suffices to check that $\mathcal{A}_\varphi$ is a ψdo. We have

$$\mathcal{A}_\varphi u(x) = \int_{\boldsymbol{V}}^{\sim}\int_{\mathcal{O}} e^{\boldsymbol{i}(F(x)-F(y)\,,\,\xi)}\,\varphi(x,y)\sigma_A\big(\,F(x),\xi\,\big)u(y)|\det\dot{F}_y||dy|_*|d\xi|_*$$

$$= \int_{\boldsymbol{V}}^{\sim}\int_{\mathcal{O}} e^{\boldsymbol{i}(x-y,T(x,y)\xi)}\,\varphi(x,y)a\big(\,F(x),\xi\,\big)u(y)|\det\dot{F}_y||dy|_*|d\xi|_*$$

$$= \int_{\boldsymbol{V}}^{\sim}\int_{\mathcal{O}} e^{\boldsymbol{i}(x-y,\eta)}\,\underbrace{\varphi(x,y)a\big(\,F(x),T(x,y)^{-1}\eta\,\big)|\det T(x,y)|^{-1}|\det\dot{F}_y|}_{\widetilde{a}(x,y,\eta)}\,u(y)|dy|_*|d\eta|_*.$$

The last equality of oscillatory integrals is justified by observing that $\widetilde{a}(x,y,\eta)\in\boldsymbol{A}^k(\mathcal{O}\times\mathcal{O}\times\boldsymbol{V})$ and then invoking the universality trick, Proposition 2.2.2. Theorem 2.4.6 now implies that $\mathcal{A}_\varphi\in\boldsymbol{\Psi}^k(\mathcal{O})$, and

$$\sigma_{\mathcal{A}}(x,\eta)\sim\sum_\alpha\frac{1}{\alpha!}\partial_\eta^\alpha D_y^\alpha\widetilde{a}(x,y,\eta)|_{y=x}.$$

We write

$$\widetilde{a}(x,y,\eta) = a(F(x),S(x,y)\eta)w(x,y),$$

where

$$S(x,y) = T(x,y)^{-1}, \;\; w(x,y) = \varphi(x,y)|\det S(x,y)||\det\dot{F}_y|.$$

Now observe that $S(x,x) = {}_x\dot{G}$ and $\partial_\eta^\alpha D_y^\alpha\widetilde{a}(x,y,\eta)|_{y=x}$ is a sum of terms of the form

$$c(x)\eta^\gamma\sigma_A^{(\beta)}(F(x),\dot{G}_x^\vee\eta),$$

where $c(x)$ depends only on F and

$$|\beta|\le 2|\alpha|, \;\; |\gamma|+|\alpha|\le|\beta|.$$

This implies that

$$|\gamma|\le|\beta|-|\alpha|\le|\beta|-|\beta|/2 = |\beta|/2,$$

and concludes the proof of Theorem 2.6.1.

□

Proof of Lemma 2.6.2. We have

$$(F(x)-F(y),\xi) = \int_0^1 \frac{d}{dt}\big(F(\,y+t(x-y)\,),\xi\,\big)dt. \tag{2.6.2}$$

We denote by $L(x,y)$ the linear operator $\boldsymbol{V}\to\boldsymbol{V}$ defined by

$$L(x,y)=\int_0^1 \dot{F}_{y_t},\ \ y_t=y+t(x-y).$$

Then $L(x,y)$ depends smoothly on x and y and we can rewrite (2.6.2) as

$$(\,F(x)-F(y),\xi\,)=(\,L(x,y)(x-y),\xi)=(x-y,L(x,y)^{\vee}\xi).$$

Observe that $L(x,x)=\dot{F}_x$. Since $\dot{F}_x\in \mathrm{GL}(\boldsymbol{V})$, $\forall x\in\mathcal{O}$, we deduce $L(x,y)\in\mathrm{GL}(\boldsymbol{V})$ for all (x,y) in a neighborhood $\mathcal{N}$ of the diagonal $\Delta_{\mathcal{O}}$. Now define $T(x,y)=L(x,y)^{\vee}$. □

Remark 2.6.3. With a little bit of extra effort one can show that

$$\sigma_{F^*A}\big(G(x),\eta\,\big)\sim\sum_\alpha\frac{1}{\alpha!}\sigma_A^{(\alpha)}\big(\,x,\dot{G}_x^{\vee}\eta)D_z^\alpha e^{\boldsymbol{i}(q_x(z),\eta)}, \tag{2.6.3}$$

where $q_x(z):=G(z)-G(x)-\dot{G}_x(z-x)$. For details we refer to [**Shu**, Thm. 4.2]. □

Corollary 2.6.4. *If $F:\mathcal{O}\to\Omega$ is a diffeomorphism, and $A\in\boldsymbol{\Psi}(\Omega)$,* non necessarily properly supported, *then $F^*A\in\boldsymbol{\Psi}(\mathcal{O})$.*

Proof. We write $A=A_0+S$ where A_0 is a proper ψdo and S is smoothing. Then $F^*A=F^*A_0+F^*S$, so it suffices to show that F^*S is smoothing, i.e., it is an integral operator with smooth kernel. This is obvious since S is such an operator. □

Observe that the diffeomorphism $F:\mathcal{O}\to\Omega$ induces a diffeomorphism

$$\widetilde{F}:T^*\mathcal{O}\to T^*\Omega,\ \ (x,\eta)\mapsto\big(\,F(x),(\dot{F}^{\vee})^{-1}\eta\,\big). \tag{2.6.4}$$

If we use the metric induced identifications $T^*\mathcal{O}\cong\mathcal{O}\times\boldsymbol{V}$, $T^*\Omega\cong\Omega\times\boldsymbol{V}$ then we can describe the diffeomorphism $\widetilde{F}$ as

$$\mathcal{O}\times\boldsymbol{V}\ni(x,\eta)\mapsto(F(x),(\dot{F}_x{}^{\vee})^{-1}\eta)=(F(x),\dot{G}_x\eta)\in\Omega\times\boldsymbol{V}.$$

If $\sigma_A\in\mathcal{S}^k(\Omega)$, then we can regard σ_A as a function on $T^*\Omega$. The asymptotic expansion (2.6.1) implies that

$$\widetilde{F}^*\sigma_A-\sigma_{F^*A}\in\mathcal{S}^{k-1}(\mathcal{O}). \tag{2.6.5}$$

For any open set $D\subset\boldsymbol{V}$, and any real number k we define

$$\Sigma^k(D):=\mathcal{S}^k(D)/\mathcal{S}^{k-1}(D),\ \ \Sigma^k_{\mathrm{phg}}(D):=\mathcal{S}^k_{\mathrm{phg}}(D)/\mathcal{S}^{k-1}_{\mathrm{phg}}(D).$$

For every $\sigma\in\mathcal{S}^k(D)$ we denote by σ^π its image in $\Sigma^k(\Omega)$, and we will refer to it as the *principal part* of σ. We can now rephrase the equality (2.6.5) as

$$(\widetilde{F}^*\sigma_A)^\pi=\sigma^\pi_{F^*A}. \tag{2.6.6}$$

Definition 2.6.5. A ψdo $A \in \mathbf{\Psi}_0(\Omega)$ is said to have order k if $A \in \mathbf{\Psi}_0^k(\Omega)$ and $\sigma_A^\pi \neq 0$. In this case the quantity σ_A^π is called the *principal symbol* of A. □

Observe that

$$\sigma_{AB}^\pi = \sigma_A^\pi \sigma_B^\pi.$$

For classical ψdo's the principal symbol can be canonically identified with a function defined on the punctured cotangent bundle

$$\widehat{T}^*\Omega := T^*\Omega - \text{zero section}.$$

Denote by $\mathcal{H}^k(\widehat{T}^*\Omega)$ the space of smooth functions $a = a(x,\xi) : \widehat{T}^*\Omega \to \mathbb{C}$ that are homogeneous of degree k in ξ. Consider a polyhomogeneous symbol

$$\sigma = \sigma(x,\xi) \in \mathcal{S}^k_{\text{phg}}(\Omega).$$

Thus σ has an asymptotic expansion

$$\sigma(x,\xi) \sim \sum_{j\geq 0} \varphi(\xi)\sigma_{k-j}(x,\xi),$$

where $\sigma_{k-j} \in \mathcal{H}^{k-j}(\widehat{T}^*\Omega)$, and $\varphi(\xi)$ is a smooth cutoff function

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \geq 2 \\ 0, & |\xi| \leq 1. \end{cases}$$

Observe that for any $\xi \neq 0$ and any $x \in \Omega$ we have

$$\sigma_k(x,\xi) = \lim_{t\to\infty} t^{-k}\sigma(x,t\xi).$$

We say that σ_k is the *leading term* of the polyhomogeneous symbol σ and we denote it by $[\sigma]$. This defines a linear map

$$\mathcal{S}^k_{\text{phg}}(\Omega) \ni \sigma \mapsto [\sigma] \in \mathcal{H}^k(\widehat{T}^*\Omega)$$

that vanishes on $\mathcal{S}^{k-1}_{\text{phg}}(\Omega)$. The induced map

$$\Sigma^k_{\text{phg}}(\Omega) \to \mathcal{H}^k(\widehat{T}^*\Omega).$$

is a linear isomorphism. In particular, we can identify $[\sigma]$ with σ^π because

$$\sigma_1^\pi = \sigma_2^\pi \Longleftrightarrow [\sigma_1] = [\sigma_2], \ \ \forall \sigma_1, \sigma_2 \in \mathcal{S}^k_{\text{phg}}(\Omega).$$

We obtain in this fashion a linear map

$$\mathbf{\Psi}^k_{\text{phg}}(\Omega) \ni A \mapsto [\sigma_A] \in \mathcal{H}^k(\widehat{T}^*\Omega),$$

We will continue to refer to it as the *principal symbol* of a classical ψdo.

Denote by $\operatorname{Diff}(\Omega)$ the group of diffeomorphisms of Ω. We have (right) actions of $\operatorname{Diff}(\Omega)$ on $\mathbf{\Psi}^k_{\text{phg}}(\Omega)$ and $\mathcal{H}^k(\widehat{T}^*\Omega)$,

$$\mathbf{\Psi}^k_{\text{phg}}(\Omega) \times \operatorname{Diff}(\Omega) \ni (A,F) \mapsto F^*A \in {\mathbf{\Psi}_{\text{phg}}}^k(\Omega),$$

$$\mathcal{H}^k(\widehat{T}^*\Omega) \times \operatorname{Diff}(\Omega) \ni (a,F) \mapsto \widetilde{F}^*a \in \mathcal{H}^k(\widehat{T}^*\Omega).$$

We can now rephrase the equality (2.6.6) in the following geometric fashion.

Corollary 2.6.6. *The principal symbol map* $\mathbf{\Psi}^k_{\text{phg}}(\Omega) \to \mathcal{H}^k(\widehat{T}^*\Omega)$ *is equivariant with respect to the canonical (right) action of the group* $\operatorname{Diff}(\Omega)$ *on* $\mathbf{\Psi}^k_{\text{phg}}(\Omega)$ *and* $\mathcal{H}^k(\widehat{T}^*\Omega)$. □

Example 2.6.7 (Symbols of differential operators)**.** Suppose

$$L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial_x^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$$

is a partial differential operator. The full symbol is the function

$$\sigma_L(x,\xi) = e^{-\boldsymbol{i}(\xi,x)} L e^{\boldsymbol{i}(\xi,x)}.$$

We would like to explain a method of computing its principal symbol

$$[\sigma_L](x,\xi) = \boldsymbol{i}^k \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha,$$

regarded as a function on $T^*\Omega$ homogeneous of degree k in the fiber coordinates ξ. This method is particularly useful when working on manifolds.

To do this define for every smooth function $f : \Omega \to \mathbb{R}$, and every partial differential operator P of order ℓ on Ω a new partial differential operator

$$\mathrm{ad}(f)P : C^\infty(\Omega) \to C^\infty(\Omega),\ \ \mathrm{ad}(f)Pu = P(fu) - fPu,\ \ \forall u \in C^\infty(\Omega).$$

If we denote by $\boldsymbol{PDO}^\ell(\Omega)$ the set of partial differential operators of order $\leq \ell$ on Ω and we set

$$\boldsymbol{PDO}(\Omega) := \bigcup_{\ell=0}^\infty \boldsymbol{PDO}^\ell(\Omega)$$

then we see that $\mathrm{ad}(f)$ defines a linear operator

$$\mathrm{ad}(f) : \boldsymbol{PDO}(\Omega) \to \boldsymbol{PDO}(\Omega)$$

such that

$$\mathrm{ad}(f)\Big(\boldsymbol{PDO}^\ell(\Omega)\Big) \subset \boldsymbol{PDO}^{\ell-1}(\Omega),\ \ \forall \ell \geq 0.$$

The operator $\mathrm{ad}(f)$ is a derivation of the algebra $\boldsymbol{PDO}((\Omega)$ in the sense that it satisfies the Leibniz rule

$$\mathrm{ad}(f)(PQ) = \big(\mathrm{ad}(f)P\big)Q + P\big(\mathrm{ad}(f)Q\ \big),\ \ \forall P,Q \in \boldsymbol{PDO}(\Omega). \tag{2.6.7}$$

If L has order k, $x_0 \in \Omega$, $\xi_0 \in T^*_{x_0}\Omega$ and $f : \Omega \to \mathbb{R}$ is a smooth function such that $df(x_0) = \xi_0$. Then $\mathrm{ad}(f)^k L$ is a zeroth order partial differential operator on Ω and thus can be identified with a smooth function $s_{f,L} : \Omega \to \mathbb{C}$. Then

$$[\sigma_L](x_0,\xi_0) = \frac{\boldsymbol{i}^k}{k!} s_{f,L}(x_0) = \frac{\boldsymbol{i}^\ell}{k!}(\mathrm{ad}(f)^\ell L)(x_0).$$

Thus we can write

$$[\sigma_L](x, df(x)) = \frac{\boldsymbol{i}^\ell}{\ell!}(\mathrm{ad}(f)^\ell L)(x),\ \ \forall f \in C^\infty(\Omega), x \in \Omega. \tag{2.6.8}$$

Equivalently, we consider the operator $e^{\boldsymbol{it}\,\mathrm{ad}(f)} : \boldsymbol{PDO} \to \boldsymbol{PDO}$. For every $P \in \boldsymbol{PDO}^k$ we obtain a polynomial in t with coefficients in $\boldsymbol{PDO}$

$$e^{\boldsymbol{it}\,\mathrm{ad}(f)}P \in \boldsymbol{PDO}[t]],\ \ \deg_t e^{\boldsymbol{it}\,\mathrm{ad}(f)}P \leq k.$$

The principal symbol of P is then the leading coefficient of this polynomial.

□

2.7. Vectorial Pseudo-Differential Operators

So far we have presented only *scalar* pseudo-differential operators, i.e., those acting on complex valued functions. Often in geometry we are faced with operators acting on smooth sections of complex vector bundles. Over $\mathbb{R}^m$ such vector bundles are trivializable, an their sections can be viewed as vector valued functions. In this sections we will briefly indicate how to extend the general theory presented so far in order to include such situations.

Suppose E_0, E_1 are complex vector spaces of dimensions r_0 and respectively r_1. If Ω is an open subset in $\boldsymbol{V}$, then we can regard the space $C^\infty(\Omega, E_j)$ of smooth functions $\Omega \to E_j$, $j = 0, 1$, as the space of smooth sections of the trivial vector bundle $\underline{E_j}_\Omega := \Omega \times E_j \to \Omega$.

Recall that

$$C^{-\infty}(\Omega, E_j) = C_0^\infty(\Omega, {E_j}^\vee)^\vee, \;\; C_0^{-\infty}(\Omega, E_j) = C^\infty(\Omega, {E_j}^\vee)^\vee.$$

Recall that we defined scalar ψdo's on Ω using their kernel which are distributions $K \in C^{-\infty}(\Omega \times \Omega)$ defined by certain oscillatory integrals. We use the same approach using kernels defined by oscillatory integrals of the form

$$K_a(x, y) = (2\pi)^{-m/2} \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x-y,\xi)} a(x, y, \xi) |d\xi|_*,$$

where the amplitude is a function

$$a : \Omega \times \Omega \times \boldsymbol{V} \to \operatorname{Hom}(E_0, E_1) \cong E_1 \otimes {E_0}^\vee$$

satisfying growth conditions of the type (2.1.3), where the norms $|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)|$ are defined in terms of Hermitian inner products on E_0 and E_1. We denote by $\boldsymbol{\mathcal{A}}(\Omega^2; E_0, E_1)$ the vector space of such amplitudes.

The arguments in the proof of Theorem 2.1.6 show that such an oscillatory integral defines a distribution

$$K_a \in C^{-\infty}(\Omega \times \Omega, E_1 \otimes {E_0}^\vee).$$

Given $a \in \boldsymbol{\mathcal{A}}(\Omega^2; E_0, E_1)$ we define

$$\mathbf{Op}(a) : C_0^\infty(\Omega, E_0) \to C^{-\infty}(\Omega, E_1)$$

via the equality

$$\langle \mathbf{Op}(a)u, v\rangle := \langle K_a, v \boxtimes u\rangle, \;\; \forall u \in C_0^\infty(\Omega, E_0), \; v \in C_0^\infty(\Omega, {E_1}^\vee). \tag{2.7.1}$$

The above equality requires some explanations. Given u, v as above we define $v \boxtimes u$ to be the function

$$v \boxtimes u \in C_0^\infty(\Omega \times \Omega, {E_1}^\vee \otimes E_0), \;\; (v \boxtimes u)(x, y) = v(x) \otimes u(y).$$

The pairing in the left-hand-side of (2.7.1) is the natural pairing between $C^{-\infty}(\Omega, E_1)$ and $C_0^\infty(\Omega, {E_1}^\vee)$ while the pairing in the right-hand-side of (2.7.1) is the natural pairing between $C^{-\infty}(\Omega^2, E_1 \otimes {E_0}^\vee)$ and $C_0^\infty(\Omega^2, {E_1}^\vee \otimes E_0)$.

Arguing exactly as in Proposition 2.2.3 we deduce that $\mathbf{Op}(a)$ induces a *continuous* linear operator

$$C_0^\infty(\Omega, E_0) \to C^\infty(\Omega, E_1).$$

The definition of the transpose of a vectorial ψdo is a bit more involved.

We recall that there exists a natural bijection $\operatorname{Hom}(E_0, E_1) \to \operatorname{Hom}({E_1}^\vee, {E_0}^\vee)$ that associates to each complex linear map $T : E_0 \to E_1$ its dual $T^\vee : {E_1}^\vee \to {E_0}^\vee$. This induces a transposition map

$$\boldsymbol{\mathcal{A}}(\Omega^2, E_0, E_1) \ni a \mapsto a^\top \in \boldsymbol{\mathcal{A}}(\Omega^2, {E_1}^\vee, {E_0}^\vee), \;\; a^\top(x, y, \xi) := a(y, x, -\xi)^\vee.$$

The continuous linear operator

$$\mathbf{Op}(a^\top) : C_0^\infty(\Omega, E_1{}^\vee) \to C^\infty(\Omega, E_0{}^\vee),$$

satisfies

$$\langle u, \mathbf{Op}(a^\top)v\rangle = \langle \mathbf{Op}(a)u, v\rangle, \;\; \forall u \in C_0^\infty(\Omega, E_0), \; v \in C_0^\infty(\Omega, E_1{}^\vee).$$

This this shows that the dual operator

$$\mathbf{Op}(a^\top)^\vee : C^\infty(\Omega, E_0{}^\vee)^\vee = C_0^{-\infty}(\Omega) \to C_0^\infty(\Omega, E_1{}^\vee)^\vee = C^{-\infty}(\Omega, E_1),$$

is an extension of

$$\mathbf{Op}(a) : C_0^\infty(\Omega, E_0) \to C^\infty(\Omega, E_1).$$

The notion of properly supported ψdo extends in an obvious fashion to vectorial ψdo's and we get a vector space $\mathbf{\Psi}_0(\Omega, E_0, E_1)$ of properly supported ψdos mapping sections of $\underline{E_0}_\Omega$ to sections of $\underline{E_1}_\Omega$. More precisely, any $A \in \mathbf{\Psi}_0(\Omega, E_0, E_1)$ induces continuous linear operators

$$A : C^\infty(\Omega, E_0) \to C^\infty(\Omega, E_1) \;\text{ and }\; A : C_0^\infty(\Omega, E_0) \to C_0^\infty(\Omega, E_1).$$

The symbol of a properly supported ψdo $A \in \mathbf{\Psi}_0(\Omega, E_0, E_1)$ is the function

$$\sigma_A : \Omega \times \boldsymbol{V} \to \mathrm{Hom}(E_0, E_1)$$

defined by

$$\sigma_A(x,\xi)\boldsymbol{u} := e^{-\boldsymbol{i}(x,\xi)} A e^{\boldsymbol{i}(x,\xi)} \underline{\boldsymbol{u}}, \;\; \forall (x,\xi,\boldsymbol{u}) \in \Omega \times \boldsymbol{V} \times E_0,$$

where $\underline{\boldsymbol{u}} : \Omega \to E_0$ is the constant function $\Omega \ni x \mapsto \boldsymbol{u} \in E_0$. The symbol admits an asymptotic expansion of the type (2.4.11). The proof is identical to the scalar case. In particular, the notion of classical ψdo extends word for word to the vector case. We obtain two spaces of matrix valued symbols

$$\mathcal{S}(\Omega, E_0, E_1) \supset \mathcal{S}_{\mathrm{phg}}(\Omega, E_0, E_1).$$

The vectorial counterpart of Theorem 2.5.1 is

$$\sigma_{A^\vee}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \sigma_A(x,-\xi)^\vee = e^{-\boldsymbol{i}(\partial_x, \partial_\xi)} \sigma_A(x,-\xi)^\vee, \tag{2.7.2}$$

while Theorem 2.5.2 generalizes word for word to the vectorial case. The formal adjoint of a properly supported ψdo $A \in \mathbf{\Psi}_0^k(\Omega, E_0, E_1)$ is defined as in the scalar case by the equality (2.5.3). The equality (2.5.5) has the vectorial counterpart

$$\sigma_{A^*}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \sigma_A(x,-\xi)^* = e^{-\boldsymbol{i}(\partial_x, \partial_\xi)} \sigma_A(x,\xi)^*, \tag{2.7.3}$$

where $\sigma_A(x,\xi) : E_1 \to E_0$ is the conjugate transpose of the linear map $\sigma_A(x,\xi) : E_0 \to E_1$.

The change in variables formula requires a bit more care since in the vectorial case there are several possible changes of variables: change of variables on Ω, and conjugation with automorphisms of the trivial bundles $\underline{E_j}_\Omega$. Since a bundle automorphism can be viewed as a ψdo of order zero we see that the conjugation of a ψdo with such automorphisms produces another ψdo. The effect of the changes of coordinates on the base of these vector bundles can be understood using the same techniques we used in the scalar case. The up-shot is: the class of vectorial ψdo's is closed under changes of coordinates on Ω and conjugations by bundle automorphisms of $\underline{E_j}_\Omega$.

The notion of principal symbol of a classical ψdo requires much more care. Again, denote by $\widehat{T}^*\Omega$ the punctured cotangent bundle of Ω and by $\pi : \widehat{T}^*\Omega \to \Omega$ the natural projection. We form the pullback bundles $\pi^* E_j := \pi^* \underline{E_j}_\Omega$, and we denote by $\mathcal{H}^k\big(\mathrm{Hom}(\pi^*E_0, \pi^*E_1)\big)$ the space of smooth sections σ of the vector bundle $\mathrm{Hom}(\pi^* E_0, \pi^* E_1) \to \widehat{T}^*\Omega$ such that, for any $x \in \Omega$, the restriction of σ to $T_x^*\Omega \setminus \{0\}$ is a homogeneous function of degree k

$$T_x^*\Omega \setminus \{0\} \ni \xi \mapsto \sigma(x,\xi) \in \mathrm{Hom}(\,\underline{E_0}_x, \underline{E_1}_x\,),$$

where $\underline{E_j}_x$ denotes the fiber over $x \in \Omega$ of the vector bundle $\underline{E_j}_\Omega$.

We fix an open set $\mathcal{O} \subset \boldsymbol{V}$, a diffeomorphism $F : \mathcal{O} \to \Omega$ and bundle isomorphisms $T_j : \underline{E_j}_\Omega \to F^*\underline{E_j}_\Omega$ covering F, i.e., the diagrams below are commutative

$$\begin{array}{ccc} F^*\underline{E_j}_\Omega & \xrightarrow{T_j} & \underline{E_j}_\Omega \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow[F]{} & \Omega \end{array}$$

For $j = 0, 1$ we then get bijections

$$F_{T_j} : C^\infty(F^*\underline{E_j}_\Omega) \to C^\infty(\underline{E_j}_\Omega), \;\; F_{T_j}u(F(x)) = T_j(x)u(x), \;\; \forall u \in C^\infty(F^*\underline{E_j}_\Omega), \;\; x \in \mathcal{O}.$$

Given $A \in \boldsymbol{\Psi}_0^k(\Omega, E_0, E_1)$ we define

$$T_1^{-1}F^*AT_0 := F_{T_1}^{-1}AF_{T_0} : C^\infty\big(F^*\underline{E_0}_\Omega\big) \to C^\infty(F^*\underline{E_1}_\Omega),$$

so that the diagram below is commutative

$$\begin{array}{ccc} C^\infty(F^*\underline{E_0}_\Omega) & \xrightarrow{T_1{}^{-1}F^*AT_0} & C^\infty(F^*\underline{E_1}_\Omega) \\ {\scriptstyle F_{T_0}}\downarrow & & \downarrow{\scriptstyle F_{T_1}} \\ C^\infty(\underline{E_0}_\Omega) & \xrightarrow[A]{} & C^\infty(\underline{E_1}_\Omega) \end{array}$$

Then

$$A \in \boldsymbol{\Psi}_0^k(\Omega, E_0, E_1) \Rightarrow T_1{}^{-1}F^*AT_0 \in \boldsymbol{\Psi}_0^k(\mathcal{O}, E_0, E_1). \tag{2.7.4a}$$

$$A \in \boldsymbol{\Psi}_{\mathrm{phg}}^k(\Omega, E_0, E_1) \Rightarrow T_1{}^{-1}F^*AT_0 \in \boldsymbol{\Psi}_{\mathrm{phg}}^k(\mathcal{O}, E_0, E_1). \tag{2.7.4b}$$

Now observe that the diffeomorphism F induces a diffeomorphism $\widetilde{F} : T^*\mathcal{O} \to T^*\Omega$ defined as in (2.6.4). The bundle isomorphisms T_j induce bundle isomorphisms

$$\widetilde{T}_j : \pi^*F^*\underline{E_j}_\Omega \to \pi^*\underline{E_j}_\Omega$$

covering $\widetilde{F}$, i.e., the diagrams below are commutative

$$\begin{array}{ccc} \pi^*F^*\underline{E_j}_\Omega & \xrightarrow{\widetilde{T}_j} & \underline{E_j}_\Omega \\ \downarrow & & \downarrow \\ \widehat{T}^*\mathcal{O} & \xrightarrow[\widetilde{F}]{} & \widehat{T}^*\Omega \end{array}$$

We thus get a linear map

$$C^\infty\big(\mathrm{Hom}(\pi^*E_0, \pi^*E_1)\big) \ni \sigma \mapsto \widetilde{T}_1^{-1}\sigma\widetilde{T}_0 \in C^\infty\big(\mathrm{Hom}(\pi^*F^*E_0, \pi^*F^*E_1)\big).$$

The change in variables formula implies that for any $A \in \mathbf{\Psi}_0^k(\Omega, E_0, E_1)$ we have

$$\sigma_{T_1^{-1}F^*AT_0} - \widetilde{T}_1^{-1}F^*\sigma_A T_0 \in \mathcal{S}^{k-1}(\mathcal{O}, E_0, E_1).$$

The above constructions define right actions of the groups $\mathrm{Diff}(\Omega)\times\mathrm{Aut}(E_0)\times\mathrm{Aut}(E_1)$ on $\mathbf{\Psi}^k_{\mathrm{phg}}(\Omega, E_0, E_1)$ and $\mathcal{H}^k\big(\mathrm{Hom}(\pi^*E_0, \pi^*E_1)\big)$, and the principal symbol map

$$\mathbf{\Psi}^k_{\mathrm{phg}}(\Omega, E_0, E_1) \ni A \mapsto [\sigma_A] \in \mathcal{H}^k\big(\mathrm{Hom}(\pi^*E_0, \pi^*E_1)\big) \tag{2.7.5}$$

is equivariant with respect to these actions. We have the equalities

$$[\sigma_{AB}] = [\sigma_A] \circ [\sigma_B], \quad [\sigma_{A^*}] = [\sigma_A]^*.$$

Example 2.7.1 (Vectorial partial differential operators)**.** Consider a vectorial partial differential operator of order ℓ

$$L = \sum_{|\alpha|\leq\ell} a_\alpha(x)\partial_x^\alpha : C^\infty(\Omega, E_0) \to C^\infty(\Omega, E_1),$$

where the coefficients a_α are smooth maps $\Omega \to \mathrm{Hom}(E_0, E_1)$. Then

$$[\sigma_L](x,\xi) = \boldsymbol{i}^\ell \sum_{|\alpha|=\ell} a_\alpha(x)\xi^\alpha.$$

We denote by $\boldsymbol{PDO}^\ell(\Omega, E_0, E_1)$ the space of partial differential operators $C^\infty(\Omega, E_0) \to C^\infty(\Omega, E_1)$ of order $\leq \ell$. As in the scalar case, any smooth function $f : \Omega \to \mathbb{R}$ defines a linear map

$$\mathrm{ad}(f) : \boldsymbol{PDO}^\ell(\Omega, E_0, E_1) \to \boldsymbol{PDO}^{\ell-1}(\Omega, E_0, E_1), \quad L \mapsto [L, m_f],$$

where m_f denote the operator of multiplication by f and $[-,-]$ denotes the commutator of two operators. For every $L \in \boldsymbol{PDO}^\ell(\Omega, E_0, E_1)$, $x \in \Omega$ we have

$$[\sigma_L](x, df(x)) = \frac{\boldsymbol{i}^\ell}{\ell!}\,\mathrm{ad}(f)^\ell L.$$

Consider by way of example the exterior derivative

$$d : \Omega^\bullet(T^*\Omega \otimes \mathbb{C}) \to \Omega^\bullet(T^*\Omega \otimes \mathbb{C}).$$

A complex valued form $\omega \in \Omega^\bullet(T^*\boldsymbol{V} \otimes \mathbb{C})$ can be viewed as a smooth section of the complex vector bundle $\Lambda^\bullet T^*\Omega \otimes \mathbb{C}$ with fiber $E_0 = E_1 = \Lambda V^\vee \otimes \mathbb{C}$. If $f : \Omega \to \mathbb{R}$ is a smooth function and $\omega \in \Omega^\bullet(T^*\Omega \otimes \mathbb{C})$ then

$$(\,\mathrm{ad}(f)d\,)\omega = d(f\omega) - fd\omega = df \wedge \omega$$

and we deduce that the principal symbol of d is given by exterior multiplication by $\boldsymbol{i}\xi$,

$$[\sigma_d](x,\xi) = \boldsymbol{i}\xi \wedge\,. \qquad \square$$

2.8. Functional properties of ψdo's

Observe that for every real k the function

$$\lambda_k(x,\xi) = \langle \xi \rangle^k = (1+|\xi|^2)^{k/2}$$

is a classical symbol of order k on Ω. Indeed, we can write

$$\lambda_k(x,\xi) = |\xi|^k (1+|\xi|^{-2})^{k/2}, \;\; \forall \xi \neq 0,$$

and we deduce that we have the following asymptotic expansion as $|\xi| \to \infty$

$$\lambda_k(x,\xi) = |\xi|^{-k} \sum_{\ell \geq 0} \binom{k/2}{\ell} |\xi|^{-2\ell}.$$

We denote by $\Lambda_k \in \boldsymbol{\Psi}^k(\boldsymbol{V})$ the ψdo with symbol $\lambda_k(x,\xi)$ given by

$$\Lambda_k u(x) = \mathcal{F}^{-1}\big(\langle \xi \rangle^k \widehat{u}(\xi) \big) = \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} \langle \xi \rangle^k \widehat{u}(\xi)\, |d\xi|_*, \;\; \forall u \in C_0^\infty(\boldsymbol{V}).$$

The operator Λ_k defines isometries

$$\Lambda_k : H^s(\boldsymbol{V}) \to H^{s-k}(\boldsymbol{V}), \;\; \forall s \in \mathbb{R}.$$

Recall that for every $s \in \mathbb{R}$ we have defined the locally convex spaces Hilbert space $H^s_{\mathrm{comp}}(\Omega)$ and $H^s_{\mathrm{loc}}(\Omega)$.

Theorem 2.8.1. *Let $a \in \mathcal{S}^\ell(\Omega)$. Then $\mathbf{Op}(a)$ induces a continuous linear operator*

$$\mathbf{Op}(a) : H^s_{\mathrm{comp}}(\Omega) \to H^{s-\ell}_{\mathrm{loc}}(\Omega),$$

for any $s \in \mathbb{R}$. More precisely, for any $\varphi \in C_0^\infty(\Omega)$ there exists a positive constant C depending only on s, a and φ such that

$$\|\varphi\, \mathbf{Op}(a) f\|_{s-\ell} \leq C \|f\|_s, \;\; \forall f \in H^s_{\mathrm{comp}}(\Omega). \tag{2.8.1}$$

Proof. According to Proposition 1.5.15 the space $C_0^\infty(\Omega)$ is dense in $H^s_{\mathrm{comp}}(\Omega)$ so it suffices to prove the inequality (2.8.1) only for $f \in C_0^\infty(\Omega)$. Our proof is inspired by the proof of [**Se**, Thm. II.1] and is based on the following classical result.

Lemma 2.8.2 (Schur). *Suppose (X,μ) is a measured spaces and*

$$K : X \times X \to \mathbb{C}$$

is a measurable function such that there exists a constant $C > 0$ so that

$$\int_X |K(x_1,z)| d\mu(z), \;\; \int_X |K(z,x_2)| d\mu(z) \leq C, \;\; \forall x_1, x_2 \in X. \tag{2.8.2}$$

Then K defines a bounded linear operator

$$T_K : L^2(X,\mu) \to L^2(X,\mu), \;\; f \mapsto (T_K f)(x) := \int_X K(x,y) f(y) d\mu(y)$$

of norm $\leq C$, i.e.,

$$\|T_K f\|_{L^2} \leq C \|f\|_{L^2}, \;\; \forall f \in L^2(X,\mu).$$

Proof. It suffices to show that for any $f, g \in L^2(X,\mu)$ we have

$$|(T_K f, g)_{L^2}| \leq C\|f\|_{L^2} \cdot \|g\|_{L^2}.$$

We have

$$|(T_K f, g)_{L^2}| = \left| \int_X \Big(\int_X K(x,y) f(y)\, d\mu(y) \Big) \overline{g(x)}\, d\mu(x) \right|$$

$$\leq \int_{X\times X} |K(x,y) f(y) \overline{g}(x)| d\mu \times d\mu$$

$$\leq \left(\int_{X\times X} |K(x,y)| \cdot |f(y)|^2\, d\mu \times d\mu \right)^{1/2} \left(\int_{X\times X} |K(x,y)| \cdot |g(x)|^2\, d\mu \times d\mu \right)^{1/2}$$

$$= \left(\int_X |f(y)|^2 \Big(\int_X |K(x,y)| d\mu(x) \Big) d\mu(y) \right)^{1/2} \cdot \left(\int_X |g(x)|^2 \Big(\int_X |K(x,y)| d\mu(y) \Big) d\mu(x) \right)^{1/2}$$

$$\stackrel{(2.8.2)}{\leq} C\|f\|_{L^2} \cdot \|g\|_{L^2}.$$

□

Observe that $\varphi\, \mathbf{Op}(a) = \mathbf{Op}(\varphi a)$. Set

$$\sigma(x,\xi) = \varphi(x) a(x,\xi) \in \mathbf{S}^\ell(\Omega).$$

Observe that σ has compact x-support, i.e., there exists a compact set $S \subset \Omega$ such that

$$\sigma(x,\xi) = 0, \quad \forall (x,\xi) \in (\Omega \setminus S) \times \boldsymbol{V}.$$

In particular, extending σ by 0 for $x \in \boldsymbol{V} \setminus \Omega$ we can regard it as a symbol $\sigma \in \mathbf{S}^\ell(\boldsymbol{V})$. We will prove that for any $s \in \mathbb{R}$ there exists $C_s > 0$ such that

$$\|\, \mathbf{Op}(\sigma) f\|_{s-\ell} \leq C_s \|f\|_s, \quad \forall f \in C_0^\infty(\boldsymbol{V}).$$

Since Λ_s defines isometries $\Lambda_s : H^t(\boldsymbol{V}) \to H^{t-s}(\boldsymbol{V})$ it suffices to show that the composition $A_s = \Lambda_{s-\ell}\, \mathbf{Op}(\sigma) \Lambda_{-s}$ defines a bounded operator $L^2(\boldsymbol{V}) \to L^2(\boldsymbol{V})$. Define

$$\widehat{\sigma}(\eta,\xi) := \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\eta)} \sigma(x,\xi)\, |dx|_*.$$

Using the support condition on σ we deduce

$$\eta^\alpha \widehat{\sigma}(\eta,\xi) = \int_{\boldsymbol{V}} D_x^\alpha \sigma(x,\xi) e^{-\boldsymbol{i}(x,\eta)}\, |dx|_*, \quad \forall \alpha, \eta.$$

This implies that for every $N > 0$, there exists $C_N > 0$, independent of ξ such that[1]

$$|\widehat{\sigma}(\eta,\xi)| \leq C_N \langle \xi \rangle^\ell \langle \eta \rangle^{-N}, \quad \forall \xi, \eta \in \boldsymbol{V}. \tag{2.8.3}$$

For $f \in C_0^\infty(\boldsymbol{V})$ we have

$$\widehat{A_s f}(\eta) = \langle \eta \rangle^{s-\ell} \boldsymbol{\mathfrak{F}}\big(\, \mathbf{Op}(\sigma) \Lambda_{-s} f\,\big)(\eta),$$

and

$$\boldsymbol{\mathfrak{F}}\big(\, \mathbf{Op}(\sigma) \Lambda_s f\,\big)(\eta) = \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\eta)} \left(\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} \sigma(x,\xi) \langle \xi \rangle^{-s} \widehat{f}(\xi)\, |d\xi|_* \right) |dx|_*$$

[1] For more precise info about the dependence of C_N on the symbol a we refer to Remark 2.8.3.

$$= \int_{\boldsymbol{V}} \left(\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi-\eta)} \sigma(x,\xi) \langle \xi \rangle^{-s} \widehat{f}(\xi)\, |dx|_* \right) |d\xi|_* = \int_{\boldsymbol{V}} \widehat{\sigma}(\eta-\xi,\xi) \langle \xi \rangle^{-s} \widehat{f}(\xi)\, |d\xi|_*.$$

Hence

$$\widehat{A_s f}(\eta) = \int_{\boldsymbol{V}} \underbrace{\widehat{\sigma}(\eta-\xi,\xi) \langle \eta \rangle^{s-\ell} \langle \xi \rangle^{-s}}_{=:K_s(\eta,\xi)} \widehat{f}(\xi)\, |d\xi|_*. \tag{2.8.4}$$

Using (2.8.3) we deduce that for any $N > 0$ there exists $C_N > 0$ such that

$$|K_s(\eta,\xi)| \leq C_N \langle \eta - \xi \rangle^{-N} \langle \eta \rangle^{s-\ell} \langle \xi \rangle^{\ell - s}.$$

Using Peetre's inequality we deduce

$$\langle \xi \rangle^{\ell-s} \leq 2^{|\ell-s|} \langle \eta \rangle^{\ell-s} \langle \eta - \xi \rangle^{|\ell-s|}$$

so that

$$|K_s(\eta,\xi)| \leq 2^{|\ell-s|} C_N \langle \eta - \xi \rangle^{|\ell-s|-N}.$$

Choosing $N := m + 1 + |\ell - s|$ we deduce

$$|K(\eta,\xi)| \leq 2^{|\ell-s|} C_N \langle \eta - \xi \rangle^{-(m+1)}.$$

If we set

$$C_{m,s} := 2^{|\ell-s|} C_N \int_{\boldsymbol{V}} \langle \xi \rangle^{-(m+1)}\, |d\xi|_*$$

we deduce from Schur's Lemma 2.8.2 that $\|\widehat{Af}\|_{L^2} \leq C_{m,s} \|\widehat{f}\|_{L^2}$. The desired conclusion follows by invoking Plancherel's theorem. □

Remark 2.8.3. Let us observe that the constant C_N in (2.8.3) can be chosen of the form

$$C = \kappa \cdot \operatorname{vol}(\operatorname{supp}\varphi) \cdot \sup\Big\{ \big| D_x^\alpha \big(\varphi(x) a(x,\xi) \big) \big| \langle \xi \rangle^{-\ell};\ \ x \in \operatorname{supp}\varphi,\ |\alpha| \leq N,\ \xi \in \boldsymbol{V} \Big\},$$

where κ is a constant that depends only on m and N. □

Theorem 2.8.4. *Suppose $A \in \boldsymbol{\Psi}_0^\ell(\Omega)$ is a properly supported ψdo of order $\leq \ell$. Then for any $\varphi \in C_0^\infty(\Omega)$ there exists $\psi \in C_0^\infty(\Omega)$ and a positive constant C such that*

$$\|\varphi A u\|_{s-\ell} \leq C \|\psi u\|_s, \ \ \forall u \in H^s_{\mathrm{loc}}(\Omega).$$

Proof. We will need the following elementary fact.

Lemma 2.8.5. *For any $\varphi \in C_0^\infty(\Omega)$ there exists $\psi \in C_0^\infty(\Omega)$ such that*

$$\varphi A \psi u = \varphi A u, \ \ \forall u \in C^{-\infty}(\Omega).$$

Proof. Let $K_{A^\vee} \in C^{-\infty}(\Omega \times \Omega)$ denote the kernel of $A^\vee$ so that, for any $u \in C^{-\infty}(\Omega)$ we have

$$\langle Au, v \rangle = \langle u, A^\vee v \rangle, \ \ \forall v \in C_0^\infty(\Omega),$$

where

$$\langle A^\vee v, w \rangle = \langle K_{A^\vee}, w \otimes u \rangle, \ \ \forall w \in C_0^\infty(\Omega),$$

and $w \otimes v(x,y) = w(x) v(y)$. Let $\varphi \in C_0^\infty(\Omega)$. Then $(\varphi A)^\vee = A^\vee \varphi$ and

$$\langle A^\vee \varphi v, w \rangle = \big\langle\, K_{A^\vee}, w \otimes (\varphi v) \,\big\rangle.$$

Fix a compact neighborhood $\mathcal{N}_\varphi$ of $\operatorname{supp}\varphi$ in Ω. The operator $A^\vee$ is properly supported so that the set

$$S_\varphi := \bigl\{ (x,y) \in \operatorname{supp} K_{A^\vee};\ \ y \in \mathcal{N}_\varphi \bigr\}$$

is compact. In particular, the image X_φ of S_φ via the projection $\Omega\times\Omega \ni (x,y) \mapsto x \in \Omega$ is a compact set. Choose a function $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in a compact neighborhood of X_φ. Then

$$\langle \varphi A\psi u, v\rangle = \langle u, \psi A^\vee \varphi v\rangle,$$

and

$$\langle \psi A^\vee \varphi v, w\rangle = \bigl\langle\, K_{A^\vee}, (\psi w)\otimes(\varphi u)\,\bigr\rangle,$$

so that

$$\langle A^\vee \varphi v, w\rangle - \langle \psi A^\vee \varphi v, w\rangle = \bigl\langle\, K_{A^\vee}, (1-\psi)w\otimes(\varphi u)\,\bigr\rangle.$$

Now observe that

$$\operatorname{supp}\bigl(\,(1-\psi)w\otimes(\varphi u)\,\bigr) \cap \operatorname{supp} K_{A^\vee} = \emptyset,$$

so that,

$$\psi A^\vee \varphi v = A^\vee \varphi v,\ \ \forall v \in C_0^\infty(\Omega)$$

and therefore $\varphi A\psi u = \varphi A u$, $\forall u \in C^{-\infty}(\Omega)$. □

Let $\varphi \in C_0^\infty(\Omega)$. Lemma 2.8.5 implies that there exists $\psi \in C_0^\infty(\Omega)$ such that $\varphi A\psi = \varphi A$. Then, for any $u \in H^s_{\mathrm{loc}}(\Omega)$ we have $\psi u \in H^s_{\mathrm{comp}}(\Omega)$. Using (2.8.1) we deduce

$$\|\varphi A u\|_s = \|\varphi A\psi u\|_s \le C\|\psi u\|_s,$$

for a constant $C > 0$ independent of u. □

Remark 2.8.6. Theorem 2.8.4 has an obvious vectorial counterpart. Its formulation and proof are identical and we leave them to the reader. □

2.9. Elliptic ψdo's

Fix complex vector spaces E_0, E_1 of dimensions r_0 and respectively r_1.

Definition 2.9.1. A symbol $a \in \mathcal{S}^k(\Omega, E_0, E_1)$ is called *elliptic* if there exists $b(x,\xi) \in \mathcal{S}^{-k}(\Omega, E_1, E_0)$ such that

$$a(x,\xi) \circledast b(x,\xi) - \mathbb{1}_{E_1} \in \mathcal{S}^{-1}(\Omega, E_1, E_1), \tag{2.9.1a}$$

$$b(x,\xi) \circledast a(x,\xi) - \mathbb{1}_{E_0} \in \mathcal{S}^{-1}(\Omega, E_0, E_0). \tag{2.9.1b}$$

A ψdo $A \in \mathbf{\Psi}^k(\Omega, E_0, E_1)$ is called *elliptic* if it is properly supported and its symbol is elliptic. □

Observe that ellipticity of a symbol $a \in \mathcal{S}^k(\Omega, E_0, E_1)$ is completely determined by its principal part $a^\pi \in \Sigma^k(\Omega, E_0, E_1)$. More precisely, we have the following immediate consequence of the definition.

Proposition 2.9.2. *A symbol $a \in \mathcal{S}^k(\Omega, E_0, E_1)$ is elliptic if and only if there exists $b \in \Sigma^{-k}(\Omega, E_1, E_0)$ such that*

$$a^\pi b = \mathbb{1}_{E_1} \in \Sigma^0(\Omega, E_1, E_1),\ \ b a^\pi = \mathbb{1}_{E_0} \in \Sigma^0(\Omega, E_0, E_1).$$ □

In particular, we see that if a is an elliptic symbol, then $\dim E_0 = \dim E_1$. Indeed, the equality (2.9.1a) and (2.9.1b) imply that for any $x \in \Omega$ there exists $C > 0$ such that for any $|\xi| > C$ the linear map

$$a(x, \xi) : E_0 \to E_1$$

is an isomorphism.

Example 2.9.3. Consider the first order partial differential operator

$$d : C^\infty(\Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee) \to C^\infty(\Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee),$$

where

$$V^\vee = \operatorname{Hom}(\boldsymbol{V}, \mathbb{R}), \;\; \Lambda^\bullet_{\mathbb{C}} \boldsymbol{V} = \bigoplus_{k=0}^{m} \Lambda^k \boldsymbol{V}^\vee \otimes \mathbb{C}.$$

The principal symbol of this operator is

$$[\sigma_d](x, \xi) = \boldsymbol{i}\xi \wedge : \Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee \to \Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee,$$

the exterior multiplication by $\boldsymbol{i}\xi \in V^\vee \otimes \mathbb{C}$. We denote this operator by $e(\boldsymbol{i}\xi)$.

The metric on $\boldsymbol{V}$ induces hermitian metrics on $\Lambda^k_{\mathbb{C}} \boldsymbol{V}^\vee$, so we can define the formal adjoint of d,

$$d^* : C^\infty(\Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee) \to C^\infty(\Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee),$$

$$(d\omega, \eta)_{L^2} = (\omega, d^*\eta)_{L^2}, \;\; \forall \omega, \eta \in C^\infty_0(\Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee).$$

Its principal symbol is

$$[\sigma_{d^*}](x, \xi) = (\boldsymbol{i}\xi \wedge)^*.$$

If we identify the covector $\xi \in \boldsymbol{V}^\vee$ with a vector $\xi_\dagger \in \boldsymbol{V}$ using the Euclidean metric on $\boldsymbol{V}$, then we see that

$$(\boldsymbol{i}\xi \wedge)^* = -\boldsymbol{i}\xi_\dagger \lrcorner, \tag{2.9.2}$$

where $\lrcorner$ denotes the contraction by a vector. To prove this note first that we can assume that $|\xi| = 1$. Next, we choose an orthonormal basis $\boldsymbol{e}^1, \dots, \boldsymbol{e}^m$ of $\boldsymbol{V}^\vee$, such that $\boldsymbol{e}^1 = \xi$. We denote by $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ the dual basis of $\boldsymbol{V}$ so that $\xi_\dagger = \boldsymbol{e}_1$. Then, a direct computation shows that for any monomials

$$\boldsymbol{e}^I := \boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_k} \in \Lambda^k \boldsymbol{V}^\vee, \;\; \boldsymbol{e}^J := \boldsymbol{e}^{j_0} \wedge \boldsymbol{e}^{j_1} \wedge \cdots \wedge \boldsymbol{e}^{j_k} \in \Lambda^k(\boldsymbol{V}^\vee)$$

we have

$$(\boldsymbol{e}^1 \wedge \boldsymbol{e}^I, \boldsymbol{e}^J) = (\boldsymbol{e}^I, \boldsymbol{e}_1 \lrcorner \, \boldsymbol{e}^J),$$

where $(-,-)$ denotes the inner product in $\Lambda^\bullet \boldsymbol{V}^\vee$. This proves (2.9.2). Set

$$L = (d + d^*)^2 = dd^* + d^*d.$$

Then

$$[\sigma_L] = \big([\sigma_d] + [\sigma_{d^*}] \big)^2 = -\big(e(\xi) - i(\xi_\dagger) \big)^2,$$

where $i(\xi_\dagger)$ denotes the operation of contraction with the vector $\xi_\dagger$. At this point we want to invoke a useful identity, usually referred to as the *Cartan identity*

$$e(\xi)i(\xi_\dagger) + i(\xi_\dagger)e(\xi)u = |\xi|^2 u, \;\; \forall u \in \Lambda^\bullet \boldsymbol{V}^\vee. \tag{2.9.3}$$

The elementary proof is left to the reader as an exercise. Observing that $e(\xi)^2 = i(\xi_\dagger)^2 = 0$ we deduce

$$[\sigma_L](x, \xi) = e(\xi)i(\xi_\dagger) + i(\xi_\dagger)e(\xi) = |\xi|^2 \mathbb{1}_{\Lambda^\bullet_{\mathbb{C}} \boldsymbol{V}^\vee}.$$

This proves that $(d + d^*)^2$ is an elliptic operator, and so is $(d + d^*)$. □

Theorem 2.9.4. *Let $A \in \boldsymbol{\Psi}_0^k(\Omega, E_0, E_1)$ and set $a = \sigma_A$. Then the following statements are equivalent.*

(a) *The operator A is elliptic.*

(b) *There exists a ψdo $B \in \boldsymbol{\Psi}_0^{-k}(\Omega, E_1, E_0)$ such that*
$$AB - \mathbb{1}, \;\; BA - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}.$$

(c) *There exists a ψdo $B \in \boldsymbol{\Psi}_0^{-k}(\Omega, E_1, E_0)$ such that*
$$BA - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}.$$

(d) *There exists a ψdo $B \in \boldsymbol{\Psi}_0^{-k}(\Omega, E_1, E_0)$ such that*
$$AB - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}.$$

Proof. Clearly (b) $\Rightarrow$ (c), (d). The implications (b), (c), (d) $\Rightarrow$ (a) follow from the composition rule (2.5.6). Thus, it suffices to show that (a) $\Rightarrow$ (b). Given that this result is key to all the other results in these lectures we will present two proofs.

1st Proof. We follow closely the approach of L. Hörmander [**H3**, Thm. 18.1.9]. Using the composition formula (2.5.6) and the assumption (a) we deduce that there exists $B \in \boldsymbol{\Psi}^{-k}(\Omega, E_1, E_0)$, and $R \in \boldsymbol{\Psi}^{-1}(\Omega, E_1, E_1)$ such that
$$AB = \mathbb{1} - R.$$
Indeed, the ellipticity of A implies that there exists $b \in \mathbf{S}^{-k}(\Omega, E_1, E_0)$ such that $ba - \mathbb{1} \in \mathbf{S}^{-1}$. If we set $B = \mathbf{Op}(a)$ then the composition formula (2.5.6) implies that $R = \mathbb{1} - AB \in \boldsymbol{\Psi}^{-1}$. Set $r = \sigma_R$.

We want to invert $\mathbb{1} - R$ using the geometric series
$$(\mathbb{1} - R)^{-1} = \sum_{n=0}^{\infty} R^n.$$

We define $C \in \boldsymbol{\Psi}^0(\Omega)$ such that
$$C \sim \sum_{k \geq 0} R^k, \;\; \text{i.e.,} \;\; C - \sum_{k=0}^{n} \mathbf{Op}(r)^k \in \boldsymbol{\Psi}^{-n-1}(\Omega), \;\; \forall n \geq 0.$$

More explicitly, we let
$$r_n(x,\xi) := \sigma_{R^n}(x,\xi) \sim \underbrace{r \circledast r \circledast \cdots \circledast r}_{n}(x,\xi) \in \mathbf{S}^{-n}(\Omega).$$

and we define
$$C = \mathbf{Op}(c), \;\; c(x,\xi) \sim \sum_{n \geq 0} r_n(x,\xi), \;\; C_n = \sum_{k=0}^{n} R^k.$$

Then $C - C_n \in \boldsymbol{\Psi}^{-n-1}(\Omega)$ and we deduce
$$ABC = ABC_n + AB(C - C_n)$$
$$= (\mathbb{1} - R)\sum_{k=0}^{n} R^n + AB(C - C_n) = \mathbb{1} - R^{n+1} + AB(C - C_n).$$

Observe that.
$$R^{n+1}, \;\; AB(C - C_n) \in \boldsymbol{\Psi}^{-n-1}(\Omega).$$

Hence, if we set $B' = BC$ then we can conclude from the above that
$$AB' - \mathbb{1} \in \boldsymbol{\Psi}^{-n} \;\;\; \forall n \geq 0.$$

If B' is not properly supported, we can modify it by a smoothing operator so it becomes properly supported.

Similarly, we can find $B'' \in \mathbf{\Psi}^{-k}(\Omega, E_1, E_0)$ such that B'' is properly supported and

$$B''A - \mathbb{1} \in \mathbf{\Psi}^{-\infty}.$$

Next observe that

$$B'' - B' - \Big(B''\big(\mathbb{1} - AB' \big) + \big(B''A - \mathbb{1} \big) B' \Big) \in \mathbf{\Psi}^{-\infty}.$$

If we let $\widetilde{B} = \frac{1}{2}(B' + B'')$, then

$$\widetilde{B} - B' \in \mathbf{\Psi}^{-\infty}, \;\; \widetilde{B} - B'' \in \mathbf{\Psi}^{-\infty},$$

and

$$A\widetilde{B} - \mathbb{1}, \;\; \widetilde{B}A - \mathbb{1} \in \mathbf{\Psi}^{-\infty}.$$

2nd Proof. This is the traditional proof. It is not as elegant as the previous argument but it has the advantage that it contains more detailed information about the operator b. For simplicity we assume that A is a classical ψdo so that its symbol a has an asymptotic expansion

$$a \sim \sum_{j \geq 0} a_{k-j},$$

where $a_{k-j}(x, \xi)$ is positively homogeneous of degree $k - j$ for $|\xi| \geq 1$.

We seek a classical ψdo B such that $BA - \mathbb{1} \in \mathbf{\Psi}^{-\infty}$. The symbol b of B has an asymptotic expansion

$$b \sim \sum_{\ell \geq 0} b_{-k-\ell},$$

where $b_{-k-\ell}(x, \xi)$ is positively homogeneous of degree $-k - \ell$ for $|\xi| \geq 1$.

Using (2.5.6) we deduce

$$\mathbb{1} = \sigma_{\mathbf{Op}(b)\,\mathbf{Op}(a)} \sim b \circledast a = \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha b \cdot D_x^\alpha a.$$

Rearranging the above sum according to the homogeneities in ξ we deduce

$$\mathbb{1} = (b \circledast a)_0 \sim b_{-k} a_k, \;\; 0 = (b \circledast a)_{-\nu} \sim \sum_{j+\ell+|\alpha|=\nu} \frac{1}{\alpha!} \partial_\xi^\alpha b_{-k-\ell} D_x^\alpha a_{k-j} \sim 0, \;\; \nu > 0. \tag{2.9.4}$$

This leads to an infinite linear system

$$\mathbb{1} = \beta_{-k} a_k^h, \tag{2.9.5a}$$

$$0 = \beta_{-k-\nu} a_k^h + \sum_{\substack{j+\ell+|\alpha|=\nu \\ \ell<\nu}} \frac{1}{\alpha!} \partial_\xi^\alpha \beta_{-k-\ell} D_x^\alpha a_{k-j}^h, \quad \nu > 0, \tag{2.9.5b}$$

where the unknown $\beta_{-k-\nu}(x, \xi)$ are positively homogeneous of degree $-k - \nu$ in ξ and $a_{k-j}^h(x, \xi)$ denotes the unique positively homogeneous function of degree $k - j$ that agrees with $a_{k-j}(x, \xi)$ for $|\xi| \geq 1$. Note that for large ν and j the functions $\beta_{-k-\nu}$ and a_{k-j} are not defined at $\xi = 0$. Its is clear that the system (2.9.5a) + (2.9.5b) has a unique solution $(\beta_{-k-\nu})_{\nu \geq 0}$, where $\beta_{-k} = (a_k^h)^{-1}$.

Let $\varphi : \mathbb{R} \to [0, \infty)$ be a smooth function such that

$$\varphi(t) = \begin{cases} 0, & |t| \leq 1/2, \\ 1, & |t| \geq 1. \end{cases}.$$

Now define

$$b_{-k-\nu}(x, \xi) = \begin{cases} \varphi(\xi)\beta_{k-\nu}(x, \xi), & \xi \neq 0, \\ 0, & \xi = 0, \end{cases} \quad \forall \nu \geq 0.$$

Note that $b_{-k-\nu} \in \mathcal{S}^{-k-\nu}_{\text{phg}}(\Omega)$. The functions $(b_{-k-\nu})$ satisfy the system (2.9.4) so that if we define B to be a ψdo with symbol b admitting the asymptotic decomposition

$$b \sim \sum_{\ell} b_{-k-\ell}$$

we deduce from (2.5.6) that $BA - \mathbb{1} \sim 0$.

Similarly, we can find an operator C such that $CA^{\vee} - \mathbb{1} \sim 0$. If we set $B' = C^{\vee}$ we deduce $AB' - \mathbb{1} \sim 0$. Arguing as in the first proof we deduce that $B \sim B'$. □

Definition 2.9.5. Let $A \in \mathbf{\Psi}^k_0(\Omega, E_0, E_1)$ be an elliptic operator. An operator $B \in \mathbf{\Psi}^{-k}_0 E, (E_1, E_0)$ such that

$$AB - \mathbb{1}, \;\; BA - \mathbb{1} \in \mathbf{\Psi}^{-\infty}$$

is called a *parametrix* of A. □

Theorem 2.9.4 has several important consequences.

Corollary 2.9.6. *Let $A \in \mathbf{\Psi}^k_0(\Omega, E_0, E_1)$ be an elliptic operator and $f \in C^{\infty}(\Omega, E_1)$. If $u \in C^{-\infty}(\Omega, E_0)$ is a distributional solution of the equation $Au = f$, then $u \in C^{\infty}(\Omega, E_0)$.*

Proof. Let B be a parametrix of A. Then $BA = \mathbb{1} + S$, where S is a smoothing operator. We deduce

$$Bf = BAu = u + Su,$$

so that $u = Bf - Su$. Since S is smoothing we deduce from Proposition 2.3.3 that $Su \in C^{\infty}$. Since $f \in C^{\infty}$ we that Bf is smooth. □

Remark 2.9.7. The result in Corollary 2.9.6 is truly remarkable. The following example may perhaps illustrate some of its hidden subtleties.

Consider the partial differential operators

$$\Delta := -\partial_x^2 - \partial_y^2, \;\; \Box := \partial_x^2 - \partial_y^2.$$

The operator Δ is elliptic, while $\Box$ is not. Corollary 2.9.6 shows that if $u \in C^{-\infty}(\mathbb{R}^2)$ satisfies $\Delta u = 0$ in the sense of distributions then in fact u is smooth, although, a priori, u may not even be differentiable. This special case is known as *Weyl's lemma*.

Things are dramatically different with the wave operator $\Box$. Consider the distribution

$$w = \frac{1}{2}\delta(x + y) + \frac{1}{2}\delta(y - x) \in C^{-\infty}(\mathbb{R}^2),$$

where the Dirac type distributions $\delta(y \pm x)$ are obtained as follows.

- Choose a smooth, compactly supported, even function $\varphi : \mathbb{R} \to [0,\infty)$ such that
$$\int_{\mathbb{R}} \varphi(t)\,|dt| = 1,$$
and set $\varphi_n(t) := n\varphi(nt)$, $n \in \mathbb{Z}_{>0}$, $t \in \mathbb{R}$. The sequence φ_n converges in $C^{-\infty}(\mathbb{R})$ to the Dirac function δ_0.
- Set
$$\delta(y \pm x) = \lim_n \varphi_n(y \pm x).$$

The distributional derivatives of $\delta(y \pm x)$ are computed using the chain rule
$$\frac{\partial}{\partial x} = \frac{d}{dt}\frac{\partial t}{\partial x},\quad \frac{\partial}{\partial y} = \frac{d}{dt}\frac{\partial t}{\partial y},$$
and a simple computation shows that $\Box w = 0$. On the other hand, w is very singular,
$$\operatorname{sing\,supp} w = \operatorname{supp} w = \big\{\,(x,y) \in \mathbb{R}^2;\;\; x^2 - y^2 = 0\,\big\}.$$

The operators Δ and $\Box$ differ by a sign, yet they have dramatically different behaviors! □

Corollary 2.9.8 (Elliptic regularity and estimates). *Let $A \in \boldsymbol{\Psi}^k_0(\Omega, E_0, E_1)$ be an elliptic operator and $f \in H^s_{\mathrm{loc}}(\Omega, E_1)$.*
(a) If $u \in C^{-\infty}(\Omega, E_0)$ and $Au \in H^s_{\mathrm{loc}}(\Omega, E_1)$ then $u \in H^{s+k}_{\mathrm{loc}}(\Omega, E_0)$.
(b) For any $\ell \in \mathbb{R}$ and any $\varphi \in C^\infty_0(\Omega)$ there exists a function $\psi \in C^\infty_0(\Omega)$ and a constant $C > 0$ such that such that
$$\|\varphi u\|_{s+k} \le C\|\psi Au\|_s + \|\psi u\|_\ell,\;\; \forall u \in H^{s+k}_{\mathrm{loc}} \cap H^\ell_{\mathrm{loc}}(\Omega, E_0). \tag{2.9.6}$$

Proof. Set $f = Au$. Let B be a parametrix of A. Then $BA = \mathbb{1} + S$, where S is a smoothing operator. We deduce as before that
$$u = Bf - Su.$$
From Theorem 2.8.4 we deduce $Bf \in H^{s+k}_{\mathrm{loc}}(\Omega, E_0)$. Moreover $Su \in H^{s+k}_{\mathrm{loc}}(\Omega, E_0)$ since $Su \in C^\infty$. This proves (a).

If $\varphi \in C^\infty_0(\Omega)$ we deduce from Theorem 2.8.4 that there exists $\psi \in C^\infty_0(\Omega)$ such that
$$\|\varphi Bf\|_{s+k} \le C\|\psi f\|_s,\;\; \|\varphi Su\|_{s+k} \le C\|\psi u\|_\ell.$$
This proves (b). □

2.10. Exercises

Exercise 2.1. Prove Theorem 2.2.5. □

Exercise 2.2. Prove Propositions 2.2.7 and 2.3.3. □

Exercise 2.3. Justify the statements marked (???) in the proof of Proposition 2.3.4. □

Exercise 2.4. Prove Proposition 2.3.2.

Exercise 2.5. Prove Lemma 2.3.6. **Hint.** Show that any proper subset admits a proper neighborhood. Next, choose a proper neighborhood $\mathcal{N}$ of C and a proper neighborhood $\mathcal{U}$ of $\mathcal{N}$. Then any function χ such that $\operatorname{supp}\chi \subset \mathcal{N}$ and $\chi|_C \equiv 1$ will do the trick. □

Exercise 2.6. Prove the equality (2.4.7). □

Exercise 2.7. Prove the equality (2.1.6) and then show that it implies (2.5.7). □

Exercise 2.8. Prove the equality (2.6.7) and then show that it implies (2.6.8). □

Exercise 2.9. Prove the identity (2.7.3). □

Exercise 2.10. Prove Cartan's identity (2.9.3). □

Exercise 2.11. Consider the distribution $\delta(y-x) \in C^{-\infty}(\mathbb{R}^2)$ defined in Remark 2.9.7.

(a) Prove that

$$\langle \delta(y-x), \varphi \rangle = \frac{1}{2}\int_{\mathbb{R}} \varphi\Big(\frac{v}{2}, \frac{v}{2}\Big)\,|dv|, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2).$$

(b) Describe the Fourier transform of $\delta(y-x)$. □

Exercise 2.12. Fix $0 < \lambda < m = \dim \boldsymbol{V}$ and consider the linear operator

$$K_\lambda : C_0^\infty(\boldsymbol{V}) \to C^\infty(\boldsymbol{V}), \quad K_\lambda u)(y) = \int_{\boldsymbol{V}} |x-y|^{-\lambda} u(y), \quad |dy|.$$

Show that K_λ is a ψdo of order $m-\lambda$ with principal symbol $C|x_1|^{m-\lambda}$, where the constant C is determined as in Exercise 1.8. □

Exercise 2.13. Let Ω be an open subset in $\mathbb{R}^m$, and Ω_1, Ω_2 be open relatively compact subset of Ω such that $\overline{\Omega}_1 \subset \Omega_2$. Fix a nonnegative integer $k \geq 0$, and denote by Δ the Laplacian

$$\Delta = -\sum_{j=1}^m \partial^2_{x_j} : C^\infty(\Omega) \to C^\infty(\Omega).$$

(a) Show that if $u \in C^{-\infty}(\Omega)$ and $\Delta u \in H^k_{\mathrm{loc}}(\Omega)$ then $u \in H^{k+2}_{\mathrm{loc}}(\Omega)$.

(b) Prove that there exists a constant $C > 0$ such that for any $u \in C^{-\infty}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega)$ such that $f = \Delta u \in H^k_{\mathrm{loc}}(\Omega)$ we have

$$\sum_{|\alpha| \leq k+2} \int_{\Omega_1} |D^\alpha u|^2\,|dx| \leq C\left(\int_{\Omega_2} |u|^2\,|dx| + \sum_{|\beta|\leq k}\int_{\Omega_2} |D^\beta f|^2\,|dx|\right).$$

Chapter 3

Pseudo-differential operators on manifolds and index theory

3.1. Pseudo-differential operators on smooth manifolds

Suppose M is a smooth, connected manifold of dimension m and $\boldsymbol{E}_0, \boldsymbol{E}_1 \to M$ are smooth complex vector bundles of ranks r_0 and respectively r_1 equipped with the following structures.

- A Riemann metric g on M with Levi-Civita connection ∇^g volume density $|dV_g|$.
- Hermitian metrics h_0, h_1 on $\boldsymbol{E}_0$ and respectively $\boldsymbol{E}_1$.
- A connection $\nabla^i = \nabla^{\boldsymbol{E}_i}$ on $\boldsymbol{E}_i$ compatible with h_i.

With these choices in place can define the locally convex topologies on the spaces of smooth sections $C_0^\infty(\boldsymbol{E}_i)$ and $C^\infty(\boldsymbol{E}_i)$. The topology on $C^\infty(\boldsymbol{E}_i)$ is given by the family of seminorms

$$\|u\|_{n,K} \sup_{x\in K, j\leq n} |(\nabla^{\boldsymbol{E}_i})^j u(x)|_{g,h_i}, \;\; u \in C^\infty(\boldsymbol{E}_i),$$

where $K \subset M$ is a compact set and $(\nabla^{\boldsymbol{E}_i})^j$ denotes the composition

$$C^\infty(\boldsymbol{E}_i) \stackrel{\nabla^{\boldsymbol{E}_i}}{\longrightarrow} C^\infty(T^*M \otimes \boldsymbol{E}_i) \stackrel{\nabla^g \otimes \nabla^{\boldsymbol{E}_i}}{\longrightarrow} \cdots \stackrel{\nabla^g \otimes \nabla^{\boldsymbol{E}_i}}{\longrightarrow} C^\infty(T^*M^{\otimes j} \otimes \boldsymbol{E}_i).$$

The space $C_0^\infty(\boldsymbol{E}_i)$ is topologized with the locally convex inductive limit topology on the union of the spaces $C_K^\infty(\boldsymbol{E}_i)$ consisting of smooth sections with support contained in the compact set K. By duality we obtain the spaces of generalized sections $C_0^{-\infty}(\boldsymbol{E}_i)$ and $C^{-\infty}(\boldsymbol{E}_i)$.

A *coordinate neighborhood* for the triplet (M, E_0, E_1) is an open set $\mathcal{O} \subset M$ together with the following data.

- A diffeomorphism
$$F : \mathcal{O} \to \Omega, \;\; \Omega \text{ open subset in } \boldsymbol{V} = \mathbb{R}^m,$$
- Complex vector spaces E_0, E_1 of dimensions r_0 and respectively r_1.

- Bundle isomorphisms $T_i : F^*\underline{E_i}_\Omega \to \boldsymbol{E}_i|_{\mathcal{O}}$, $i = 0, 1$.

We will use the symbol $(\mathcal{O}, \Omega, F, T_i, E_i)$ to label such a coordinate neighborhood, and we will refer to $\mathcal{O}$ as the *domain* of the coordinate neighborhood.

Definition 3.1.1. A linear map $A : C_0^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$ is said to be a ψdo (respectively pdo) of order $\leq k$ if and only if, for any coordinate neighborhood $(\mathcal{O}, \Omega, F, T_i, E_i)$ the linear map

$$A_{\mathcal{O}} : C_0^\infty(\Omega, E_0) \to C^\infty(\Omega, E_1)$$

given by the composition

$$u \stackrel{T_0F^*}{\to} T_0F^*(u) \stackrel{A}{\to} AT_0F^*u \stackrel{T_1^{-1}|_{\mathcal{O}}}{\longrightarrow} T_1^{-1}\big(AT_0F^*u\big)|_{\mathcal{O}} \stackrel{(F^*)^{-1}}{\longrightarrow} (F^*)^{-1}T_1^{-1}\big(AT_0F^*(u)\big)|_{\mathcal{O}}$$

is a *classical* ψdo in $\boldsymbol{\Psi}^k_{\mathrm{phg}}(\Omega, E_0, E_1)$ (respectively a partial differential operator of order $\leq k$). We denote by $\boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ the space of pseudodifferential operators $A : C_0^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$ of order $\leq k$. When $\boldsymbol{E}_0 = \boldsymbol{E}_1 = \boldsymbol{E}$ we will use the simpler notation $\boldsymbol{\Psi}(\boldsymbol{E}) := \boldsymbol{\Psi}(\boldsymbol{E}, \boldsymbol{E})$. □

Remark 3.1.2. (a) Observe that if $(\mathcal{O}, \Omega, F, T_i, E_i)$ and $(\mathcal{O}, \widetilde{\Omega}, \widetilde{F}, \widetilde{T}_i, \widetilde{E}_i)$ are two coordinate neighborhoods with identical domain then the change in variables formula (2.7.4b) implies that

$$A_{\mathcal{O}} \;\text{ is a classical } \psi\text{do} \Longleftrightarrow A_{\widetilde{\mathcal{O}}} \;\text{ is a classical } \psi\text{do}.$$

(b) We must draw attention to a rather subtle point. If the manifold M in the above definition happens to be an open subset of the Euclidean vector space $\boldsymbol{V}$ and $\boldsymbol{E}_0, \boldsymbol{E}_1$ are the trivial, $\boldsymbol{E}_i = \underline{E_i}_M$, then the class operators that are pseudo-differential in the sense of Definition 3.1.1 is a priori more restrictive than the class of classical ψdo's in the sense of Chapter 2.

Indeed, a linear operator $A : C_0^\infty(M, E_0) \to C^\infty(M, E_1)$ which is a classical ψdo in the sense of Chapter 2 is a ψdo in the sense of Definition 3.1.1 if and only if, for any open subset $\mathcal{O} \subset M$ the linear map

$$C_0^\infty(\mathcal{O}, E_0) \ni u \mapsto A_{\mathcal{O}}u := (Au)|_{\mathcal{O}} \in C^\infty(\mathcal{O})$$

is also a classical ψdo in the sense of Chapter 2.

Let us show that in fact these two classes of ψdo's coincide. Suppose A is a classical ψdo as defined in the previous chapter. If A is smoothing then clearly $A_{\mathcal{O}}$ is also smoothing for any open $\mathcal{O} \subset M$.

If A is properly supported, we denote by $\sigma_A(x, \xi)$ its total symbol so that $A = \mathbf{Op}(\sigma_A)$, i.e.,

$$Au(x) = \int_{\boldsymbol{V}}^{\sim} e^{\boldsymbol{i}(x,\xi)}\sigma_A(x, \xi)\widehat{u}(\xi)\,|d\xi|_*, \;\; \forall u \in C_0^\infty(M, E_0).$$

This shows that if $\mathcal{O} \subset M$ is open, then $A_{\mathcal{O}} = \mathbf{Op}(\sigma_A|_{\mathcal{O}})$, where $\sigma_A|_{\mathcal{O}} := \sigma_A|_{\mathcal{O}\times\boldsymbol{V}}$. Thus $A_{\mathcal{O}}$ is a classical ψdo in the sense of Chapter 2. The general case reduces to these two since any classical ψdo is a sum of a properly supported classical ψdo and a smoothing operator. Thus, when M is an open subset of a vector space $\boldsymbol{V}$, the class of operators introduced in Definition 3.1.1 coincides with the space of classical ψdo's defined in the previous chapter. □

Remark 3.1.3. The definition of a ψdo has a built-in subtlety that we want to address. More precisely we want to discuus the following isue. Given a ψdo $A \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ and a smooth compactly supported compactly supported section $u \in C_0^\infty(\boldsymbol{E})$ express Au in terms of the operators $A_{\mathcal{O}}$ entering into the definition of A as a ψdo.

We need to introduce a language that will be useful in other instance. define a *coordinate region* of M to be an open subset $\mathcal{O}$ of M satisfying the following properties.

- The set $\mathcal{O}$ is precompact and has finitely many connected components such that their closures are disjoint.
- Each component of $\mathcal{O}$ admits an open neighborhood diffeomorphic to an m-dimensional open ball.

Note three things.

(i) Any connected open subset contained in a geodesic ball of M is a coordinate region. We use the *normal coordinates* on that geodesic ball to coordinatize the respective component of $\mathcal{O}$.

(ii) The restriction of any bundle to a coordinate region is trivializable. Indeed, over a geodesic ball we will trivialize $\boldsymbol{E}$ using the parallel transport along the radii defined by the hermitian connection ∇ on $\boldsymbol{E}$.

For any compact subset $K \subset M$ we let $\mathbf{inj}(K)$ denote the infimum of injectivity radii of points in K.

Suppose $u \in C_0^\infty(\boldsymbol{E}_0)$, $x_0 \in M$, $r < \frac{1}{3}\,\mathbf{inj}(x_0)$. How do we describe the restriction of of Au to the open ball $B_r(x_0)$ in some local coordinates on this ball?

Set $K := \operatorname{supp} u \cup \boldsymbol{cl}\big(\, B_r(x_0)\,\big)$, $\rho := \mathbf{inj}(K)$. We can now construct a finite family of smooth functions $\eta_i \in C_0^\infty(M)$, $i \in I$ with the following properties.

- For any $i \in I$ the support of η_i is contained in a geodesic ball centered at a point in K and of radius $r_i < \frac{1}{3}\rho$.
- The function $\sum_{i\in I} \eta_i$ is identically 1 on a neighborhood $\mathcal{N}$ of K.

Define

$$v := \sum_{i,j\in I} \eta_i A(\eta_i u).$$

Observe that $v = Au$ on $\mathcal{N}$ so that $(Au)|_{B_r(x_0)} = v|_{B_r(x_0)}$. We set $v_{ij} := \eta_i A(\eta_j)u$ and we observe that

$$(Au)|_{B_r(x_0)} = \sum_{i,j} (v_{ij})|_{B_r(x_0)}$$

Thus, we only need to know how to compute $(v_{ij})|_{B_r(x_0)}$.

The set $\operatorname{supp}\eta_i \cup \operatorname{supp}\eta_j \cup B_r(x_0)$ is contained in a coordinate region. This is the case because each component of this set is contained either in a ball of radius $< \frac{2\rho}{3}$ centered at a point in K, or in a ball of radius $r + \frac{2\rho}{3} < \mathbf{inj}(x_0)$ centered at x_0. In both cases these geodesic balls are diffeomorphic to Euclidean balls.

Let $\mathcal{O}_{ij}$ be a coordinate region containing $\operatorname{supp}\eta_i \cup \operatorname{supp}\eta_j \cup B_r(x_0)$. Choose local coordinates in on this region and fix trivializations of $\boldsymbol{E}_0|_{\mathcal{O}_{ij}}$ and $\boldsymbol{E}_1|_{\mathcal{O}_{ij}}$. We can thus identify $\mathcal{O}_{ij}$ with an open set Ω_{ij} in $\mathbb{E}^m$, and the sections of $\boldsymbol{E}_0|_{\mathcal{O}_{ij}}$ and $\boldsymbol{E}_1|_{\mathcal{O}_{ij}}$ with maps from Ω_{ij} to vector spaces E_0 and E_1. The operator

$$C_0^\infty(\boldsymbol{E}_0|_{\mathcal{O}_{ij}}) \ni w \mapsto (Aw)|_{\mathcal{O}_{ij}} \in C^\infty(\boldsymbol{E}_1|_{\mathcal{O}_{ij}})$$

can be identified with a ψdo $A_{ij} \in \boldsymbol{\Psi}^k(\Omega_{ij}, E_0, E_1)$. Then the function $(\eta_i A(\eta_j u)|_{\mathcal{O}_{ij}}$ can be identified with the function $\eta_i A_{ij}(\eta_j u)$.

Remark 3.1.4. Perhaps this is a good place to stop and comment a bit about the differences between differential operator and pseudo-differential operators.

First, let us point out that the differential operators on manifolds admit a simple *intrinsic* definition. Denote by $\mathcal{L}(\boldsymbol{E}_0, \boldsymbol{E}_1)$ the space of linear operators $C_0^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$. For any smooth function $f \in C^\infty(M)$ we define a linear map

$$\operatorname{ad}(f) : \mathcal{L}(\boldsymbol{E}_0, \boldsymbol{E}_1) \to \mathcal{L}(\boldsymbol{E}_0, \boldsymbol{E}_1), \;\; T \mapsto \operatorname{ad}(f)T = M_f T - T M_f,$$

where M_f denotes the operation of multiplication by f. If $\operatorname{ad}(f)T = 0$, for any $f \in C_0^\infty(M)$ then T is a bundle morphism $T : \boldsymbol{E}_0 \to \boldsymbol{E}_1$, or equivalently, a partial differential operator of order zero. We can now define inductively the space of $\boldsymbol{PDO}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ of partial differential operators of order $\leq k$ from sections of $\boldsymbol{E}_0$ to sections of $\boldsymbol{E}_1$. More precisely

$$T \in \boldsymbol{PDO}^k(\boldsymbol{E}_0, \boldsymbol{E}_1) \stackrel{\text{def}}{\Longleftrightarrow} \operatorname{ad}(f)T \in \boldsymbol{PDO}^{k-1}(\boldsymbol{E}_0, \boldsymbol{E}_1), \; . \, \forall f \in C_0^\infty(M).$$

In particular, if $L \in \boldsymbol{PDO}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$, then for any $f \in C_0^\infty(M)$ we have

$$\operatorname{ad}(f)^k L \in \boldsymbol{PDO}^0(\boldsymbol{E}_0, \boldsymbol{E}_1).$$

This bundle morphism determines the principal symbol of L, more precisely, we have

$$[\sigma_L]\big(\, x, df(x)\,\big) = \frac{\boldsymbol{i}^k}{k!}\big(\operatorname{ad}(f)^k L\,\big)_x.$$

In the beautiful paper [H65] L. Hörmander gives an *intrinsic* definition of a pseudo-differential operator. More precisely, a continuous linear operator $P : C_0^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$ is a pseudo-differential operator of order k if for any $f \in C_0^\infty(\boldsymbol{E}_0)$, and any $g \in C^\infty(M)$ such that $dg \neq 0$ on $\operatorname{supp} f$ there is an asymptotic expansion

$$e^{-\boldsymbol{i}tg} P(e^{\boldsymbol{i}tg} f) \sim \sum_{j=0}^\infty P_j(f,g) t^{k-j}, \;\; t \to \infty, \;\; P_j(f,g) \in C^\infty(\boldsymbol{E}_1),$$

which has the following property: for every integer $N > 0$, for every compact set $\mathcal{K}$ of smooth functions g such that $dg \neq 0$ on $\operatorname{supp} f$ the error

$$t^{k-N}\Big(\, e^{-\boldsymbol{i}tg} P(e^{\boldsymbol{i}tg} f) - \sum_{j=0}^{N-1} P_j(f,g)\,\Big)$$

belongs to a bounded set of $C^\infty(\boldsymbol{E}_1)$, when $t > 1$ and $g \in \mathcal{K}$. A subset $\mathcal{B} \subset C^\infty(\boldsymbol{E}_1)$ is called bounded if for any compact set $S \subset M$, and any $n > 0$ we have

$$\sup\big\{\, \big| {}^{\boldsymbol{E}_1}\nabla^{\otimes j} u(x) \big|_{h_1}; \;\; x \in S, \;\; j \leq n, \;\; u \in \mathcal{B} \,\big\} < \infty. \qquad \square$$

Arguing as in previous chapter we deduce that any ψdo $A \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ defines a continuous linear operator

$$A : C_0^{-\infty}(\boldsymbol{E}_0) \to C^{-\infty}(\boldsymbol{E}_1).$$

A ψdo $A \in \boldsymbol{\Psi}(\boldsymbol{E}_0, \boldsymbol{E}_1)$ has a Schwartz kernel $K_A \in C^{-\infty}(\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee)$ characterized by the equality

$$\langle K_A, v \boxtimes u \rangle = \int_M \langle Au, v \rangle_{\boldsymbol{E}_1} \, |dV_g|, \;\; \forall u \in C_0^\infty(\boldsymbol{E}_0), \;\; v \in C_0^\infty(\boldsymbol{E}_1{}^\vee),$$

where

$$\langle -, - \rangle_{\boldsymbol{E}_i} : C^\infty(\boldsymbol{E}_i{}^\vee) \times C^\infty(\boldsymbol{E}_i) \to C^\infty(\underline{\mathbb{C}}_M)$$

is the natural bilinear pairing between a bundle and its dual.

The *transpose* or *dual* of A is the continuous linear operator $A^\vee : C_0^\infty(\boldsymbol{E}_1{}^\vee) \to C^{-\infty}(\boldsymbol{E}_0{}^\vee)$ with Schwartz kernel $K_{A^\vee} \in C^{-\infty}(\boldsymbol{E}_0{}^\vee \boxtimes \boldsymbol{E}_1)$ given by the equality

$$\langle K_{A^\vee}, u \boxtimes v\rangle = \langle K_A, v \boxtimes u\rangle, \;\; \forall u \in C_0^\infty(\boldsymbol{E}_0), \;\; v \in C_0^\infty(\boldsymbol{E}_1{}^\vee).$$

The arguments in the previous chapter show that $A^\vee$ is also a ψdo, and defines a continuous linear operator

$$A^\vee : C_0^\infty(\boldsymbol{E}_1{}^\vee) \to C^\infty(\boldsymbol{E}_0{}^\vee)$$

uniquely determined by the equality

$$\int_M \langle A^\vee u, v\rangle_{\boldsymbol{E}_0}\, |dV_g| = \int_M \langle u, Av\rangle_{\boldsymbol{E}_1}\, |dV_g|, \;\; \forall u \in C_0^\infty(\boldsymbol{E}_1{}^\vee), \;\; v \in C_0^\infty(\boldsymbol{E}_0).$$

If we fix hermitian metrics h_j on $\boldsymbol{E}_j$ we obtain complex *conjugate* linear bundle isomorphisms $I_{h_j} : \boldsymbol{E}_j \to \boldsymbol{E}_j{}^\vee$. These isomorphisms transport the dual $A^\vee$ to an operator

$$A^* = I_{h_0}^{-1} A^\vee I_{h_1} : C_0^\infty(\boldsymbol{E}_1) \to C^\infty(\boldsymbol{E}_0)$$

called the *formal adjoint* of A.

If $\boldsymbol{E}_0 = \boldsymbol{E}_1 = \underline{\mathbb{C}}_M$, then the action of A^* on a smooth, compactly supported function $f : M \to \mathbb{C}$ is given by

$$A^* f = \overline{A^\vee \overline{f}}.$$

The operator is said to be *properly supported* if the Schwartz kernel is properly supported. We denote by $\boldsymbol{\Psi}_0(\boldsymbol{E}_0, \boldsymbol{E}_1)$ the vector space of properly supported ψdo's. As in the previous chapter one can prove that any properly supported ψdo $A \in \boldsymbol{\Psi}_0(\boldsymbol{E}_0, \boldsymbol{E}_1)$ induces continuous linear operators

$$C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1), \;\; C_0^\infty(\boldsymbol{E}_0) \to C_0^\infty(\boldsymbol{E}_1).$$

Arguing exactly as in the proof of Proposition 2.3.7 we obtain the following result.

Proposition 3.1.5. *Let $A \in \boldsymbol{\Psi}(\boldsymbol{E}_0, \boldsymbol{E}_1)$. Then there exists a properly supported ψdo $A_0 \in \boldsymbol{\Psi}(\boldsymbol{E}_0, \boldsymbol{E}_1)$ such that $A - A_0$ is smoothing, i.e., its Schwartz kernel is a smooth section of $\boldsymbol{E}_1 \boxtimes \boldsymbol{E}_0{}^\vee$.*

Denote by $\hat{T}^*M$ denote the punctured cotangent bundle of M, i.e., the cotangent bundle with the zero section removed. Let $\hat{\pi} : \hat{T}^*M \to M$ denote the canonical projection. We define $\mathcal{H}^k(M, \boldsymbol{E}_0, \boldsymbol{E}_1)$ the space of bundle morphisms

$$S : \hat{\pi}^*\boldsymbol{E}_0 \to \hat{\pi}^*\boldsymbol{E}_1$$

such that, for any $x \in M$, $\xi \in T_x^*M \setminus \{0\}$ and $t > 0$ we have

$$S(x, t\xi) = t^k S(x, \xi) \in \operatorname{Hom}\big(\boldsymbol{E}_0(x), \boldsymbol{E}_1(x)\big).$$

The equivariance of the principal symbol map (2.7.5) discussed at the end of Section 2.7 shows that every properly supported ψdo $A \in \boldsymbol{\Psi}_0^k(M, \boldsymbol{E}_0, \boldsymbol{E}_1)$ has a well defined principal symbol $[\sigma_A] \in \mathcal{H}^k(M, \boldsymbol{E}_0, \boldsymbol{E}_1)$. More precisely, for $x \in M$ and $\xi \in T_x^*M \setminus \{0\}$ we define $[\sigma_A](x, \xi) : \boldsymbol{E}_0(x) \to \boldsymbol{E}_1(x)$ as follows.

- Fix a coordinate neighborhood $(\mathcal{O}, \Omega, F, T_i, E_i)$ such that $x \in \mathcal{O}$.
- If $\dot{F} : T_x\mathcal{O} \to T_{F(x)}\mathcal{O}$ denotes the differential of F at x and $\eta = (\dot{F}^\vee)^{-1}(\xi)$,

  $$[\sigma_A](x, \xi) = T_1(x)[\sigma_{A_\Omega}](F(x), \eta) T_0^{-1}(x),$$

  where

  $$A_\Omega := (F^*)^{-1} T_1^{-1}\big(A T_0 F^* \big)|_{\mathcal{O}}.$$

Proposition 3.1.6. *Suppose $\boldsymbol{E}_0, \boldsymbol{E}_1, \boldsymbol{E}_2$ are complex vector bundles over M. If $A_i \in \boldsymbol{\Psi}_0(\boldsymbol{E}_i, \boldsymbol{E}_{i+1})$, $i = 0, 1$, then $A_1 \circ A_0 \in \boldsymbol{\Psi}_0(\boldsymbol{E}_0, \boldsymbol{E}_2)$, $A_0^* \in \boldsymbol{\Psi}_0(E_1, E_0)$. Moreover*

$$[\sigma_{A_1 \circ A_0}] = [\sigma_{A_1} \circ [\sigma_{A_0}], \;\; [\sigma_{A_0^*}] = [\sigma_{A_0}]^*. \tag{3.1.1}$$

Proof. The only non-obvious statements are that the operators $A_1 \circ A_0$ and A_0^* are ψdo's. We will prove only the first statement. It suffices to show that for any smooth compactly supported functions $\eta, \varphi \in C_0^\infty(M)$ the operator $\eta A_1 A_0 \varphi$ is a ψdo. Since A_1, A_0 are properly supported, for any compact $K \subset M$ there exist compacts K_{A_0} and K_{A_0} such that for any $u_0 \in C^\infty(\boldsymbol{E}_0)$ and $u_1 \in C^\infty(\boldsymbol{E}_1)$ such that $\operatorname{supp} u_0, \operatorname{supp} u_1 \in K$ then

$$\operatorname{supp} A_i u_i \subset K_{A_i}, \;\; i = 0, 1.$$

Consider the compact set

$$K = \operatorname{supp} \eta \cup \operatorname{supp} \varphi \cup (\operatorname{supp} \varphi)_{A_0} \cup \big((\operatorname{supp} \varphi)_{A_0} \big)_{A_1}.$$

We construct a finite collection $(\psi_i)_{i \in I}$ of smooth, compactly supported functions on M such that following hold.

- The function $\psi = \sum_i \psi_i$ is identically 1 on a pre-compact open neighborhood $\mathcal{N}$ of K.
- For any $i_1, i_2, i_3, i_4 \in I$ the union of the supports of $\psi_{i_1}, \dots, \psi_{i_4}$ is contained in a coordinate region of M.

We do this as follows. Fix open precompact neighborhoods $\mathcal{O} \Supset \mathcal{N}$ of K. Then there exists a number $\delta > 0$ such that any open subset of M of diameter $< \delta$ that intersects $\mathcal{N}$ is contained in a coordinate region. It suffices to take δ smaller than the distance form $\mathcal{N}$ to $M \setminus \mathcal{O}$ and the injectivity radius of any point in the closure of $\mathcal{O}$. Observe that the union of any four geodesic balls of radius $< \delta/8$ centered at a point in K is contained in a coordinate region, because each connected component of such a set has diameter $< \delta$. Now choose a finite open cover $(B_i)_{i \in I}$ of the closure $\overline{\mathcal{N}}$ of $\mathcal{N}$ by geodesic balls of radii $< \delta/16$ and centered at points in the closure of $\mathcal{N}$. Set

$$B_* := M \setminus \overline{\mathcal{N}}, \;\; I_* = I \sqcup \{*\}.$$

Choose a partition of unity $(\psi_j)_{j \in I_*}$ subordinated to the cover $(B_j)_{j \in \boldsymbol{J}}$. The collection $(\psi_i)_{i \in I}$ has all the claimed properties. Observe that

$$A_1 = \sum_{i,j \in I_*} \psi_i A_1 \psi_j, \;\; A_0 = \sum_{k,\ell \in I_*} \psi_k A_0 \psi_\ell,$$

and

$$\eta A_0 A_1 \varphi = \sum_{i,j,k,\ell \in I} \underbrace{\eta \psi_i A_1 \psi_j \psi_k A_0 \psi_\ell \varphi}_{T_{i,j,k,\ell}}$$

If we set $B_{i,j,k,\ell} = B_i \cup \dots \cup B_\ell$, then by construction this is a coordinate region. In this coordinate region the operators $\psi_k A_0 \psi_\ell \varphi$ and $\eta \psi_i A_1 \psi_j$ are ψdo's, and the results in the previous chapter imply that $T_{i,j,k,\ell}$ is a ψdo. □

3.2. Elliptic ψdo's on manifolds

Definition 3.2.1. A ψdo $A \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ is said to be *elliptic* if it is properly supported and its principal symbol $[\sigma_A] : \pi^* \boldsymbol{E}_0 \to \pi^* \boldsymbol{E}_1$ defines an isomorphism of complex vector bundles over $T_0^* M$. □

We have the following counterpart of Theorem 2.9.4.

Theorem 3.2.2. *Let $A \in \boldsymbol{\Psi}_0^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$. Then the following statements are equivalent.*

(a) *The operator A is elliptic.*

(b) *There exists a ψdo $B \in \boldsymbol{\Psi}_0^{-k}(\boldsymbol{E}_1, \boldsymbol{E}_0)$ such that*

$$AB - \mathbb{1}, \;\; BA - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}.$$

(c) *There exists a ψdo $B \in \boldsymbol{\Psi}_0^{-k}(\boldsymbol{E}_1, \boldsymbol{E}_0)$ such that*

$$BA - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}.$$

(d) *There exists a ψdo $B \in \boldsymbol{\Psi}_0^{-k}(\boldsymbol{E}_1, \boldsymbol{E}_0)$ such that*

$$AB - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}.$$

Proof. Clearly (b) $\Rightarrow$ (c), (d). The implications (b), (c), (d) $\Rightarrow$ (a) follow from the composition rule (3.1.1). Thus, it suffices to show that (a) $\Rightarrow$ (b).

Choose a locally finite open cover $(\mathcal{O}_i)_{i\in I}$ of M by pre-compact coordinate regions. We set

$$A_i : A_{\mathcal{O}_i} : C_0^\infty(\boldsymbol{E}_0|_{\mathcal{O}_i}) \to C^\infty(\boldsymbol{E}_1|_{\mathcal{O}_i}).$$

Let A_i' be a properly supported ψdo on $\mathcal{O}_i$ such that $S_i = A_i - A_i'$ is smoothing. Invoking Theorem 2.9.4 we can find $B_i \in \boldsymbol{\Psi}_0^{-k}(\mathcal{O}_i, \boldsymbol{E}_i|_{\mathcal{O}_i}, \boldsymbol{E}_0|_{\mathcal{O}_i}$ such that $B_i A_i' - \mathbb{1} = B_i S_i \in \boldsymbol{\Psi}^{-\infty}$. Using Proposition 2.3.4 we deduce $B_i A_i - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}$.

Let $(\eta_i)_{i\in I}$, $\eta_i \in C_0^\infty(\mathcal{O}_i)$ be a partition of unity subordinated to the cover $(\mathcal{O}_i)_{i\in I}$. Next, choose $\varphi_i \in C_0^\infty(\mathcal{O}_i)$ such that $\varphi_i \equiv 1$ on an open neighborhood $\mathcal{N}_i$ of $\operatorname{supp} \eta_i$ in $\mathcal{O}_i$. Since the collection $(\mathcal{O}_i)_{i\in I}$ is locally finite, so is the collection $(\mathcal{N}_i)_{i\in I}$. For $u \in C_0^\infty(\boldsymbol{E}_1)$ define

$$Bu := \sum_i \eta_i B_i \varphi_i u|_{\mathcal{O}_i}.$$

Let us show that B is a ψdo, i.e., for any coordinate neighborhood with domain $\mathcal{O}$ the operator $B_{\mathcal{O}}$ (defined as in Remark 3.1.2(a)) is a ψdo. We will use Corollary 2.4.8 so it suffices to show that for any $\eta, \varphi \in C_0^\infty(\mathcal{O})$ the operator $\eta B_{\mathcal{O}} \varphi \in \boldsymbol{\Psi}^{-k}(\mathcal{O}, \boldsymbol{E}_1, \boldsymbol{E}_0)$ is a ψdo. There exists a finite set $I_\eta \subset I$ such that

$$\eta B_{\mathcal{O}} \varphi u = \sum_{i \in I_\eta} \eta \eta_i B_i \varphi_i \varphi u, \;\; \forall u \in C_0^\infty(\boldsymbol{E}_1|_{\mathcal{O}}).$$

Note that $\eta\eta_i B_i \varphi_i \varphi \in \boldsymbol{\Psi}_0^{-k}(\mathcal{O}, \boldsymbol{E}_1, \boldsymbol{E}_0)$, $\forall i \in I_{\varphi,\eta}$.

We want to prove that $BA - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}$. We will show that given $i_0 \in I$ and $x \in \mathcal{O}_{i_0}$ there exists a small neighborhood of $\mathcal{N}_x$ of $x$in $\mathcal{O}_{i_0}$ such that

$$\sigma_{BA}|_{\mathcal{N}_x} \sim \mathbb{1},$$

where the symbols are computed using the given trivializations and local coordinates over $\mathcal{O}_{i_0}$.

Since the collection $(\mathcal{N}_i)_{i\in I}$ is locally finite there exists a small open neighborhood $\mathcal{N}_x$ of $x \in \mathcal{O}_{i_0}$ such that the set

$$I_x := \{ i \in I; \;\; \mathcal{N}_i \cap \mathcal{N}_x \neq \emptyset \}$$

is finite. Note that

$$\sum_{i \in I_x} \eta_i(y) = \varphi_j(y) = 1, \;\; \forall y \in \mathcal{N}_x, \;\; \forall j \in I_x.$$

Hence, on $\mathcal{N}_x$, we have

$$\sigma_{BA} = \sum_{i \in I_x} \eta_i \big(\sigma_B \circledast \sigma_A \big) \sim \mathbb{1}.$$

□

Definition 3.2.3. If $A \in \boldsymbol{\Psi}_0^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ is an elliptic operator, then a parametrix of A is an operator $B \in \boldsymbol{\Psi}_0^{-k}(M, \boldsymbol{E}_1, \boldsymbol{E}_0)$ such that

$$AB - \mathbb{1}_{\boldsymbol{E}_1} \in \boldsymbol{\Psi}^{-\infty}, \;\; BA - \mathbb{1}_{\boldsymbol{E}_0} \in \boldsymbol{\Psi}^{-\infty}.$$ □

Arguing as in the proof of Corollary 2.9.6 we obtain the following result.

Corollary 3.2.4. *If $A \in \boldsymbol{\Psi}_0^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ is an elliptic ψdo and $u \in C^{-\infty}(\boldsymbol{E}_0)$ is such that $Au \in C^\infty(\boldsymbol{E}_1)$, then $u \in C^\infty(\boldsymbol{E}_0)$.* □

The method of construction of the parametrix presented in the proof of Theorem 3.2.2 yields the following more general result.

Corollary 3.2.5. *For any element $S \in \mathcal{H}^k(M, \boldsymbol{E}_0, \boldsymbol{E}_1)$ there exists a properly supported operator $T \in \boldsymbol{\Psi}_0^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ whose principal symbol is S, $[\sigma_T] = S$.*

Proof. Consider again the open cover $(\mathcal{O}_i)_{i \in I}$ of M and the functions $\eta_i, \varphi_i \in C_0^\infty(\mathcal{O}_i)$ used in the proof of Theorem 3.2.2. Then on the coordinate neighborhood $\mathcal{O}_i$ we can find an operator $T \in \boldsymbol{\Psi}_0^k(\mathcal{O}_i, \boldsymbol{E}_0, \boldsymbol{E}_1)$ such that $[\sigma_{T_i}] = S|_{\mathcal{O}_i}$. We deduce again that the operator

$$T = \sum_i \eta_i T_i \varphi_i$$

is pseudo-differential and its principal symbol is S. □

We conclude this section with a couple of of classical examples of elliptic operators that have found numerous applications in geometry and topology.

Definition 3.2.6. Suppose M is a smooth connected m-dimensional manifold , g is a Riemann metric on M, $\boldsymbol{E}$ is a complex vector bundle on M of rank r and h is a Hermitian metric on $\boldsymbol{E}$. A *Laplacian-type* (or generalized Laplacian) operator on $\boldsymbol{E}$ is a second order partial differential operator $L : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ such that the following hold.

(a) $L = L^*$.

(b) For any $x \in M$, $\xi \in T_x^*M$ we have $[\sigma_L](x, \xi) = |\xi|_g^2 \mathbb{1}_{\boldsymbol{E}_x}$.

□

From the definition we deduce that the generalized Laplacians are elliptic operators.

Example 3.2.7. Suppose (M, g) is a connected, smooth Riemann manifold of dimension m. Denote by $\Omega_{\mathbb{C}}^k(M)$ the space of smooth, complex valued differential forms of degree k on M. We set

$$\Omega_{\mathbb{C}}^\bullet(M) = \bigoplus_{k=0}^m \Omega_{\mathbb{C}}^k(M) = C^\infty\Big(\bigoplus_{k=0}^m \Lambda^k T^*M \otimes \mathbb{C} \Big).$$

The exterior derivative defines a first order operator

$$d : \Omega^\bullet_{\mathbb{C}}(M) \to \Omega^\bullet_{\mathbb{C}}(M).$$

Observe that for any $x \in M$ and $\xi \in T^*_x M$ we have

$$[\sigma_d](x, \xi) = e(\boldsymbol{i}\xi)$$

where $e(\boldsymbol{i}\xi) : \oplus_k \Lambda^k T^*_x M \otimes \mathbb{C} \to \oplus_k \Lambda^k T^*_x M \otimes \mathbb{C}$ denotes the operation of exterior multiplication with the complex covector $\boldsymbol{i}\xi$. We form the Hodge-DeRham operator

$$D := d + d^* : \Omega^\bullet_{\mathbb{C}}(M) \to \Omega^\bullet_{\mathbb{C}}(M).$$

From (2.9.2) we deduce that

$$\sigma_{d^*}(\xi) = \sigma_d(\xi)^* = e(\boldsymbol{i}\xi)^* = \boldsymbol{i}\xi_\dagger \lrcorner,$$

where $\xi_\dagger \lrcorner$ denotes the contraction with the vector $\xi_\dagger$ dual to ξ with respect to the metric g. The Cartan identity (2.9.3) then implies that the operator

$$D^2 = (d + d^*)^2 = dd^* + d^*d : \Omega^\bullet_{\mathbb{C}}(M) \to \Omega^\bullet_{\mathbb{C}}(M)$$

is a generalized Laplacian. From the definition it follows that

$$D^2\Big(\,\Omega^k_{\mathbb{C}}(M)\,\Big) \subset \Omega^k_{\mathbb{C}}(M), \;\; \forall k \geq 0.$$

Thus, D^2 decomposes as a direct sum of generalized Laplacians

$$D^2 = \bigoplus_{k=0}^{m} \Delta_k, \;\; \Delta_k := D^2|_{\Omega^k_{\mathbb{C}}(M)}.$$

The operator Δ_0 acts on smooth functions

$$\Delta_0 = d^*d : C^\infty(M) \to C^\infty(M).$$

It is called the *scalar Laplacian* of the Riemann manifold (M, g).

Definition 3.2.8. Suppose (M, g) is a smooth, connected m-dimensional manifold and $(\boldsymbol{E}_0, h_0)$, $(\boldsymbol{E}_1, h_1)$ are two complex vector bundles of the same rank r equipped with Riemann metrics. A first order partial differential operator $\mathcal{D} : C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$ is said to be a *Dirac-type* operators if the differential operators

$$\mathcal{D}^*\mathcal{D} : C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_0) \;\text{ and }\; \mathcal{D}\mathcal{D}^* : C^\infty(\boldsymbol{E}_1) \to C^\infty(\boldsymbol{E}_1)$$

are Laplacian-type operators. □

Clearly the Dirac type operators are elliptic. The computations in Example 3.2.7 show that the Hodge-DeRham operator is a Dirac-type operator.

Definition 3.2.9. Suppose (M, g) is a smooth, connected m-dimensional manifold and $(\widehat{\boldsymbol{E}}, h)$ is a complex vector bundle equipped with a Hermitian metric. A *super-symmetric* Dirac-type operator on $\widehat{\boldsymbol{E}}$ is a pair $(\widehat{\mathcal{D}}, \Gamma)$ where $\Gamma : \widehat{\boldsymbol{E}} \to \widehat{\boldsymbol{E}}$ is a unitary automorphism of $\widehat{\boldsymbol{E}}$ such that $\Gamma^2 = \mathbb{1}$, and $\widehat{\mathcal{D}} : C^\infty(\widehat{E}) \to C^\infty(\widehat{\boldsymbol{E}})$ is a Dirac-type operator such that

$$\widehat{\mathcal{D}}^* = \widehat{\mathcal{D}}, \;\; \widehat{\mathcal{D}}\Gamma + \Gamma\widehat{\mathcal{D}} = 0.$$

The involution Γ is called the *chirality operator* associated to the super-symmetric Dirac-type operator. □

To every Dirac-type operator $\mathcal{D} : C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$ we can associate a canonical super-symmetric Dirac type operator $(\widehat{\mathcal{D}}, \Gamma)$ on $\widehat{\boldsymbol{E}} := \boldsymbol{E}_0 \oplus \boldsymbol{E}_1$, where Γ and $\widehat{\mathcal{D}}$ are given by the block decompositions

$$\Gamma := \begin{bmatrix} \mathbb{1}_{\boldsymbol{E}_0} & 0 \\ 0 & -\mathbb{1}_{\boldsymbol{E}_1} \end{bmatrix} : \begin{array}{c} \boldsymbol{E}_0 \\ \oplus \\ \boldsymbol{E}_1 \end{array} \to \begin{array}{c} \boldsymbol{E}_0 \\ \oplus \\ \boldsymbol{E}_1 \end{array}, \quad \widehat{\mathcal{D}} = \begin{bmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{bmatrix} : \begin{array}{c} C^\infty(\boldsymbol{E}_0) \\ \oplus \\ C^\infty(\boldsymbol{E}_1) \end{array} \to \begin{array}{c} C^\infty(\boldsymbol{E}_0) \\ \oplus \\ C^\infty(\boldsymbol{E}_1) \end{array}.$$

Conversely, if $(\widehat{\mathcal{D}}, \Gamma)$ is a super-symmetric Dirac-type operator on $\widehat{\boldsymbol{E}}$ then chiral operator induces an orthogonal direct sum decomposition

$$\widehat{\boldsymbol{E}} = \boldsymbol{E}_+ \oplus \boldsymbol{E}_-,$$

where $\boldsymbol{E}_\pm$ is the ± 1-eigenbundle of Γ, $\boldsymbol{E}_\pm := \ker(\pm\mathbb{1} - \Gamma)$. Since $\widehat{\mathcal{D}}$ anti-commutes with Γ we deduce that

$$\widehat{\mathcal{D}}\Big(C^\infty(\boldsymbol{E}_\pm) \Big) \subset C^\infty(\boldsymbol{E}_\mp).$$

The induced differential operators $\mathcal{D}_\pm : C^\infty(\boldsymbol{E}_\pm) \to C^\infty(\boldsymbol{E}_\mp)$ satisfy

$$\mathcal{D}_+^* = \mathcal{D}_-^*$$

since $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}^*$. This proves that $\widehat{\mathcal{D}}_+$ is a Dirac-type operator and $\widehat{\mathcal{D}}$ is the super-symmetric Dirac-type operator associated to $\mathcal{D}_+$. The operator $\mathcal{D}_+$ is called the *even* Dirac-type operator determined by $\widehat{\mathcal{D}}$.

Example 3.2.10. Consider the Hodge-DeRham operator

$$D : \Omega^\bullet_{\mathbb{C}}(M) \to \Omega^\bullet_{\mathbb{C}}(M),$$

on the m-dimensional smooth Riemann manifold (M, g). Define

$$\epsilon : \bigoplus_{k=0}^m \Lambda^k_{\mathbb{C}} T^*M \to \bigoplus_{k=0}^m \Lambda^k_{\mathbb{C}} T^*M, \quad \epsilon|_{\Lambda^k_{\mathbb{C}}T^*M} = (-1)^k \mathbb{1}_{\Lambda^k_{\mathbb{C}}T^*M}.$$

Thus, if α is a differential form of degree k on M then $\epsilon(\alpha) = (-1)^k\alpha$. Clearly D anti-commutes with ϵ so (D, ϵ) is a super-symmetric Dirac-type operator on $\Lambda^\bullet_{\mathbb{C}} T^*M$. We will refer to it as the *Gauss-Bonnet operator*.

Suppose now that M is also *oriented* and *even dimensional*, $m = 2m_0$. We then have a Hodge $*$-operator

$$* : \Lambda^k T^*M \to \Lambda^{2m_0-k} T^*M$$

uniquely determined by the equality

$$\alpha(x) \wedge (*\beta)(x) = \big(\alpha(x), \beta(x) \big)_g \, dV_g(x), \ \ \forall \alpha, \beta \in \Omega^k(M), \ \ x \in M$$

where $dV_g \in \Omega^{2m_0}(M)$ denotes the volume *form* determined by the metric g and the *orientation* on M. We extend $*$ by complex linearity to an unitary bundle isomorphism

$$* : \Lambda^k_{\mathbb{C}} T^*M \to \Lambda^{m-k}_{\mathbb{C}} T^*M.$$

This operator satisfies the identity [**N**, Prop. 2.2.70]

$$*(*\alpha) = (-1)^k \alpha, \ \ \forall \alpha \in \Omega^k_{\mathbb{C}}(M),$$

The adjoint of the exterior derivative d can be expressed using the Hodge operator via the classical equality, [**N**, Lemma 4.1.49]

$$d^* = - * d * .$$

Now define the *Hodge chirality operator*

$$\Gamma_M = \oplus_{k=0}^{2m_0}\Gamma_M^k,\;\; \Gamma_M : \Lambda^k_{\mathbb{C}}T^*M \to \Lambda^{2m_0-k}_{\mathbb{C}}T^*M,\;\; \Gamma_M^k\alpha = \boldsymbol{i}^{\mu(k)} * \alpha,\;\; \mu(k) = k(k-1) + m_0.$$

Observe that if $\alpha \in \Omega^k_{\mathbb{C}} * M)$ then

$$\Gamma_M^2\alpha = \boldsymbol{i}^{\mu(k)+\mu(2m_0-k)+k}\alpha = \alpha$$

because $\mu(k) + \mu(2m_0 - k) \equiv 2k^2 \bmod 4$.

$$D * \alpha = (d - *d*) * \alpha = d * \alpha - (-1)^k * d\alpha$$

Next, for $\alpha \in \Omega^k_{\mathbb{C}}(M)$ we have

$$D\Gamma_M\alpha = \boldsymbol{i}^{\mu(k)}(d * \alpha - (-1)^k * d\alpha),$$

and

$$\Gamma_M D\alpha = \boldsymbol{i}^{\mu(k+1)} * d\alpha - \boldsymbol{i}^{\mu(k-1)}(-1)^{2m_0-k+1} d * \alpha$$

Now observe that for any ℓ we have $\boldsymbol{i}^{\mu(\ell+1)} = (-1)^{\ell+1}\boldsymbol{i}^{\mu(\ell)}$ which shows that D anti-commutes with Γ_M. The resulting super-symmetric Dirac-type operator $D, \Gamma_M)$ is called the *signature operator*. □

3.3. Sobolev spaces on manifolds

Suppose M is a smooth connected, m-dimensional manifold equipped with a Riemann metric, $\boldsymbol{E} \to M$ is a smooth complex vector bundle of rank r equipped with a hermitian metric h and compatible connection. We denote by ∇^g the Levi-Civita connection on the various bundles of tensors, and by $|dV_g|$ the volume density on M determined by g.

We define $L^2_{\mathrm{loc}}(\boldsymbol{E})$ to be the vector space of Borel measurable sections $u : M \to \boldsymbol{E}$ such that, for any compact subset $K \subset M$ we have

$$\int_K |u(x)|_h^2\,|dV_g(x)| < \infty.$$

Equivalently, a Borel measurable section $u : M \to \boldsymbol{E}$ belongs to $L^2_{\mathrm{loc}}(\boldsymbol{E})$ if and only if

$$\int_M |\varphi u|^2 |dV_g| < \infty,\;\; \forall \varphi \in C_0^\infty(M).$$

We define

$$H^s_{\mathrm{loc}}(\boldsymbol{E}) := \bigl\{ u \in C^{-\infty}(\boldsymbol{E});\;\; Au \in L^2_{\mathrm{loc}}(\boldsymbol{E}),\;\; \forall A \in \boldsymbol{\Psi}^k_0(\boldsymbol{E}) \bigr\}.$$

Finally we define

$$H^s_{\mathrm{comp}}(\boldsymbol{E}) := \bigl\{ u \in H^s_{\mathrm{loc}}(\boldsymbol{E});\;\; \operatorname{supp} u \;\text{ is compact} \bigr\}.$$

Observe that if M happens to be an open subset of an Euclidean vector space $\boldsymbol{V}$ of dimension m, and $\boldsymbol{E}$ is the trivial vector bundle $\boldsymbol{E} = \underline{E}_M$, then Theorem 2.8.4 shows that the spaces $H^s_{\mathrm{loc}}(\underline{E}_M)$ and $H^s_{\mathrm{comp}}(\underline{E}_M)$ defined above coincide with the space $H^s_{\mathrm{loc}}(M, E)$ defined in Section 1.5. To ease the burden of notation we will assume that $\boldsymbol{E}$ is the trivial complex line bundle $\underline{\mathbb{C}}_M$ over M, so that the (generalized) sections of $\boldsymbol{E}$ are (generalized) functions on M. The general situation can be safely left to the reader. We set

$$H^s_{\mathrm{loc}}(M) := H^s_{\mathrm{loc}}(\underline{\mathbb{C}}_M),\;\; H^s_{\mathrm{comp}}(M) := H^s_{\mathrm{comp}}(\underline{\mathbb{C}}_M).$$

We want to equip $H^s_{\text{comp}}(M)$ and $H^s_{\text{loc}}(M)$ with a locally convex topologies. The construction will require some additional choices, but the resulting topologies will be independent of these choices. We begin by defining a structure of Hilbert space on the vector spaces

$$H^s(K) := \big\{ u \in H^s_{\text{loc}}(M);\ \ \operatorname{supp} u \subset K \big\},$$

where $K \subset M$ is an arbitrary compact subset. Choose a finite open cover of K by precompact coordinate neighborhoods $(\mathcal{O}_i)_{i\in I}$ and let $(\varphi_i)_{i\in I}$ be a partition of unity subordinated to $\mathcal{O}_i$. On particular $\varphi_i \in C_0^\infty(\mathcal{O}_i)$. The local coordinates $F_i : \mathcal{O}_i \to \Omega_i$ allows us to identify $\mathcal{O}_i$ with an open subset $\Omega_i \subset \boldsymbol{V}$, while for any $w \in C_0^{-\infty}(\mathcal{O}_i, \boldsymbol{E})$ we can identify w with the compactly supported distribution $(F_i^{-1}) * w \in C_0^{-\infty}(\Omega_i, \mathbb{C}^r)$. For simplicity we set $G_i := F_i^{-1}$. Given $u, v \in H^s(K)$ we define

$$(u,v)_{s,K} = \sum_{i\in I} \big(\, (G_i)^*(\varphi_i u), (G_i)^*(\varphi_i v)\,\big)_s$$

$$= \sum_{i\in I} \int_{\boldsymbol{V}} \mathfrak{F}[(G_i)^*(\varphi_i u)](\xi) \,\cdot\, \overline{\mathfrak{F}[(G_i)^*(\varphi_i v)](x))}(1+|\xi|^2)^s\,|d\xi|.$$

so that

$$\|u\|^2_{s,K} = \sum_{i\in I} \|(G_i)^*(\varphi_i u)\|^2_{s,\boldsymbol{V}}.$$

The norm $\|-\|_{s,K}$ depends on the choice Ξ consisting of a finite open cover $(\mathcal{O}_i)$ consisting of precompact sets, local coordinates F_i on $\mathcal{O}_i$, and a partition of unity $(\varphi_i)_{i\in I}$ subordinated to $(\mathcal{O}_i)_{i\in I}$. Thus, it is more appropriate to denote this norm by $\|-\|_{s,\Xi}$. We want to prove that for any two such choices Ξ, $\widetilde{\Xi}$, and any $s \in \mathbb{R}$ there exists a constant $C = C(s, \Xi, \widetilde{\Xi}) > 0$ such that

$$\|u\|_{s,\Xi} \le C\|u\|_{s,\widetilde{\Xi}},\ \ \forall u \in H^s(K).$$

This boils down to proving the following result.

Proposition 3.3.1. *Suppose Ω is an open precompact subset of $\boldsymbol{V}$, and $(\Omega_i)_{i\in I}$ finite collection of open precompact subsets of $\boldsymbol{V}$ such that*

$$\Omega \subset \bigcup_{i\in I} \Omega_i.$$

For every $i \in I$ we fix a diffeomorphism $F_i : \Omega_i \to D_i$ where D_i is also a subset of $\boldsymbol{V}$. Then, for any $\varphi \in C_0^\infty(\Omega)$ and any partition of unity $(\eta_i)_{i\in I}$, $\eta_i \in C_0^\infty(\Omega_i)$, there exists a constant $C > 0$ such that, for any $u \in H^s_{\text{loc}}(\Omega)$ we have

$$\|\varphi u\|_s \le C \sum_{i\in I} \|(G_i)^*(\eta_i \varphi u)\|_s,$$

where $G_i = F_i^{-1}$.

Proof. We have

$$\varphi u = \sum_{i\in I} \varphi \eta_i u$$

so that

$$\|\varphi u\|_s \le \sum_{i\in I} \|\varphi \eta_i u\|_s$$

We conclude by invoking Proposition 1.5.17. □

The natural topology on $H^s_{\rm comp}(M)$ is the finest locally convex topology such that all the inclusions

$$H^s(K) \hookrightarrow H^s_{\rm comp}(M), \;\; K \subset M \text{ compact}$$

are continuous. We equip $H^s_{\rm loc}(M)$ with the topology given by the family of seminorms

$$p_\varphi : H^s_{\rm loc}(M) \to \mathbb{R}, \;\; p_\varphi(u) = \|\varphi u\|_{s, \operatorname{supp}\varphi}, \;\; \varphi \in C_0^\infty(M).$$

The embedding theorems in Section 1.5 imply the following result.

We can define Hölder spaces of sections in a similar fashion. If ℓ is a nonnegative integer and $\alpha \in (0,1)$ then a $C^{\ell,\alpha}_{\rm loc}(\boldsymbol{E})$ consists of C^ℓ-sections of $\boldsymbol{E}$ such that for any coordinate region $\mathcal{O}$ we have the restriction

$$\|u|_{\mathcal{O}}\|_{\ell,\alpha} < \infty,$$

where the above norm is constructed using normal coordinates on the components of $\mathcal{O}$ and trivializing the bundle by radial parallel transport. If $K \subset M$ is compact and $u \in C^{\ell,\alpha}{\rm loc}(\boldsymbol{E})$ is supported on K then we define

$$\|u\|_{\ell,\alpha} = \sum_i \|\varphi_i u\|_{\ell,\alpha},$$

where $(\varphi)_{i\in I}$ is a partition of unity subordinated to a finite cover of K by coordinate regions, and the norms $\|vfi_i u\|_{\ell,\alpha}$ are determined using local coordinates and trivializations of $\boldsymbol{E}|_{\mathcal{O}_i}$. This norm depends on the various spaces, but the induced Banach space topology of section $u \in C^{\ell,\alpha}(\boldsymbol{E})$, $\operatorname{supp} u \subset K$, is independent of these choices.

Theorem 3.3.2. *Suppose M is a smooth, connected, m-dimensional manifold equipped with a smooth metric and $\boldsymbol{E} \to M$ is a smooth complex vector bundle of rank r equipped with a hermitian metric and compatible connection.*

(a) Let k be a positive integer, $\mu \in (0,1)$ and $s > \mu + k + m/2$. Fix a function $\varphi \in C_0^\infty(M)$. Then

$$H^s_{\rm loc}(\boldsymbol{E}) \subset C^{k,\mu}(\boldsymbol{E}),$$

and there exists a positive constant C such that for any $u \in H^s_{\rm loc}(M, \boldsymbol{E})$ we have

$$\|\varphi u\|_{k,\mu} \leq C\|\varphi u\|_{s,\operatorname{supp}\varphi}.$$

(b) For any real numbers $t > s$, and any compact set $K \subset M$ the inclusion $H^t(K, \boldsymbol{E}) \to H^s(\boldsymbol{E})$ is compact, i.e., any sequence $(f_n)_{n\geq 1} \subset H^t(K, \boldsymbol{E})$ that is bounded in the $\| - \|_{t,K}$ norm contains a subsequence that converges in the $\| - \|_s$-norm. □

If M is a *compact* manifold then

$$H^s_{\rm loc}(\boldsymbol{E}) = H^s_{\rm comp}(M, \boldsymbol{E}) = H^s(\boldsymbol{E}), \;\; \forall s \in \mathbb{R}$$

and we obtain the following consequence of Theorem 3.3.2.

Corollary 3.3.3 (Embedding theorems)**.** *(a) Suppose M is a compact manifold, k be a positive integer, $\mu \in (0,1)$ and $s > \mu + k + m/2$. Then $H^s(\boldsymbol{E})$ embeds continuously in the Banach space $C^{k,\mu}(\boldsymbol{E})$.*

(b) If $t > s$ then the natural inclusion of Hilbert spaces $H^t(\boldsymbol{E}) \hookrightarrow H^s(\boldsymbol{E})$ is a compact, continuous operator, i.e., the sets that are bounded in the $\| - \|_t$-norm are precompact in the $\| - \|_s$-norm. □

Remark 3.3.4. If g is a Riemann metric on the compact manifold M, $\boldsymbol{E} \to M$ is a smooth complex vector bundle on M, h a hermitian metric on E, and ∇ is a connection on E compatible with the metric h, then for any nonnegative k the topology of $H^k(\boldsymbol{E})$ is defined by the norm

$$\|u\|_k = \Big(\sum_{j=0}^{k} \int_M |\nabla^j u(x)|^2_{h,g} \, |dV_g(x)| \Big)^{1/2}$$

where $\nabla^j : C^\infty(\boldsymbol{E}) \to C^\infty((T^*M)^{\otimes j} \otimes \boldsymbol{E})$ is defined as in (1.4.1) and $| - |_{h,g}$ denotes the induced metric on $(T^*M)^{\otimes j} \otimes \boldsymbol{E}$. □

✍ **Notational convention.** When working on manifolds the various Sobolev spaces $H^s(M)$ could be confused with various cohomology groups. To eliminate this confusion we will use the alternate notation $L^{s,2}(M)$ to denote the spaces $H^s(M)$. Thus $L^{s,2}$ stands for functions (sections) that have weak derivatives up to order s which belong to L^2. We keep the superscript 2 since there exist spaces $L^{s,p}$ for any $s \in \mathbb{R}$, $p \in [1, \infty]$.

From Theorem 2.8.4 we deduce immediately the following important continuity result.

Theorem 3.3.5. *Suppose $A \in \boldsymbol{\Psi}^k_0(M, \boldsymbol{E}_0, \boldsymbol{E}_1)$ is a properly supported ψdo of order $\leq k$ and $s \in \mathbb{R}$. Then for any $\varphi \in C^\infty_0(M)$ there exists $\psi \in C^\infty_0(M)$ and a positive constant C such that*

$$\|\varphi A u\|_{s, \operatorname{supp} \varphi} \leq C \|\psi u\|_{s+k, \operatorname{supp} \psi}, \quad \forall u \in L^{s+k,2}_{\mathrm{loc}}(\boldsymbol{E}_0).$$ □

Later we will need the following consequence.

Corollary 3.3.6. *Suppose M is a compact manifold of dimension m, $\boldsymbol{E}_0, \boldsymbol{E}_1 \to M$ are smooth complex vector bundles of ranks r_0 and respectively r_1, and $A \in \boldsymbol{\Psi}^{-k}(\boldsymbol{E}_0, \boldsymbol{E}_1)$, $k > 0$ Then for any $s \in \mathbb{R}$ the operator A induces a continuous compact map*

$$A : L^{s,2}(\boldsymbol{E}_0) \to L^{s,2}(\boldsymbol{E}_1).$$

Proof. We know that A induces a continuous map $L^{s,2}(\boldsymbol{E}_0) \to L^{s+k,2}(\boldsymbol{E}_1)$. Since the inclusion $L^{s+k,2}(\boldsymbol{E}_1) \hookrightarrow L^{s,2}(\boldsymbol{E}_1)$ is compact we deduce that the composition

$$L^{s,2}(\boldsymbol{E}_0) \xrightarrow{A} L^{s+k,2}(\boldsymbol{E}_1) \hookrightarrow L^{s,2}(\boldsymbol{E}_1)$$

is compact. □

The elliptic regularity and estimates (Corollary 2.9.8) have obvious counterparts for ψdo's on manifolds. Their formulations can be safely left to the reader.

3.4. Fredholm operators

We want to survey here a few more or less classical facts of functional analysis that will play a key part in the sequel. For simplicity we will restrict ourselves to a Hilbert space context.

Definition 3.4.1. Let H_0, H_1 be two (separable) complex Hilbert spaces.

(a) A continuous linear operator $T : H_0 \to H_1$ is called *Fredholm* if the following hold.

(a1) The kernel of T is finite dimensional.

(a2) The range of T is a closed subspace $\operatorname{ran}(T) \subset H_1$.

(a3) The cokernel of T is finite dimensional, i.e., the orthogonal complement of $\operatorname{ran}(T)$ in H_1 is finite dimensional.

(b) The *index* of a Fredholm operator $T : H_0 \to H_1$ is the integer

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{ran}(T)^{\perp}.$$

(c) We denote by $\boldsymbol{Fred}(H_0, H_1)$ the space of Fredholm operators $T_0 \to T_1$. If $H_0 = H_1 = H$ we use the simpler notation $\boldsymbol{Fred}(H) = \boldsymbol{Fred}(H, H)$.

Example 3.4.2. Consider the Hilbert space ℓ_2 of sequences of complex numbers $\underline{x} = (x_n)_{n \geq 0}$ such that

$$\sum_{n \geq 0} |x_n|^2 < \infty.$$

For every integer k we define the shift map

$$S_k : \ell_2 \to \ell_2, \quad S\underline{x} = \underline{y}, \quad y_n = \begin{cases} x_{n+k}, & n+k \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then S_k is a Fredholm operator and $\operatorname{ind} S_k = k$. □

We have the following important characterization of Fredholm operators.

Theorem 3.4.3. *Suppose H_0, H_1 are separable complex Hilbert spaces and $T : H_0 \to H_1$ is a continuous linear operator. Then the following statements are equivalent.*

(a) *The operator T is Fredholm.*

(b) *The adjoint operator $T^* : H_1 \to H_0$ is Fredholm.*

(c) *There exist a continuous linear operators $Q : H_1 \to H_0$ such that the operators $TQ - \mathbb{1}_{H_1}$ and $QT - \mathbb{1}_{H_0}$ are compact.*

Proof. (a) $\Longleftrightarrow$ (b). This follows from Banach's closed range theorem (see [**Br**, II] or [**Y**, VII.5]) which states that if $T : X \to Y$ is a continuous operator between two Hilbert spaces the following conditions are equivalent.

- The range of T is closed.
- The range of T^* is closed.
- $\operatorname{ran}(T) = (\ker T^*)^{\perp}$.
- $\operatorname{ran}(T^*) = (\ker T)^{\perp}$.

(a) $\Longrightarrow$ (c) Let $V := \operatorname{ran}(T) \subset H_1$ and $U := (\ker T)^{\perp}$; see Figure 3.1.

Then the induced map $T|_U : U \to V$ is bijective, and the open mapping theorem implies that its inverse S is continuous. Define $Q : H_1 \to H_0$ by

$$Qx = \begin{cases} Sx, & x \in V \\ 0, & x \in V^{\perp}. \end{cases}$$

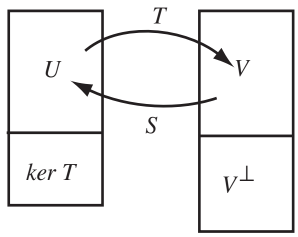

Figure 3.1. *Decomposing H_0 and H_1.*

If we denote by P_0 the orthogonal projection onto $\ker T$, and by P_1 the orthogonal projection onto $V^\perp$, then P_0, P_1 are compact because they have finite dimensional ranges and moreover

$$QT = \mathbb{1}_{H_0} - P_0, \;\; TQ = \mathbb{1}_{H_1} - P_1.$$

(c) $\Longrightarrow$ (a) Let $Q : H_1 \to H_0$ be a continuous linear operator such that $K_0 = QT - \mathbb{1}_{H_0}$ and $K_1 = TQ - \mathbb{1}_{H_1}$ are compact.

Let us first prove that $\dim \ker T < \infty$. This follows from the following result.

Lemma 3.4.4. *Any bounded sequence in* $\ker T$ *admits a convergent subsequence.*

Proof. Let $(x_n)_{n\geq 0}$ be a bounded sequence in $\ker T$. Hence

$$-K_0 x_n = -QTx_n + x_n = x_n.$$

Since the operator $-K_0$ is compact and the sequence $(x_n)_{n\geq 0}$ is bounded we deduce that the sequence $-K_0 x_n$ admits a convergent subsequence. □

The above lemma implies that $\ker T$ is locally compact and therefore (see [**Br**, Thm. Vi.5] or [**RSz**, §77, 89]) it must be finite dimensional. Since $K_1^* = Q^*T^* - \mathbb{1}_{H_1}$ is compact we deduce as above that $\ker T^*$ is also finite dimensional.

Let us now show that $\operatorname{ran}(T)$ is closed. We denote by $\widetilde{T}$ the restriction of T to $U = (\ker T)^\perp$ and we observe that $\widetilde{T}$ is one-to-one and $\operatorname{ran}(\widetilde{T}) = \operatorname{ran}(T)$ so it suffices to prove that $\operatorname{ran}(\widetilde{T})$ is closed. If we denote by P_U the orthogonal projection onto U we observe that the operator $\widetilde{Q} = P_U Q$ satisfies

$$\widetilde{Q}\widetilde{T} = \mathbb{1}_U + \widetilde{K}_0, \;\; \widetilde{K}_0 := P_U K_0|_U$$

so that $\widetilde{Q}\widetilde{T} - \mathbb{1}_U$ is compact.

Lemma 3.4.5. *There exists $C > 0$ such that*

$$\|u\| \leq c\|\widetilde{T}u\|, \;\; \forall u \in U.$$

Proof. We argue by contradiction. We assume that there exists a sequence $(u_n)_{n\geq 0}$ in U such that

$$\|u_n\| = 1 \;\text{ and }\; \widetilde{T}u_n \to 0. \tag{3.4.1}$$

Then

$$u_n + \widetilde{K}_0 u_n = \widetilde{Q}\widetilde{T}u_n \to 0$$

Since (u_n) is bounded and $\widetilde{K}_0$ is compact we deduce that a subsequence $\widetilde{K}_0 u_{n_k}$ of $\widetilde{K}_0 u_n$ is convergent. From the above equality we deduce that u_{n_k} is also convergent to an element u_*. Moreover

$$\|u_*\| = \lim \|u_{n_k}\| = 1 \neq 0.$$

Using this in (3.4.1) we deduce that $\widetilde{T} u_* = 0$. Thus $\ker \widetilde{T} \neq 0$. This contradicts the fact that $\widetilde{T}$ is one-to-one. □

Suppose $y_n = \widetilde{T} u_n$ converges to y. We need to prove that there exists $u \in U$ such that $y = Tu$. Using Lemma 3.4.5 we deduce that there exists $C > 0$ such that

$$\|u_n - u_m\| \leq \|\widetilde{T}(u_n - u_m)\| \leq C\|y_n - y_m\|, \;\; \forall m, n \geq 0.$$

The sequence (y_n) is Cauchy and we deduce from the above inequality that the sequence (u_n) is also Cauchy and thus converges to some $u \in U$. Clearly $y = \widetilde{T} u$. This proves that $\operatorname{ran}(T)$ is closed.

From the closed graph range theorem we deduce that $\operatorname{ran}(T)^\perp = ((\ker T^*)^\perp)^\perp = \ker T^*$ so that $\dim \operatorname{ran}(T)^\perp < \infty$. This completes the proof of Theorem 3.4.3. □

We record for later use some consequences of the above proof.

Corollary 3.4.6. *If $T : H_0 \to H_1$ then so is its adjoint and moreover*

$$\operatorname{ind} T = \dim \ker T - \dim \ker T^* = -\operatorname{ind} T^*. \qquad \square$$

Corollary 3.4.7. *If $T : H_0 \to H_1$ is a Fredholm operator then there exists a constant $C > 0$ such that*

$$\|x\|_{H_0} \leq C\|Tx\|_{H_1}, \;\; \forall x \in H_0, \;\; x \perp \ker T.$$

In particular, if T is injective, then there exists a constant $C > 0$ such that

$$\|x\|_{H_0} \leq C\|Tx\|_{H_1}, \;\; \forall x \in H_0. \qquad \square$$

Definition 3.4.8. A *quasi-inverse* of the continuous linear operator $T : H_0 \to H_1$ is a continuous linear operator $Q : H_1 \to H_0$ satisfying condition (c) in Theorem 3.4.3, i.e., the operators

$$QT - \mathbb{1}_{H_0} \;\text{ and }\; TQ - \mathbb{1}_{H_1}$$

are compact. □

Corollary 3.4.9. *If $S, T : H_0 \to H_1$ are continuous linear operators and $K = T - S$ is compact, then T is Fredholm if and only if S is Fredholm.*

Proof. Suppose T is Fredholm. If Q is a quasi-inverse of T then $QT - \mathbb{1}$ and $TQ - \mathbb{1}$ are compact. We observe that $S = T - K$ so that

$$QS - \mathbb{1} = Q(T - K) - \mathbb{1} = QT - \mathbb{1} - QK.$$

Since K is compact we deduce QK is compact as well so that $QS - \mathbb{1}$ is compact. A similar argument shows that $SQ - \mathbb{1}$ is compact so that Q is a quasi-inverse of S so that S is Fredholm. □

Corollary 3.4.10. *Suppose $T : H_0 \to H_1$ is a continuous linear operator, U, V are finite dimensional complex Hermitian vector spaces an $A : U \to H_1$ and $B : H_0 \to V$ are continuous linear operators. Define*

$$T_A : H_0 \oplus U \to H_1 \;\; \text{and} \;\; T^B : H_0 \to H_1 \oplus V$$

by

$$T_A(x \oplus u) = Tx + Au, \;\; T^B x = (Tx) \oplus (Bx), \;\; \forall x \in H_0, \;\; u \in U.$$

Then the following statements are equivalent.

(a) *The operator T is Fredholm.*

(b) *The operator T_A is Fredholm.*

(c) *The operator T^B is Fredholm.*

Proof. Set $T_0 = T_A$, $A = 0$ and $T^0 = T^B$, $B = 0$. Clearly

$$T \;\text{ is Fredholm} \Longleftrightarrow T_0 \;\text{ is Fredholm} \Longleftrightarrow T^0 \;\text{ is Fredholm}.$$

To conclude we observe that for any B the operator $T^B - T^0$ is compact because it has finite dimensional range $\subset V$. The equivalence (a) $\Longleftrightarrow$ (c) now follows from Corollary 3.4.9.

Observe that $(T_A)^* = (T^*)^{A^*}$ and since T_A is Fredholm if and only if its adjoint is we see that the equivalence (a) $\Longleftrightarrow$ (b) is a special case of the equivalence (a) $\Longleftrightarrow$ (c). □

We denote by $\boldsymbol{B}(H_0, H_1)$ the vector space of continuous (or equivalently bounded) linear operators $T : H_0 \to H_1$. This is a Banach space with respect to the operator norm

$$\|T\| = \sup_{x \in H_0, \;\; \|x\|=1\|} \|Tx\|, \;\; \forall T \in \boldsymbol{B}(H_0, H_1).$$

The space $\boldsymbol{Fred}(H_0, H_1)$ is a subset of $\boldsymbol{B}(H_0, H_1)$. We have the following important result.

Theorem 3.4.11. *The space $\boldsymbol{Fred}(H_0, H_1)$ is an* open *subset of $\boldsymbol{B}(H_0, H_1)$ and the index function*

$$\text{ind} : \boldsymbol{Fred}(H_0, H_1) \to \mathbb{Z}$$

is continuous.

Proof. We have to prove that for any operator $T_0 \in \boldsymbol{Fred}(H_0, H_1)$ there exists $\varepsilon > 0$ such that if $T \in \boldsymbol{B}(H_0, H_1)$ and $\|T - T_0\| < \varepsilon$ then

$$T \in \boldsymbol{Fred}(H_0, H_1) \tag{3.4.2a}$$

$$\text{ind}\, T = \text{ind}\, T_0. \tag{3.4.2b}$$

Both statements above are consequences of the following fundamental fact whose proof is left to the reader as an exercise.

Lemma 3.4.12. *The set $\boldsymbol{B}_*(H_0, H_1)$ of invertible continuous linear operators $H_0 \to H_1$ is open in $\boldsymbol{B}(H_0, H_1)$.* □

Denote by $P_0 : H_0 \to \ker T_0$ the orthogonal projection onto $\ker T_0$ and by I_0 the natural inclusion $\ker T_0^* \hookrightarrow H_1$. Define

$$\widetilde{H}_0 := H_0 \oplus \ker T_0^*, \;\; \widetilde{H}_1 := H_1 \oplus \ker T_0,$$

and for every $T \in \boldsymbol{B}(H_0, H_1)$ define $\widetilde{T} : \widetilde{H}_0 \to \widetilde{H}_1$ by

$$\widetilde{T} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} T & I_0 \\ P_0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ u \end{bmatrix}, \;\; \forall x \in H_0, \; u \in \ker T_0^*.$$

Observe that

$$\|\widetilde{T}(x \oplus u)\| = \|Tx + u\| + \|P_0 x\| \leq \big(1 + \|T\|\big)\|x \oplus u\|, \;\; \forall x \in H_0, \;\; u \in \ker T_0^*,$$

so that $\widetilde{T} \in \boldsymbol{B}(\widetilde{H}_0, \widetilde{H}_1)$. Note also that for any $S, T \in \boldsymbol{B}(H_0, H_1)$ we have

$$\|\widetilde{T} - \widetilde{S}\| \leq \|T - S\|$$

so that the map

$$\boldsymbol{B}(H_0, H_1) \ni T \mapsto \widetilde{T} \in \boldsymbol{B}(\widetilde{H}_0, \widetilde{H}_1)$$

is continuous. Corollary 3.4.10 implies that T is Fredholm if and only if $\widetilde{T}$ is Fredholm.

Now observe that $\widetilde{T}_0$ is one-to-one and onto so that $\widetilde{T}_0 \in \boldsymbol{B}_*(H_0, H_1)$. Since $\boldsymbol{B}_*(\widetilde{H}_0, \widetilde{H}_1)$ is open in $\boldsymbol{B}(\widetilde{H}_0, \widetilde{H}_1)$ we deduce that if T is sufficiently close to T_0 we have

$$\widetilde{T} \in \boldsymbol{B}_*(\widetilde{H}_0, \widetilde{H}_1) \subset \boldsymbol{Fred}(\widetilde{H}_0, \widetilde{H}_1)$$

so that $T \in \boldsymbol{Fred}(H_0, H_1)$. This proves (3.4.2a).

To prove (3.4.2b) it suffices to show that the map $T \mapsto \operatorname{ind} T$ is lower semicontinuous, i.e.,

$$\operatorname{ind} T \leq \liminf_{T \to T_0} \operatorname{ind} T \tag{3.4.3}$$

Indeed (3.4.3) implies

$$-\operatorname{ind} T_0 = \operatorname{ind} T_0^* \leq \liminf_{T^* \to T_0^*} \operatorname{ind} T^* = -\limsup_{T \to T_0} \operatorname{ind} T$$

so that

$$\limsup_{T \to T_0} \operatorname{ind} T \leq \operatorname{ind} T \leq \liminf_{T \to T_0} \operatorname{ind} T$$

which clearly implies (3.4.2b).

To prove (3.4.3) we will show that if T is sufficiently close to T_0, then there exists an injection

$$\ker T^* \oplus \ker T_0 \to \ker T \oplus \ker T_0^*$$

so that

$$\dim \ker T^* + \dim \ker T_0 \leq \dim \ker T + \dim \ker T_0^* + \operatorname{ind} T_0 \leq \operatorname{ind} T \;\Leftrightarrow\; \operatorname{ind} T_0 \leq \operatorname{ind} T.$$

Let T sufficiently close to T_0 so that $\widetilde{T}$ is invertible. Set

$$V_0 := \ker T_0,\;\; V := \ker T,\;\; U := \ker T^*,\;\; U_0 := \ker T_0^*,\;\; X := V^\perp \subset H_0,\;\; Y := U^\perp \subset H_1.$$

Then $\widetilde{T}$ is a linear continuous bijective map (see Figure 3.2)

$$X \oplus V \oplus U_0 \to Y \oplus U \oplus V_0.$$

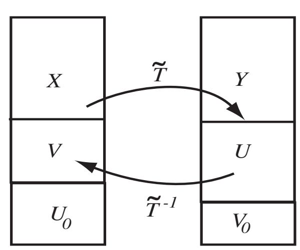

Figure 3.2. *Visualizing $\widetilde{T}$.*

Its inverse defines three continuous linear maps

$$Y \oplus U \oplus V_0 \ni (y, u, v_0) \mapsto x(y, u, v_0) \in X,$$
$$Y \oplus U \oplus V_0 \ni (y, u, v_0) \mapsto v(y, u, v_0) \in V,$$
$$Y \oplus U \oplus V_0 \ni (y, u, v_0) \mapsto u_0(y, u, v_0) \in U_0.$$

We claim that the induced map

$$U \oplus V_0 \ni (u, v_0) \stackrel{L_T}{\longmapsto} \big(v(0, u, v_0),\ u_0(0, u, v_0) \big) \in V \oplus U_0$$

is one-to-one. Suppose $(u, v_0) \in \ker L_T$, i.e.,

$$v(0, u, v_0) = 0, \quad u_0(0, u, v_0) = 0.$$

Set $x = x(0, u, v_0)$. We deduce that

$$\widetilde{T}(x, 0, 0) = (0, u, v_0) \iff Tx = 0, \quad P_0 x = u_0.$$

The induced map $T : X \to Y = \operatorname{ran}(T)$ is bijective so that $x = 0$. Hence

$$x(0, u, v_0) = v(0, u, v_0) = u_0(0, u, v_0) = 0,$$

i.e., $\widetilde{T}^{-1}(0, u, v_0) = 0$. Since $\widetilde{T}^{-1}$ is one-to-one we deduce $u = v_0 = 0$, i.e., $\ker L_T = 0$. □

Corollary 3.4.13. *Suppose $T, S \in \boldsymbol{Fred}(H_0, H_1)$ and $T - S$ is compact. Then* $\operatorname{ind} T = \operatorname{ind} S$.

Proof. Observe that for any $t \in \mathbb{R}$ the operator $A_t = S + t(T - S)$ is Fredholm. Then the map

$$[0, 1] \in t \mapsto \operatorname{ind}(A_t) \in \mathbb{Z}$$

is constant so that $\operatorname{ind} S = \operatorname{ind} A_0 = \operatorname{ind} A_1 = \operatorname{ind} T$. □

Corollary 3.4.14. *If $T_0 \in \boldsymbol{Fred}(H_0, H_1)$, $T_1 \in \boldsymbol{Fred}(H_1, H_2)$ then $T_1 T_0 \in \boldsymbol{Fred}(H_0, H_2)$ and*

$$\operatorname{ind}(T_1 T_0) = \operatorname{ind}(T_1) + \operatorname{ind} T_0. \tag{3.4.4}$$

Proof. Let $Q_1 \in \boldsymbol{B}(H_2, H_1)$ be a quasi-inverse of T_1 and $Q_0 \in \boldsymbol{B}(H_1, H_0)$ be a quasi-inverse of T_0. then

$$Q_0 Q_1 T_1 T_0 = Q_0(\mathbb{1} + \text{compact}) T_0 = \mathbb{1} + \text{compact}.$$

Similarly $T_1 T_0 Q_0 Q_1 = \mathbb{1} + \text{compact}$. Hence $Q_0 Q_1$ is a quasi-inverse of $T_1 T_0$ so that $T_1 T_0$ is Fredholm. To prove the equality (3.4.4) we use the elegant argument in [**H3**, Cor. 19.1.7]. Define

$$A_t = \begin{bmatrix} \mathbb{1}_{H_1} & 0 \\ 0 & T_1 \end{bmatrix} \cdot \begin{bmatrix} (\cos t)\mathbb{1}_{H_1} & (-\sin t)\mathbb{1}_{H_1} \\ (\sin t)\mathbb{1}_{H_1} & (\cos t)\mathbb{1}_{H_1} \end{bmatrix} \cdot \begin{bmatrix} T_0 & 0 \\ 0 & \mathbb{1}_{H_1} \end{bmatrix} \in \boldsymbol{B}(H_0 \oplus H_1, H_1 \oplus H_2).$$

Observe that A_t is Fredholm for any t because the middle operator in the above product is invertible for any t. Moreover,

$$A_0 = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix}, \quad A_{t=\pi/2} = \begin{bmatrix} 0 & -\mathbb{1}_{H_1} \\ T_1 T_0 & 0 \end{bmatrix}$$

and

$$\operatorname{ind} T_0 + \operatorname{ind} T_1 = \operatorname{ind} A_0 = \operatorname{ind} A_{\pi/2} = \operatorname{ind}(T_1 T_0).$$

□

3.5. Elliptic operators on compact manifolds

Throughout this section we fix a smooth, compact connected manifold M of dimension m and a Riemann metric g on M.

Let $\boldsymbol{E}_0, \boldsymbol{E}_1 \to M$ be two smooth, complex vector bundles over M. Fix metrics h_i and compatible connections on $\boldsymbol{E}_i$ so we can define the Hilbert spaces $L^{s,2}(\boldsymbol{E}_i)$.

Theorem 3.5.1. *Suppose $A \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ is an elliptic operator. Then the following hold.*

(a) For any $s \in \mathbb{R}$ the induced continuous linear operator

$$A_s : L^{s+k,2}(\boldsymbol{E}_0) \to L^{s,2}(\boldsymbol{E}_1)$$

is Fredholm and its index is independent of s. We denote this index by $\operatorname{ind} A$.
(b) If $B \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ and $[\sigma_B] = [\sigma_A]$ then B is elliptic and $\operatorname{ind} B = \operatorname{ind} A$.

Proof. (a) Let $Q \in \boldsymbol{\Psi}^{-k}(\boldsymbol{E}_1, \boldsymbol{E}_0)$ be a parametrix of A then the induced operator

$$Q_s : L^{s,2}(\boldsymbol{E}_1) \to L^{s+k,2}(\boldsymbol{E}_0)$$

is a quasi-inverse of A_s. Indeed $QA - \mathbb{1}$ is a smoothing operator, thus has negative order, and invoking Corollary 3.3.6 we conclude that the induced operator

$$Q_sA_s - \mathbb{1} : L^{s+k,2}(\boldsymbol{E}_0) \to L^{s+k,2}(\boldsymbol{E}_0)$$

is compact. In a similar fashion we conclude that $A_sQ_s - \mathbb{1}$ is a compact operator. This proves that A_s is Fredholm.

To prove that $\operatorname{ind} A_s$ is independent of s observe that Corollary 2.9.6 implies that

$$\ker A_s = \big\{ u \in C^\infty(\boldsymbol{E}_0);\;\; Au = 0 \big\} =: \ker A.$$

To show that $\dim \operatorname{coker} A_s$ is independent of s we will prove that it is isomorphic to

$$\ker A^* = \big\{ u \in C^\infty(\boldsymbol{E}_0);\;\; Au = 0 \big\},$$

where A^* denotes the formal[1] adjoint of A. This requires a bit of foundational contortionism.

First let us explain how to extend the Duality Principle (Theorem 1.5.5) to Sobolev spaces of sections of smooth bundles over compact smooth manifolds. Let $\boldsymbol{E}$ be a complex vector bundle over M. Observe that we have a bilinear pairing

$$\langle\langle -, - \rangle\rangle : C^\infty(\boldsymbol{E}^\vee) \times C^\infty(\boldsymbol{E}) \to \mathbb{C}, \;\; (u, v) \mapsto \langle\langle u, v \rangle\rangle := \int_M \langle u, v \rangle_E \, |dV_g| \in \mathbb{C},$$

where

$$\langle -, - \rangle_E : C^\infty(\boldsymbol{E}^\vee) \times C^\infty(\boldsymbol{E}) \to C^\infty(M)$$

is the natural pairing between a bundle and its dual. From the inequality (1.5.2) we deduce that there exists a constant $C > 0$ such that

$$\langle\langle u, v \rangle\rangle \leq C \|u\|_{-s} \|v\|_{s,\boldsymbol{E}_1}, \;\; \forall (u, v) \in C^\infty(\boldsymbol{E}^\vee) \times C^\infty(\boldsymbol{E}). \tag{3.5.1}$$

For any $u \in C^\infty(\boldsymbol{E}^\vee)$ we denote by $\mathcal{L}_{\boldsymbol{E}}(u)$ the linear map

$$\langle\langle u, - \rangle\rangle : C^\infty(\boldsymbol{E}) \to \mathbb{C}.$$

[1] ***Not to be confused*** with the adjoint of the operator A_s acting between the Hilbert spaces $L^{s+k,2}$ and $L^{s,2}$! This confusion seems to appear in a large part of the literature on ψdo's that I have consulted.

The inequality (3.5.1) shows that we have a natural map

$$\mathcal{L}_{\boldsymbol{E}} : C^\infty(\boldsymbol{E}^\vee) \ni u \mapsto \mathcal{L}_{\boldsymbol{E}}(u) \in L^{s,2}(\boldsymbol{E})^\vee, \;\; \langle \mathcal{L}(u), v\rangle = \langle\!\langle u, v\rangle\!\rangle,$$

where $\langle -, -\rangle$ denotes the natural pairing between a Banach space and its dual. Note that

$$\|\mathcal{L}_{\boldsymbol{E}}(u)\| \leq C\|u\|_{-s}, \;\; \forall u.$$

Since $C^\infty(\boldsymbol{E}^\vee)$ is dense in $L^{-s,2}(\boldsymbol{E}^\vee)$ we deduce that $\mathcal{L}$ defines a continuous map

$$\mathcal{L}_{\boldsymbol{E}} : L^{-s,2}(\boldsymbol{E}^\vee) \to L^{s,2}(M, \boldsymbol{E})^\vee.$$

Proposition 3.5.2 (Duality trick)**.** *The continuous map*

$$\mathcal{L} : L^{-s,2}(\boldsymbol{E}^\vee) \to L^{s,2}(\boldsymbol{E})^\vee.$$

is bijective so $L^{-s,2}(\boldsymbol{E}^\vee)$ is isomorphic as a topological vector space to the dual of $L^{s,2}(M, \boldsymbol{E})$. □

The proof is elementary, and reduces via finite partitions of unity to the Duality Principle in Theorem 1.5.5 and (1.5.7).

The operator $A_s : L^{s+k,2}(\boldsymbol{E}_0) \to L^{s,2}(\boldsymbol{E}_1)$ has a dual

$$(A_s)^\vee : L^{s,2}(\boldsymbol{E}_1)^\vee \to L^{s+k,2}(\boldsymbol{E}_0)^\vee.$$

The operator A_s has closed range and the Banach space version of the closed range theorem [**Y**, VII.5] implies that

$$\operatorname{ran}(A_s) = \ker({A_s}^\vee)^\perp := \big\{\, v \in L^{s,2}(\boldsymbol{E}_1); \;\; \langle w, u\rangle = 0; \;\; \forall w \in \ker {A_s}^\vee \,\big\}. \tag{3.5.2}$$

This proves that

$$\operatorname{coker} A_s \cong \ker(A_s)^\vee.$$

Consider the dual ψdo $A^\vee \in \boldsymbol{\Psi}^k({\boldsymbol{E}_1}^\vee, {\boldsymbol{E}_0}^\vee)$ defined by

$$\langle\langle A^\vee u, v\rangle\rangle = \langle\langle u, Av\rangle\rangle, \;\; \forall u, v \;\text{ smooth.}$$

Let us observe that we have a commutative diagram

$$\begin{array}{ccc}
L^{-s,2}({\boldsymbol{E}_1}^\vee) & \xrightarrow{(A^\vee)_{-s-k}} & L^{-s-k,2}({\boldsymbol{E}_0}^\vee) \\
\Big\downarrow \mathcal{L}_{\boldsymbol{E}_1} & & \Big\downarrow \mathcal{L}_{\boldsymbol{E}_0} \\
L^{s,2}(\boldsymbol{E}_1)^\vee & \xrightarrow[(A_s)^\vee]{} & L^{s+k,2}(\boldsymbol{E}_0)^\vee
\end{array}$$

Indeed, for $u \in C^\infty({\boldsymbol{E}_1}^\vee)$ and $v \in C^\infty(\boldsymbol{E}_1)$ we have

$$\big\langle\, \mathcal{L}_{\boldsymbol{E}_0}\big(\,(A^\vee)_{-s-k}\,u\,\big), v\,\big\rangle = \langle\!\langle A^\vee u, v\rangle\!\rangle = \langle\!\langle u, Av\rangle\!\rangle$$
$$= \langle\!\langle u, A_s v\rangle\!\rangle = \langle \mathcal{L}_{\boldsymbol{E}_1}(u), A_s v\rangle = \langle\, (A_s)^\vee \mathcal{L}_{\boldsymbol{E}_1}(u), v\,\rangle.$$

Since the spaces of smooth sections are dense in all Sobolev spaces we deduce that the above equality holds for all $u \in L^{-s,2}(\boldsymbol{E}_1)$ and $v \in L^{s,2}(\boldsymbol{E}_0)$ thus establishing the commutativity of the above diagram.

Proposition 3.5.2 shows that the maps $\mathcal{L}_{\boldsymbol{E}_i}$ are bijective which implies that

$$\ker(A_s)^\vee \cong \ker(A^\vee)_s.$$

Since $A^{\vee}$ is also elliptic we deduce from Corollary 2.9.6 that

$$\ker(A^{\vee})_s \subset C^{\infty}, \text{ i.e., } \ker(A^{\vee})_s = \{ u \in C^{\infty}(\boldsymbol{E}_1{}^{\vee}); \; A^{\vee}u = 0 \}.$$

Since $A^{\vee}$ is conjugate to the formal adjoint A^* via the conjugate linear isomorphism $I_h : \boldsymbol{E}^{\vee} \to \boldsymbol{E}$ induced by the metric h we deduce that

$$I_h\big(\ker A^{\vee}\big) = \ker A^*. \tag{3.5.3}$$

In any case this shows that $\dim \operatorname{coker}(A_s)$ is independent of s. This proves (a).

To prove (b) consider a ψdo $B \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ that has the same principal symbol as A. Then B is elliptic and

$$B - A \in \boldsymbol{\Psi}^{k-1}(\boldsymbol{E}_0, \boldsymbol{E}_0).$$

Thus $B - A$ induces a continuous operator $L^{s+k,2} \to L^{s+1,2}$ and since the embedding $L^{s+1,2} \to L^{s,2}$ is compact we deduce that the operator $B_s - A_s : L^{s+k,2} \to L^{s,2}$ is compact. This shows that

$$\operatorname{ind} A_s = \operatorname{ind} B_s, \;\; \forall s.$$

□

Let us mention a useful consequence of the above proof.

Corollary 3.5.3. *Suppose $A \in \boldsymbol{\Psi}^k(\boldsymbol{E}_0, \boldsymbol{E}_1)$ is an elliptic operator. Set*

$$\operatorname{ran}_{L^2} A := \operatorname{ran}\big(A : L^{k,2}(\boldsymbol{E}_0) \to L^2(\boldsymbol{E}_1) \big).$$

Then $\operatorname{ran}_{L^2} A$ coincides with the orthogonal complement in $L^2(\boldsymbol{E}_1)$ of the kernel of A^*.

Proof. This follows from the following key observation. If $\mathcal{I}_h : \boldsymbol{E}^{\vee} \to \boldsymbol{E}$ is the natural conjugate linear isomorphism defined by a hermitian metric on the vector bundle $\boldsymbol{E}$ and $\mathcal{R} : L^2(\boldsymbol{E})^{\vee} \to L^2(\boldsymbol{E})$ is the conjugate linear isomorphism induced by the Riesz representation theorem then the composition

$$L^2(\boldsymbol{E}) \xrightarrow{\mathcal{I}_h^{-1}} L^2(\boldsymbol{E}^{\vee}) \xrightarrow{\mathcal{L}_{\boldsymbol{E}}} L^2(\boldsymbol{E})^{\vee} \xrightarrow{\mathcal{R}} L^2(\boldsymbol{E})$$

is the identity map. □

If (M, g) is a Riemann manifold, we denote by $S_g(T^*M)$ the unit sphere bundle

$$S_g(T^*M); = \{ (x, \xi); \; x \in M, \;\; \xi \in T^*_x M, \;\; |\xi|_g = 1 \}.$$

Observe that if $\sigma_0, \sigma_1 \in \mathcal{H}^k(M, \boldsymbol{E}_0, \boldsymbol{E}_1)$ then

$$\sigma_0 = \sigma_1 \Longleftrightarrow \sigma_0|_{S_g(T^*M)} = \sigma_1|_{S_g(T^*M)}. \tag{3.5.4}$$

We have the following generalization of Theorem 3.5.1(b)

Proposition 3.5.4. *Consider two elliptic operators $A_0 \in \boldsymbol{\Psi}^{k_0}(\boldsymbol{E}_0, \boldsymbol{E}_1)$, $A_1 \in \boldsymbol{\Psi}^{k_1}(\boldsymbol{E}_0, \boldsymbol{E}_1)$. If*

$$[\sigma_{A_0}]|_{S_g(T^*M)} = [\sigma_{A_1}]|_{S_g(T^*M)}$$

then $\operatorname{ind} A_0 = \operatorname{ind} A_1$.

Proof. If A_0 and A_1 have the same orders, $k_0 = k_1$, then the conclusion follows from (3.5.4) and Theorem 3.5.1(b). Assume $r = k_1 - k_0 > 0$. Using Corollary 3.2.5 we deduce that there exists

$$S \in \boldsymbol{\Psi}^r(\boldsymbol{E}_1) \text{ such that } [\sigma_S] = |\xi|^r \cdot \mathbb{1}_{\boldsymbol{E}_1}.$$

Set

$$\Lambda_r := \frac{1}{2}(S + S^*) \in \boldsymbol{\Psi}^r(\boldsymbol{E}_1).$$

Then $\Lambda_r = \Lambda_r^*$ and $[\sigma_{\Lambda_r}](x,\xi) = |\xi|^r \cdot \mathbb{1}_{\boldsymbol{E}_1}$. Thus, Λ_r is elliptic and

$$\operatorname{ind}\Lambda_r = \dim\ker\Lambda_r - \dim\ker\Lambda_r^* = 0.$$

Set $B_1 := \Lambda_r \circ A_0 \in \boldsymbol{\Psi}^{k_1}(\boldsymbol{E}_0, \boldsymbol{E}_1)$. Then A_1 and B_1 have the same order and $[\sigma_{A_1}]|_{S_g(T^*M)} = [\sigma_{B_1}|_{S_g(T^*M)}$. Hence

$$\operatorname{ind} A_1 = \operatorname{ind} B_1 = \operatorname{ind}\Lambda_r + \operatorname{ind} A_0 = \operatorname{ind} A_0.$$

□

We denote by $\Psi\mathrm{Ell}(\boldsymbol{E}_0, \boldsymbol{E}_1)$ the space of elliptic pseudodifferential operators $C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$. We denote by $\widetilde{\boldsymbol{E}}_i$, the pullbacks to $S_g(T^*M)$ of the vector bundles $\boldsymbol{E}_i$, $i = 0, 1$, and we denote by $\operatorname{Iso}(\widetilde{\boldsymbol{E}}_0, \widetilde{\boldsymbol{E}}_1)$ the space of bundle isomorphisms $\boldsymbol{E}_0 \to \widetilde{\boldsymbol{E}}_1$. The principal symbol map induces a surjection

$$[\sigma] : \Psi\mathrm{Ell}(\boldsymbol{E}_0, \boldsymbol{E}_1) \to \operatorname{Iso}(\widetilde{\boldsymbol{E}}_0, \widetilde{\boldsymbol{E}}_1), \quad A \mapsto [\sigma_A]|_{S_g(T^*M)},$$

while Proposition 3.5.4 implies that there exists a map $\operatorname{ind}_a : \operatorname{Iso}(\widetilde{\boldsymbol{E}}_0, \widetilde{\boldsymbol{E}}_1) \to \mathbb{Z}$ such that the diagram below is commutative

$$\begin{array}{ccc} \Psi\mathrm{Ell}(\boldsymbol{E}_0, \boldsymbol{E}_1) & \xrightarrow{\operatorname{ind}} & \mathbb{Z} \\ \Big\downarrow{\scriptstyle [\sigma]} & \nearrow{\scriptstyle \operatorname{ind}_a} & \\ \operatorname{Iso}(\widetilde{\boldsymbol{E}}_0, \widetilde{\boldsymbol{E}}_1) & & \end{array}$$

The map ind_a is called the *analytic index* and one can prove that it is continuous with respect to a natural topology on $\operatorname{Iso}(\widetilde{\boldsymbol{E}}_0, \widetilde{\boldsymbol{E}}_1)$.

Suppose $(\widehat{\mathcal{D}}, \Gamma)$ is a super-symmetric Dirac-type operator on a Hermitian vector bundle $(\widehat{\boldsymbol{E}}, h)$ over the compact, m-dimensional, Riemann manifold M. As explained on page 88, the chiral operator Γ induces an orthogonal bundle decomposition $\widehat{\boldsymbol{E}} = \boldsymbol{E}_+ \oplus \boldsymbol{E}_-$ and Dirac-type operators

$$\mathcal{D}_\pm : C^\infty(\boldsymbol{E}_\pm) \to C^\infty(\boldsymbol{E}_\mp) \text{ such that } \mathcal{D}_- = \mathcal{D}_+^*.$$

The operator $\mathcal{D}_+$ is elliptic and its index is called the *index of the super-symmetric Dirac-type operator* $\widehat{\mathcal{D}}$.

3.6. Spectral decomposition of elliptic selfadjoint partial differential operators on compact manifolds

Throughout this section we fix a smooth, compact Riemann manifold (M, g) of dimension m and a complex vector bundle $\boldsymbol{E} \to M$ of rank r equipped with a Hermitian metric h.

Let $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ be an elliptic partial differential operator of order k. We assume that A is formally self-adjoint, i.e., $A = A^*$. The operator defines an unbounded[2] operator

$$\widetilde{A} : \mathrm{Dom}(\widetilde{A}) \subset L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E}),$$

with domain $\mathrm{Dom}(\widetilde{A}) = L^{k,2}(\boldsymbol{E})$, defined by

$$L^{k,2}(\boldsymbol{E}) \ni u \mapsto Au \in L^2(\boldsymbol{E}).$$

We will refer to $\widetilde{A}$ as the *analytic realization* of A.

Proposition 3.6.1. *The operator $\widetilde{A}$ is closed, and selfadjoint, i.e., the following hold.*

(a) *The graph $\widetilde{A}$ is closed in $L^2(\boldsymbol{E}) \times L^2(\boldsymbol{E})$.*

(b) *For any $u, v \in L^{k,2}(\boldsymbol{E})$ we have $(Au, v)_{L^2} = (u, Av)_{L^2}$.*

(c) *If $v \in L^2(\boldsymbol{E})$ and there exists $C > 0$ such that*

$$|(Au, v)_{L^2}| \leq C\|u\|_{L^2}, \;\; \forall u \in L^{k,2}(\boldsymbol{E}),$$

then $v \in \mathrm{Dom}(\widetilde{A}) = L^{k,2}(\boldsymbol{E})$.

Proof. Part (b) follows from the equality $A = A^*$. To prove (a) we need to show that if $(u_n)_{n\geq 0}$ is a sequence in $L^{k,2}(\boldsymbol{E})$ such that there exist $(u, v) \in L^2(\boldsymbol{E}) \times L^2(\boldsymbol{E})$ so that

$$\lim_{n\to\infty} \big(\|u_n - u\|_{L^2} + \|Au_n - v\|_{L^2} \big) = 0$$

then $u \in L^{k,2}(\boldsymbol{E})$ and $v = Au$. This follows from the elliptic estimates. Indeed, there exists a constant $C > 0$ such that for any $n, n, \geq 0$ we have

$$\|u_n - u_{n'}\|_{L^{k,2}} \leq C\big(\|Au_n - Au_{n'}\|_{L^2} + \|u_n - u_{n'}\|_{L^2} \big).$$

Since the sequences (u_n) and (Au_n) are Cauchy in the L^2-norm we deduce from the above inequality that the sequence (u_n) is also Cauchy in the $L^{k,2}$-norm. This implies that $u_n \to u$ in $L^{k,2}$ and thus $Au_n \to Au = v$ in $L^{k,2}$.

Part (c) follows from elliptic regularity. Denote by I_h the conjugate linear bundle isometry $I_h : \boldsymbol{E} \to \boldsymbol{E}^\vee$. For any $u \in L^{k,2}(\boldsymbol{E})$ we have

$$(Au, v)_{L^2} = \langle\!\langle Au, I_h v\rangle\!\rangle = \langle\!\langle u, A^\vee I_h v\rangle\!\rangle.$$

Set $w := A^\vee I_h v$. A priori, all that we know is that $w \in L^2(\boldsymbol{E}^\vee)$ so that $w \in L^{-k,2}(\boldsymbol{E}^\vee)$. On the other hand we know that

$$|\langle\!\langle u, w\rangle\!\rangle| \leq C\|u\|_{L^2}, \;\; \forall u \in L^{k,2}(\boldsymbol{E}).$$

Hence, the linear map

$$L^{k,2}(\boldsymbol{E}) \ni u \mapsto \langle\!\langle u, w\rangle\!\rangle \in \mathbb{C}$$

is continuous with respect to the L^2-topology. Since $L^{k,2}(\boldsymbol{E})$ is dense in $L^2(\boldsymbol{E})$ we deduce that the above linear functional extends to a continuous linear functional $L^2(\boldsymbol{E}) \to \mathbb{C}$. From the Riesz representation theorem this implies that $I_h^{-1}w = I_h^{-1}A^\vee I_h v = A^* v \in L^2(\boldsymbol{E})$. Since A^* is elliptic, we deduce that $A^* v \in L^{k,2}(\boldsymbol{E})$.

□

[2] For basic facts about unbounded operators we refer to [**Br**, II.6], [**K**, Chap 3, §5], [**ReSi**, VIII].

Definition 3.6.2. Suppose $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ is an elliptic, partial differential operator of order k.

(a) The *resolvent set* of A is the subset $\rho(A) \subset \mathbb{C}$ consisting of complex numbers λ such that the induced operator $\lambda - A : L^{k,2}(\boldsymbol{E}) \to L^2(\boldsymbol{E})$ is bijective. The *spectrum* of A is the subset

$$\operatorname{spec}(A) := \mathbb{C} \setminus \rho(A) \subset \mathbb{C}.$$

(b) The complex number λ is said to be an *eigenvalue* of A if the operator $(\lambda - A) : L^{k,2}(\boldsymbol{E}) \to L^2(\boldsymbol{E})$ has nontrivial kernel. The sections in this kernel are called *eigensections* of A corresponding to the eigenvalue λ. We denote by $\operatorname{spec}_e(A)$ the collection of all the eigenvalues of A. □

Observe two things. First, the resolvent set of A is open so that $\operatorname{spec}(A)$ is a closed subset of $\mathbb{C}$. Second, for any $\lambda \in \mathbb{C}$ the operator $\lambda - A$ is also elliptic so that $\ker(\lambda - A) \subset C^\infty(\boldsymbol{E})$ and $\dim \ker(\lambda - A) < \infty$. This dimension is called the multiplicity of λ with respect to A. Observe that λ is an eigenvalue of A if and only if its multiplicity with respect to A is positive.

Theorem 3.6.3 (Spectral decomposition)**.** *Suppose (M, g) is a smooth, compact Riemann manifold of dimension m and $(\boldsymbol{E}, h)$ is a smooth complex vector bundle of rank r over M equipped with a Hermitian metric. Let $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ be a formally self-adjoint elliptic pdo of order k. Then the following hold.*

(a) *The spectrum of A is real, i.e.,* $\operatorname{spec}(A) \subset \mathbb{R}$

(b) *The spectrum of A is a* discrete *subset of* $\mathbb{R}$ *consisting only of eigenvalues, i.e.,*

$$\operatorname{spec}(A) = \operatorname{spec}_e(A).$$

(c) *There exists a Hilbert basis $(\boldsymbol{\phi}_n)_{n\in\mathbb{Z}}$ of $L^2(\boldsymbol{E})$ consisting of eigensections $\boldsymbol{\phi}_n \in C^\infty(\boldsymbol{E})$ of A. If λ_n is the eigenvalue corresponding to $\boldsymbol{\phi}_n$ then*

$$\operatorname{spec}(A) = \{\lambda_n;\ \ n \in \mathbb{Z}\}.$$

We will refer to such a basis as a spectral basis *of $L^2(\boldsymbol{E})$ relative to the operator A.*

(d) *If $u \in L^2(\boldsymbol{E})$ decomposes along a spectral basis $(\boldsymbol{\phi}_n)_{n\in\mathbb{Z}}$ as a series*

$$u = \sum_{n\in\mathbb{Z}} u_n \boldsymbol{\phi}_n,\ \ u_n \in \mathbb{C},\ \ \sum_n |u_n|^2 < \infty,$$

then $u \in L^{k,2}(\boldsymbol{E})$ if and only if

$$\sum_n |\lambda_n u_n|^2 < \infty.$$

In this case Au has the decomposition

$$Au = \sum_n \lambda_n u_n \boldsymbol{\phi}_n.$$

Proof. To prove (a) it suffices to show that for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the induced operator $\lambda - A : L^{k,2}(\boldsymbol{E}) \to L^2(\boldsymbol{E})$ is bijective. Observe that $\lambda - A$ is an elliptic operator that has the same symbol as A so that

$$\operatorname{ind}(\lambda - A) = \operatorname{ind} A = 0,$$

where the last equality is due to the fact that $A = A^*$. Thus

$$\lambda \in \rho(A) \Longleftrightarrow \ker(\lambda - A) = 0. \tag{3.6.1}$$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $u \in \ker(\lambda - A)$ then we have

$$Au = \lambda u \Rightarrow (u, \lambda u)_{L^2} = (u, Au)_{L^2} = (Au, u)_{L^2} = (\lambda u, u)_{L^2}.$$

Hence $\bar{\lambda}\|u\|_{L^2}^2 = \lambda\|u\|_{L^2}^2$ and since λ is not real we deduce $u = 0$. This proves (a).

To prove (b) let us first observe that (3.6.1) implies that the spectrum of A consists only of eigenvalues. Let us show that $\operatorname{spec}(A)$ is a discrete subset of A. Fix $\lambda_0 \in \operatorname{spec}(A)$. We need to prove that there exists $\varepsilon > 0$ such that if $0 < |\lambda - \lambda_0| < \varepsilon$, then $\lambda \in \rho(A)$, i.e., $\ker(\lambda - A) = 0$.

We argue by contradiction. Suppose that there exists $\lambda_n \to \lambda_0$, $\lambda_n \neq \lambda$ such that $\ker(\lambda_n - A) \neq 0$. Choose $u_n \in \ker(\lambda_n - A)$ such that $\|u_n\|_{L^2} = 1$. Observe first that

$$(\lambda A)u_n = (\lambda - \lambda_n)u_n$$

which implies that

$$u_n \in \operatorname{ran}\big(L^{k,2}(\boldsymbol{E}) \stackrel{\lambda - A}{\longrightarrow} L^2(\boldsymbol{E}) \big).$$

Since $(\lambda - A)^* = \lambda - A$ we deduce from Corollary 3.5.3 that

$$(u_n, v)_{L^2} = 0, \;\; \forall v \in \ker(\lambda - A), \;\; \forall n. \tag{3.6.2}$$

From the elliptic estimates we deduce that there exists $C > 0$ such that

$$\|u_n\|_{L^{k,2}} \leq C\big(\|Au_n\|_{L^2} + \|u_n\|_{L^2} \big) = C(|\lambda_n| + 1).$$

This proves that the sequence (u_n) is bounded in $L^{k,2}(\boldsymbol{E})$. Using the fact that the inclusion $L^{k,2}(\boldsymbol{E}) \hookrightarrow L^2(\boldsymbol{E})$ is compact we conclude that a subsequence of (u_n) converges in the norm L^2. Let (u_{n_j}) be this subsequence, and denote by u its L^2 limit. Note that $\|u\|_{L^2} = 1$.

Using the elliptic estimates again we deduce that

$$\|u_{n_i} - u_{n_j}\|_{L^{k,2}} \leq C\big(\|Au_{n_i} - Au_{n_j}\|_{L^2} + \|u_{n_i} - u_{n_j}\|_{L^2} \big)$$
$$= C\big(\|\lambda_{n_i}u_{n_i} - \lambda_{n_j}u_{n_j}\|_{L^2} + \|u_{n_i} - u_{n_j}\|_{L^2} \big).$$

The sequences (u_{n_i}) and $(\lambda_{n_i}u_{n_i})$ are Cauchy in the L^2 norm and so we conclude from the above inequality that the sequence (u_{n_i}) is convergent in the $L^{k,2}$ norm to u. Passing to limit in the equality $Au_{n_i} = \lambda_{n_i}u_{n_i}$ we deduce that $Au = \lambda u$. Hence

$$u \in \ker(\lambda - A) \;\text{ and }\; \|u\|_{L^2} = 1.$$

Finally, using (3.6.2) we deduce $(u_{n_i}, u)_{L^2} = 0, \forall i$. Passing to limit in the last equality we reach the contradiction $1 = \|u\|_{L^2}^2 = 0$.

To prove (c) observe first that since $\operatorname{spec}(A)$ is a discrete subset of $\mathbb{R}$ there exists $c_0 \in \rho(A) \cap \mathbb{R}$. We deduce that $c_0 - A : L^{k,2}(\boldsymbol{E}) \to L^2(\boldsymbol{E})$ is continuous and bijective. By the open mapping theorem, its inverse $(c_0 - A)^{-1} : L^2(\boldsymbol{E}) \to L^{k,2}(\boldsymbol{E})$ is continuous. The resulting operator

$$R(c_0, A) : L^2(\boldsymbol{E}) \stackrel{(\lambda_0 - A)^{-1}}{\longrightarrow} L^{k,2}(\boldsymbol{E}) \hookrightarrow L^2(\boldsymbol{E})$$

is compact since the inclusion $L^{k,2}(\boldsymbol{E}) \hookrightarrow L^2(\boldsymbol{E})$ is compact. Since $A = A^*$ we deduce that $R(c_0, A)$ is also self-adjoint as a bounded operator $L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E})$.

Invoking the spectral theorem for compact selfadjoint operators on Hilbert spaces ([**Br**, VI.4], [**K**, V.3]) we deduce that there exists a Hilbert basis $(\boldsymbol{\phi}_n)$ consisting of eigen-sections of $R(c_0, A)$. The spectrum of $R(c_0, A)$ has a unique accumulation point, the origin, and any nonzero number in the spectrum of $R(c_0, A)$ is an eigenvalue with finite multiplicity. Moreover we have an orthogonal decomposition

$$L^2(\boldsymbol{E}) = \bigoplus_{\mu} \ker\big(\mu - R(c_0, A)\big).$$

If μ is an eigenvalue of $R(c_0, A)$ then $\mu \neq 0$ since $R(c_0, A)$ is injective. Moreover, if $\boldsymbol{\phi}$ is an eigenvector of $R(c_0, A)$ corresponding to μ then

$$R(c_0, A)\boldsymbol{\phi} = \mu\boldsymbol{\phi} \Longleftrightarrow \phi = \mu(c_0 - A)\boldsymbol{\phi} \Longleftrightarrow A\boldsymbol{\phi} = (c_0 - \mu^{-1})\boldsymbol{\phi}.$$

This proves (c).

To prove (d) fix a spectral basis $(\boldsymbol{\phi}_n)$ of $L^2(\boldsymbol{E})$ and denote by λ_n the eigenvalue corresponding to $\boldsymbol{\phi}_n$. Fix $c_0 \in \rho(A) \cap \mathbb{R}$ and for every $\lambda \in \mathbb{R}$ set

$$\mu(\lambda) = \frac{1}{c_0 - \lambda}.$$

so that λ is an eigenvalue of A if and only if $\mu(\lambda)$ is an eigenvalue of $R(c_0, A)$.

Let $u \in L^{k,2}(\boldsymbol{E})$,

$$u = \sum_n u_n \boldsymbol{\phi}_n, \;\; u_n \in \mathbb{C}, \;\; \sum_n |u_n|^2 < \infty$$

Set $v = Au$ so that $c_0 u - v = (c_0 - A)u$. We can write

$$v = \sum_n v_n \boldsymbol{\phi}_n, \;\; v_n \in \mathbb{C}, \;\; \sum_n |v_n|^2 < \infty.$$

Note that

$$c_0 u - v = (c_0 - A)u \Longleftrightarrow u = R(c_0, A)(c_0 u - v) = \sum_n \mu(\lambda_n)(c_0 u_n - v_n)\boldsymbol{\phi}_n,$$

and we deduce

$$(c_0 - \lambda_n)u_n = (c_0 u_n - v_n), \;\; \text{i.e.,} \;\; \lambda_n u_n = v_n, \;\; \forall n.$$

This implies that

$$\sum_n |\lambda_n u_n|^2 < \infty.$$

Conversely, let

$$u = \sum_n x_n \boldsymbol{\phi}_n \in L^2(\boldsymbol{E}) \;\; \text{such that} \;\; \sum_n |\lambda_n x_n|^2 < \infty.$$

We want to prove that $u \in L^{k,2}(\boldsymbol{E})$. Define

$$v := \sum_n \lambda_n x_n \boldsymbol{\phi}_n \in L^2(\boldsymbol{E}).$$

For any positive integer ν we set

$$u_\nu := \sum_{|n| \leq \nu} x_n \boldsymbol{\phi}_n, \;\; v_\nu := \sum_{|n| \leq \nu} \lambda_n x_n \boldsymbol{\phi}_n.$$

Then $Au_\nu = v_\nu$ and

$$\lim_{\nu \to \infty} \big(\|u_\nu - u\|_{L^2} + \|v_\nu - v\|_{L^2} \big) = 0.$$

Invoking Proposition 3.6.1(a) we deduce $u \in L^{k,2}(\boldsymbol{E})$ and $v = Au$. □

Example 3.6.4. Let us consider a simple example when M is the unit circle and $\boldsymbol{E}$ is the trivial complex line bundle. The operator

$$A = -\boldsymbol{i}\frac{d}{d\theta}C^{\infty}(S^1) \to C^{\infty}(S^1),$$

is elliptic and self-adjoint and its spectrum is

$$\operatorname{spec}(A) = \mathbb{Z}, \;\; \ker(n - A)\operatorname{span}\left(e_n(\theta) = e^{\boldsymbol{i}n\theta}\right).$$

The collection

$$\boldsymbol{\phi}_n(\theta) = \frac{1}{(2\pi)^{1/2}}e_n(\theta), \;\; n \in \mathbb{Z}$$

is a spectral basis relative to A. The decomposition of a function $u \in L^2(S^1)$ determined by this basis is note other than the Fourier decomposition of u,

$$u = \sum_{n\in\mathbb{Z}} \widehat{u}(n)e_n(\theta), \;\; \widehat{u}(n) := \frac{1}{(2\pi)^{1/2}}\int_0^{2\pi} u(\theta)e^{-\boldsymbol{i}n\theta}\,d\theta.$$

Observe that $u \in L^{1,2}(S^1)$ if and only if

$$\sum_{n\in\mathbb{Z}} |n\widehat{u}(n)|^2 < \infty. \qquad \square$$

3.7. Hodge theory

Recall that a (cochain) complex of vector spaces is a sequence $(E^n, d_n)_{n\geq 0}$ of complex vector spaces E_n and linear operators $d_n : E^n \to E^{n+1}$ such that

$$d_{n+1}d_n = 0, \;\; \forall n \geq 0. \tag{3.7.1}$$

The complex is said to have finite length if $E^n = 0$ for all $n \gg 0$. Note that (3.7.1) implies that for any $n \geq 0$ we have

$$\operatorname{ran}(d_{n-1}) \subset \ker d_n,$$

where we set $d_{-1} := 0$. The elements in $Z^n(E^\bullet) := \ker d_n$ are called *cocycles* (of degree n) while the elements in $B^n(E_\bullet) := \operatorname{ran}(d_{n-1})$ are called *coboundaries* (of degree n. ,

The *cohomology* of a complex $(E_\bullet, d) = (E^n, d_n)$ is the vector space

$$H^\bullet(E_\bullet, d) := \bigoplus_{n\geq} H^n(E_\bullet, d), \;\; H^n(E_\bullet, d) := \frac{\ker d_n}{\operatorname{ran} d_{n-1}}, \;\; \forall n \geq 0,$$

The complex is called *acyclic* if $H^n(E_\bullet) = 0$, for all $n \geq 0$.

We declare two cocycle $z_0, z_1 \in Z^n(E_\bullet)$ *cohomologous*, and we write this $z_0 \sim z_1$, if there exists $u \in E^{n-1}$ such that

$$z_0 - z_1 = du,$$

We see that the binary relation "$\sim$" is an equivalence relation on the space of cycles and $H^n(E_\bullet)$ can be identified with the space of cohomology classes of cocycles of degree n. For a cocycle z we denote by $[z]$ its cohomology class.

Example 3.7.1 (Baby Hodge theory)**.** We want to discuss a special, finite dimensional case of Hodge theory for two reasons. First, we get to see the main ideas in the proof, unencumbered by technicalities. The second reason is that we need the finite dimensional version to establish some important technical facts about elliptic complexes.

Suppose $(E_\bullet, d) = (E^n, d_n)_{n\geq 0}$ is a cochain complex of *finite dimensional* complex vector spaces and

$$E^n = 0, \;\; \forall n > N.$$

Fix an a Hermitian inner product h_n on each E_n. We can now define adjoints $d_n^* : E^n \to E^{n-1}$. Set

$$E^\bullet := \bigoplus_{n=0}^{N} E^n.$$

The operators d_n and the metrics h_n define operators

$$d = \oplus_n d_n : E^\bullet \to E^\bullet,$$

and a metric $h = \oplus_n h_n$ on $E^\bullet$. The adjoint of d is the operator $\oplus_n d_n^*$. The condition (3.7.1) can be rewritten simply as $d^2 = 0$. Define

$$\boldsymbol{H}^n(E^\bullet, h) := \big\{\, u \in E^n;\;\; d_n u = d_{n-1}^* u = 0 \,\big\}, \;\; \boldsymbol{H}^\bullet(E^\bullet, h) := \bigoplus_n \boldsymbol{H}^n(E^\bullet, h).$$

The elements in $\boldsymbol{H}^n(E^\bullet, h)$ are called *harmonic* (with respect to the metric h). We have a natural map

$$\boldsymbol{H}^n(E^\bullet, h) \to H^n(E^\bullet), \;\; u \mapsto [u] \tag{3.7.2}$$

which associates to each harmonic element its cohomology class. Hodge theorem states that this map is an isomorphism of vector spaces. This is a consequence of the *Hodge decomposition theorem* which states that the subspaces $\boldsymbol{H}^n(E^\bullet, h)$, $\operatorname{ran}(d_{n-1})$, $\operatorname{ran}(d_{n+1}^*)$ of E^n are mutually orthogonal and we have a direct sum decomposition

$$E^n = \boldsymbol{H}^n(E^\bullet, h) \oplus \operatorname{ran}(d_{n-1}) \oplus \operatorname{ran}(d_{n+1}^*). \tag{3.7.3}$$

Let us verify the orthogonality statement. Denote by $(-,-)$ the hermitian inner product h. Let $u \in E^n = \boldsymbol{H}^n(E^\bullet, h)$, $y_0 \in \operatorname{ran}(d_{n-1})$ and $y_1 \in \operatorname{ran}(d_{n+1}^*)$. Then there exist $x_0 \in E^{n-1}$ and $x_1 \in E^{n+1}$ such that

$$y_0 = dx_0, \;\; y_1 = dx_1.$$

Then

$$(u, y_0) = (u, dx_0) = (d^* u, x_0) = 0, \;\; (u, y_1) = (u, d^* x_1) = (du, x_1) = 0,$$
$$(y_0, y_1) = (dx_0, d^* x_1) = (d^2 x_0, x_1) = 0.$$

To prove the decomposition (3.7.3) we consider the selfadjoint operator $d + d^* : E^\bullet \to E^\bullet$. Note first that

$$(d + d^*)x = 0 \Longleftrightarrow dx = x^* x = 0. \tag{3.7.4}$$

Indeed, we have

$$0 = (dx + d^* x, dx) = |dx|^2 + ((d^*)^2 x, x) = |dx|^2,$$

and similarly,

$$0 = (dx + d^* x, d^* x) = |d^* x|^2 + (d^2 x, x) = |d^* x|^2.$$

Then

$$E^\bullet = \operatorname{ran}(d + d^*) \oplus \operatorname{ran}(d + d^*)^\perp$$
$$\stackrel{(3.7.4)}{=} \operatorname{ran}(d + d^*) \oplus \ker(d + d^*) = \operatorname{ran}(d + d^*) \oplus \boldsymbol{H}^\bullet(E, h).$$

In the above string of equalities the key role is played by the equality

$$\operatorname{ran}(d+d^*)^{\perp} = \ker(d+d^*)$$

which in the finite dimensional context follows by elementary methods, while in the infinite dimensional context is a consequence of the highly nontrivial closed range theorem.

It is now easy to prove that the map (3.7.2) is an isomorphism. Indeed if z is a harmonic element cohomologous to 0 then

$$z \in \boldsymbol{H}^{\bullet}(E^{\bullet}, h) \cap \operatorname{ran}(d) = \{0\}.$$

This proves that (3.7.2) is injective. To prove the surjectivity, consider a cohomology class c and a cocyle z such that $[z] = c$. Using the Hodge decomposition we can write

$$z = z_0 + du + d^*v, \quad z_0 \in \boldsymbol{H}^{\bullet}(E^{\bullet}, h).$$

From the equality $dz = 0$ we conclude $dd^*v = 0$ so that

$$0 =)dd^*v, v) = |d^*v|^2.$$

Thus $z = z_0 + du$ so that $[z_0] = [z] = c$.

The operator $\Delta_h := (d+d^*)^2$ is called the *Laplacian* of the complex determined by the metric h. From the conditions $d^2 = (d^*)^2 = 0$ we deduce that

$$\Delta_h = (d+d^*)^2 = dd^* + d^*d.$$

Observe that

$$\boldsymbol{H}^{\bullet}(E^{\bullet}, h) = \ker(d+d^*) = \ker \Delta_h.$$

The first equality follows from (3.7.4). The inclusion

$$\ker(d+d^*) \subset \ker(d+d^*)^2 = \ker \Delta_h$$

is obvious. To prove the opposite inclusion let $u \in \ker \Delta_h$. Then

$$0 = (\Delta_h u, u) = (dd^*u, u) + (d^*du, u) = |d^*u|^2 + |du|^2.$$

Let us observe a simple consequence of the above facts. More precisely, we see that

$$\text{the complex } (E^{\bullet}, d) \text{ is acyclic} \Longleftrightarrow d + d^* : E^{\bullet} \to E^{\bullet} \text{ is a linear isomorphism.} \tag{3.7.5}$$

□

Definition 3.7.2. A *complex* of ψdo's on a smooth manifold M is a finite sequence of smooth complex vector bundles $(\boldsymbol{E}_k)_{0 \leq k \leq N}$ over M and first order[3] properly supported ψdo's

$$A_k \in \boldsymbol{\Psi}_0^1(\boldsymbol{E}_k, \boldsymbol{E}_{k+1}), \quad 0 \leq k \leq N-1,$$

such that the following hold $A_k \circ A_{k-1} = 0, \forall 1 \leq k \leq N-1$.

The complex is called *elliptic* if for any $x \in M$ and any $\xi \in T_x^*M \setminus \{0\}$ the finite dimensional *symbol complex*

$$0 \to \boldsymbol{E}_0(x) \stackrel{[\sigma_{A_0}](x,\xi)}{\longrightarrow} \boldsymbol{E}_1(x) \stackrel{[\sigma_{A_1}](x,\xi)}{\longrightarrow} \cdots \stackrel{[\sigma_{A_{N-1}}](x,\xi)}{\longrightarrow} \boldsymbol{E}_N(x) \to 0$$

is acyclic. □

[3] The restriction on the order is not really necessary, but this is what one encounters in concrete applications.

Suppose that $(\, A_k \in \boldsymbol{\Psi}^1(\boldsymbol{E}_k, \boldsymbol{E}_{k+1})\,)_{0\leq k\leq N-1}$ is a complex of ψdos. We fix a Riemann metric g on M, and Hermitian metrics h_k on the vector bundles $\boldsymbol{E}_k$ so we can define the formal adjoints $A_k^* \in \boldsymbol{\Psi}^1(\boldsymbol{E}_{k+1}, \boldsymbol{E}_k)$. Now form the direct sums

$$\boldsymbol{E}_\bullet = \bigoplus_{k=0}^{N} \boldsymbol{E}_k, \quad h_\bullet = \oplus_{k=0}^N h_k, \quad A_\bullet = \oplus_{k=0}^{N-1} A_k.$$

Then

$$A_\bullet, A_\bullet^* \in \boldsymbol{\Psi}^1(\boldsymbol{E}_\bullet).$$

Proposition 3.7.3. *The complex of ψdo's $(\, A_k \in \boldsymbol{\Psi}^1(\boldsymbol{E}_k, \boldsymbol{E}_{k+1})\,)_{0\leq k\leq N-1}$ is elliptic if and only if the operator $A_\bullet + A_\bullet^*$ is elliptic.*

Proof. This is a consequence of the baby Hodge theory, more precisely (3.7.5). □

Suppose $(\, A_k \in \boldsymbol{\Psi}^1(\boldsymbol{E}_k, \boldsymbol{E}_{k+1})\,)_{0\leq k\leq N-1}$ is a complex of ψdo's. Its space of *cocycles* is the vector space

$$Z^k(A_\bullet) := \ker\big(\, C^\infty(\boldsymbol{E}_k) \xrightarrow{A_k} C^\infty(\boldsymbol{E}_{k+1}\,)\big),$$

its the space of *coboundaries* is

$$B^k(A_\bullet) := \operatorname{ran}\big(\, C^\infty(\boldsymbol{E}_{k-1}) \xrightarrow{A_{k-1}} C^\infty(\boldsymbol{E}_k\,)\big),$$

and its degree k-*cohomology* space is

$$H^k(A_\bullet) := Z^k(A_\bullet)/B_k(A_\bullet).$$

Theorem 3.7.4 (Hodge Decomposition)**.** *Suppose $(\, A_k \in \boldsymbol{\Psi}^1(\boldsymbol{E}_k, \boldsymbol{E}_{k+1})\,)_{0\leq k\leq N-1}$ is an elliptic complex of ψdo's over the* compact *manifold M. Fix a Riemann metric on M, Hermitian metrics h_k and compatible connections on $\boldsymbol{E}_k$. Set*

$$\boldsymbol{H}^k(A_\bullet, g, h_\bullet) := \big\{ u \in C^\infty(\boldsymbol{E}_k);\;\; A_k u = A_{k-1}^* u = 0 \,\big\},$$

$$\operatorname{ran}_{L^2} A_k := \operatorname{ran}\Big(\, L^{1,2}(\boldsymbol{E}_k) \xrightarrow{A_k} L^2(\boldsymbol{E}_{k+1})\,\Big),$$

$$\operatorname{ran}_{L^2} A_k^* := \operatorname{ran}\Big(\, L^{1,2}(\boldsymbol{E}_{k+1}) \xrightarrow{A_k^*} L^2(\boldsymbol{E}_k)\,\Big).$$

Then the following hold.

(a) The spaces $\boldsymbol{H}^k(A_\bullet, g, h_\bullet)$, $\operatorname{ran}_{L^2} A_{k-1}$, $\operatorname{ran}_{L^2} A_k^*$ are closed in $L^2(\boldsymbol{E}_k)$, they are mutually orthogonal and we have a direct sum decomposition

$$L^2(\boldsymbol{E}_k) = \boldsymbol{H}^k(A_\bullet, g, h_\bullet) \oplus \operatorname{ran}_{L^2} A_{k-1} \oplus \operatorname{ran}_{L^2} A_{k-1}.$$

(b) The space $\boldsymbol{H}^k(A_\bullet, g, h_\bullet)$ is finite dimensional *and the natural map*

$$\boldsymbol{H}^k(A_\bullet, g, h_\bullet) \to H^k(A_\bullet)$$

is a linear isomorphism.

Proof. From Proposition 3.7.3 we deduce that the operator

$$\mathcal{D}_A = A_\bullet + A_\bullet^* : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E}).$$

is elliptic. Arguing as in the proof of (3.7.4) we deduce that

$$(A_\bullet + A_\bullet^*)u = 0 \Longleftrightarrow A_\bullet u = A_\bullet^* u = 0.$$

This implies that

$$\boldsymbol{H}^k(A_\bullet, g, h_\bullet) = \{u \in C^\infty(\boldsymbol{E}_k);\ \ \mathcal{D}_A u = 0\} \ \text{ and } \ \ker \mathcal{D}_A = \bigoplus_k \boldsymbol{H}^k(A_\bullet, g, h_\bullet).$$

Since $\mathcal{D}_A$ is elliptic we deduce that the space $\boldsymbol{H}^k(A_\bullet, g, h_\bullet)$ is finite dimensional and is also equal to

$$\{u \in C^{-\infty}(\boldsymbol{E}_k);\ \ \mathcal{D}_A u = 0\}.$$

The operator $\mathcal{D}_A$ induces a Fredholm operator

$$\mathcal{D}_A : L^{1,2}(\boldsymbol{E}) \to L^2(\boldsymbol{E}).$$

Therefore its range $\operatorname{ran}_{L^2}(\mathcal{D}_A)$ is closed and, according to Corollary 3.5.3, it is equal to the L^2-orthogonal complement of the kernel of $\mathcal{D}_A^* = \mathcal{D}_A$. In particular, we have an orthogonal decomposition

$$L^2(\boldsymbol{E}_k) = \boldsymbol{H}^k(A_\bullet, g, h_\bullet) \oplus \Big(\operatorname{ran}_{L^2}(\mathcal{D}_A) \cap L^2(\boldsymbol{E}_k)\Big).$$

Clearly

$$\operatorname{ran}_{L^2}(\mathcal{D}_A) \cap L^2(\boldsymbol{E}_k) = \operatorname{ran}_{L^2} A_{k-1} \oplus \operatorname{ran}_{L^2} A_k^*,$$

so to prove (a) it suffices to show that the subspaces $\operatorname{ran}_{L^2} A_{k-1}$, $\operatorname{ran}_{L^2} A_k^*$ are closed and orthogonal to each other.

The orthogonality is immediate. Indeed, let $u \in \operatorname{ran}_{L^2} A_k$ and $v \in \operatorname{ran}_{L^2} A_k^*$. Then there exist $u' \in L^{1,2}(\boldsymbol{E}_{k-1})$ and $v' \in L^{1,2}(\boldsymbol{E}_{k+1})$ such that

$$u = A_{k-1}u', \ \ v = A_k^* v'.$$

then

$$(u, v)_{L^2} = (A_{k-1}u', A_k^* v') = (A_k A_{k-1} u, v) = 0 \ \text{ since } \ A_k A_{k-1} = 0.$$

To prove that $\operatorname{ran}_{L^2} A_{k-1}$ is closed we consider a sequence $u_n \in \operatorname{ran}_{L^2} A_{k-1}$ that converges in the L^2-norm to some $u \in L^2(\boldsymbol{E}_k)$. Observe that $u_n \in \operatorname{ran}_{L^2} \mathcal{D}_A \cap L^2(\boldsymbol{E}_k)$, and since $\operatorname{ran}_{L^2} \mathcal{D}_A$ is closed, there exists $v = v_{k-1} \oplus v_{k+1} \in L^{1,2}(\boldsymbol{E}_{k-1} \oplus \boldsymbol{E}_{k+1})$ such that

$$u = \mathcal{D}_A v = A v_{k-1} + A^* v_{k+1}.$$

Since $u_n \perp \operatorname{ran}_{L^2} A_k^*$ we deduce by passing to limit that $u \perp \operatorname{ran}_{L^2} A_k^*$. Hence

$$0 = (u, A_k^* v_{k+1})_{L^2} = (A v_{k-1} + A^* v_{k+1}, A_k^* v_{k+1})_{L^2} = \|A_k^* v_{k+1}\|_{L^2}^2.$$

Thus

$$u = A_{k-1} v_{k-1} \in \operatorname{ran}_{L^2} A_{k-1}.$$

In a similar fashion we prove that $\operatorname{ran}_{L^2} A_k$ is closed in $L^2(\boldsymbol{E}_k)$. This completes part (a) of the theorem.

To prove part (b) we need to show that the natural map

$$\boldsymbol{H}^k(A_\bullet, g, h_\bullet) \to H^k(A_\bullet) \tag{3.7.6}$$

is both injective an surjective. Both facts are consequences of the Hodge decomposition in part (a). Consider a cohomology class $x \in H^k(A_\bullet)$ represented by a smooth section $u \in C^\infty(\boldsymbol{E}_k)$ such that $A_k u = 0$. We decompose

$$u = u_0 + A_{k-1}u' + A_k^* u''$$

where

$$u_0 \in \boldsymbol{H}^k(A_\bullet, g, h_\bullet), \ \ u' \in L^{1,2}(\boldsymbol{E}_{k-1}), \ \ u'' \in L^{1,2}(\boldsymbol{E}_{k+1}).$$

Then $A_k u_0 = AkA_{k-1}u'$ so that

$$0 = A_k u = A_k A_k^* u'' \Rightarrow 0 = (u, A_k A_k^* u'')_{L^2} = \|A_k^* u''\|_{L^2}^2.$$

Hence $u = u_0 + A_k u'$ so that u is cohomologous to u_0 and therefore the class x is also represented by the element $u_0 \in \boldsymbol{H}^k(A_\bullet, g, h_\bullet)$. This proves the surjectivity of the morphism (3.7.6).

To prove that this is also injective, consider $u_0 \in \boldsymbol{H}^k(A_\bullet, g, h_\bullet)$ that is cohomologous to 0. Thus $u_0 \in \operatorname{ran}_{L^2} A_{k-1}$. It follows that $u_0 = 0$ since

$$\boldsymbol{H}^k(A_\bullet, g, h_\bullet) \cap \operatorname{ran}_{L^2} A_{k-1} = 0.$$

□

Definition 3.7.5. The spaces $\boldsymbol{H}^k(A_\bullet, g, h_\bullet)$ defined in Theorem 3.7.4 are called the spaces of *harmonic* sections determined by the complex and the metrics g and $h_\bullet$. □

Example 3.7.6 (Classical Hodge theory)**.** Suppose (M, g) is a compact, connected, smooth Riemann manifold of dimension m. Denote by $\Omega^k_{\mathbb{C}}(M)$ the space of smooth, complex valued differential forms of degree k on M. Consider the DeRham complex

$$0 \to \Omega^0_{\mathbb{C}}(M) \xrightarrow{d} \Omega^1_{\mathbb{C}}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_{\mathbb{C}}(M) \to 0.$$

As we have seen in Example 3.2.7 the Hodge-DeRham operator $D = d + d^* : \Omega^\bullet_{\mathbb{C}}(M) \to \Omega^\bullet_{\mathbb{C}}(M)$ is elliptic so that the DeRham complex is an elliptic complex. It thus leads to an (orthogonal) Hodge decomposition

$$\Omega^k_{\mathbb{C}}(M) = d\Omega^{k-1}_{\mathbb{C}}(M) \oplus d^*\Omega^{p+1}_{\mathbb{C}}(M) \oplus \boldsymbol{H}^k(M, g),$$

where $\boldsymbol{H}^k(M, g)$ is the space of harmonic k-forms, i.e., k-forms α which are both closed and co-closed

$$d\alpha = d^*\alpha = 0.$$

The space $\boldsymbol{H}^k(M, g)$ is *finite dimensional*, it depends on the metric g but its dimension is independent of g. We deduce that the k-th DeRham cohomology space

$$H^k_{DR}(M) := \frac{\ker\big(\,\Omega^k_{\mathbb{C}}(M) \xrightarrow{d} \Omega^{k+1}(M)\,\big)}{\operatorname{ran}\big(\,\Omega^{k-1}_{\mathbb{C}}(M) \xrightarrow{d} \Omega^k_{\mathbb{C}}(M)\,\big)}$$

is finite dimensional. Its (complex) dimension is equal with to the k-th Betti number of the cohomology of M with rational coefficients.

The index of the Gauss-Bonnet operator (D, ε) is the index of the elliptic operator

$$D : \Omega^{even}_{\mathbb{C}}(M) \to \Omega^{odd}_{\mathbb{C}}(M), \;\; \Omega^{even/odd}_{\mathbb{C}} = \bigoplus_{k \equiv 0/1 \bmod 2} \Omega^k_{\mathbb{C}}.$$

Hodge theory now implies that the index of the Gauss-Bonnet operator is the integer

$$\sum_{k \geq 0} (-1)^k \dim_{\mathbb{C}} H^k_{DR}(M) = \text{the Euler characteristic of } M \text{ with rational coefficients.}$$

□

3.8. Exercises

Exercise 3.1. Suppose that H_0, H_1 are two complex separable Hilbert spaces. Prove that the set $\boldsymbol{B}_*(H_0, H_1) \subset \boldsymbol{B}(H_0, H_1)$ of invertible continuous linear operators $H_0 \to H_1$ is open. □

Exercise 3.2. We say that two operators $T_0, T_1 \in \boldsymbol{Fred}(H)$ are homotopic in $\boldsymbol{Fred}(H)$ if there exists a continuous map

$$[0,1] \ni t \mapsto T(t) \in \boldsymbol{Fred}(H)$$

such that $T(0) = T_0$, $T(1) = T_1$. Prove that if $T_0, T_1 \in \boldsymbol{Fred}(H)$ then the following two conditions are equivalent

(a) The operators T_0 and T_1 are homotopic in $\boldsymbol{Fred}(H)$.

(a) $\operatorname{ind} T_0 = \operatorname{ind} T_1$.

Hint: You need to use the fact that the group $\mathrm{GL}(H)$ of continuous, bijective maps $H \to H$ is connected.[4] □

Exercise 3.3 (Poincaré)**.** Suppose that M is a compact oriented manifold. Prove that for every Riemann metric g on M there exists a positive constant $C = C(g) > 0$ such that

$$\int_M |du|_g^2 \, |dV_g| \leq C \int_\Omega |u|^2 \, |dV_g|, \;\; \forall u \in C^\infty(M), \;\; \int_M u \, |dV_g| = 0.$$

Hint: Use Corollary 3.4.7 for the Fredholm operator $\Delta : L^{2,2}(M) \to L^2(M)$. □

Exercise 3.4 (The Dirichlet Principle)**.** Let (M, g) be a compact Riemannian manifold, and $f \in L^2(M, |dV_g|)$. Define

$$J : L^{1,2}(M) \to \mathbb{R}, \;\; J(u) = \frac{1}{2} \int_M \big(|du|^2 + |u|^2 \big) + \int_M \mathbf{Re}(u\bar{f}) \, |dV_g.$$

(a) Prove that $J_0 := \inf_u J(u) > -\infty$.

(b) Show that if $J(u_0) = J_0$ then u_0 is a distributional solution of the equation

$$\Delta_g u + u = f.$$

Conclude that there exists at most one u_0 such that $J(u_0) = J_0$.

(c) Show that there exists at least one function $u_0 \in L^{1,2}(M)$ such that $J(u_0) = J_0$. **Hint:** Have a look at [**N**, Thm. 10.3.15, Prop. 10.3.20]. □

Exercise 3.5. Consider the complex $(E^\bullet, d)$ from Example 3.7.1 equipped with the hermitian metric h. Let $u \in E^n$ be a cocycle, i.e., $du = 0$. The cohomology class of u can be identified with the affine subspace

$$S_u = \big\{ u' \in E^n; \;\; u' - u \in \operatorname{ran}(d_{n-1}) \big\} = \big\{ u + dv; \;\; v \in E^{n-1} \big\}.$$

Denote by $[u]_h$ the element in S_u of closest to the origin. Prove $[u]_h$ is harmonic. Moreover, if u' is another cocycle, then u is cohomologous to u' if and only if $[u]_h = [u']_h$. □

[4] A stronger result is true. Namely, a theorem of N. Kuiper states that the group $\mathrm{GL}(H)$ is *contractible*. The connectedness of $\mathrm{GL}(H)$ can be proved much faster using a bit of functional calculus. First one proves that the natural inclusion of the unitary group $U(H)$ in $\mathrm{GL}(H)$ is a homotopy equivalence with homotopy inverse the map $\mathrm{GL}(H) \ni T \mapsto T(T^*T)^{-1/2} \in U(H)$. To prove the connectivity of $U(H)$ we can use Stone's theorem [**RSz**, §137] which states that for any $S \in U(H)$ there exists a (possibly unbounded) selfadjoint operator A such that $S = e^{iA}$. Then $t \mapsto S_t = e^{itA}$, $t \in [0,1]$, is a continuous path in $U(H)$ from $\mathbb{1}$ to S.

Chapter 4

The heat kernel

4.1. A look ahead

This is a rather technical chapter, and to help the reader endure the analytical work to come, we thought it would help if we outline the main goal and the strategy for achieving it.

As in the last part of the previous chapter we will work on a smooth, compact Riemann manifold (M, g) of dimension M. We fix a smooth complex vector bundle $\boldsymbol{E} \to M$ over M and a Hermitian metric h on it. Suppose $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ is formally self-adjoint elliptic operator of order k. We also assume it is *positive*, i.e.,

$$(Au, u)_{L^2} > 0, \ \ \forall u \in C^\infty(\boldsymbol{E}) \setminus \{0\}.$$

It is very easy to produce such operators. Start with and elliptic partial differential operator $L : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{F})$, where $\boldsymbol{F} \to M$ is another smooth complex Hermitian vector bundle. Then the operator

$$A = L^*L + \mathbb{1}$$

is elliptic, formally self-adjoint and positive .

Fix a spectral basis $(\boldsymbol{\phi}_n)_{n\geq 0}$ of $L^2(\boldsymbol{E})$ with respect to A, where the eigenvalue corresponding to $\boldsymbol{\phi}_n$ is λ_n. Then $\lambda_n \geq 1$ for any $n \geq 0$. We may assume that the eigenvalues are thus labeled so that

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots ...$$

In the sequence (λ_n) each eigenvalue of A appears as many times as its multiplicity. The main goal of this chapter is to gain a better understanding of the behavior of $\lambda_n \to \infty$.

To achieve this we consider the bounded operator

$$e^{-tA} : L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E}), \ \ e^{-tA}\left(\sum_n u_n \boldsymbol{\phi}^n\right) = \sum_n e^{-t\lambda_n} u_n \boldsymbol{\phi}_n.$$

We want to prove that for any $t > 0$ this is a trace class operator, i.e.,

$$\boldsymbol{Tr}\, e^{-tA} := \sum_n e^{-t\lambda_n} < 0,$$

and then investigate the behavior of $\boldsymbol{Tr}\, e^{-tA}$ as $t \searrow 0$.

To achieve this we will express e^{-tA} as an operator valued integral over a contour γ_R, $0 < R$, of the type depicted in Figure 4.1. In this figure, the two linear branches are described by

$$\arg\lambda = \pm\frac{\pi}{4}, \quad |\lambda| \geq R.$$

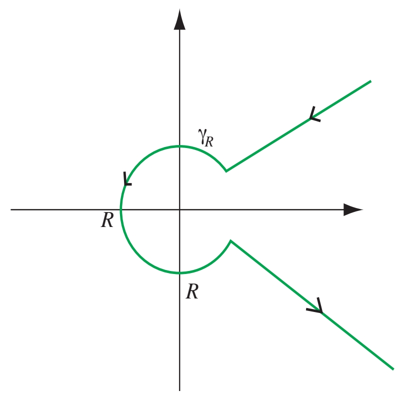

Figure 4.1. *The contour* γ_R.

More precisely, under the "right circumstances" we have[1]

$$e^{-tA} = \frac{(-1)^n n!}{2\pi \boldsymbol{i} t^n}\int_{\gamma_R} e^{-t\lambda}(\lambda - A)^{-(n+1)}\,d\lambda = \frac{1}{2\pi\boldsymbol{i}t^n}\int_{\gamma_R} e^{-t\lambda}\partial_\lambda^n(\lambda - A)^{-1}, \quad n \in \mathbb{Z}_{\geq 0}. \tag{4.1.1}$$

The right circumstances alluded to above guarantee the following things.

(i) The inverse $(\lambda - A)^{-1}$ exists for any $\lambda \in \gamma_R$.

(ii) The improper integral in (4.1.1) is convergent, i.e., we have some control on the norm $\|(\lambda - A)^{-(n+1)}\|$ for large $|\lambda|$.

To prove the existence of $(\lambda - A)^{-1}$ we use the same idea in the construction of a parametrix of an elliptic operator. More precisely we will construct a family of ψdo's B_λ such that

$$R_\lambda = A_\lambda B_\lambda - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty}(\boldsymbol{E}) \tag{4.1.2}$$

such that

$$\|R_\lambda\| = O(|\lambda|^{-p}) \ \text{ as } \ |\lambda| \to \infty, \tag{4.1.3}$$

for some $p > 0$. This show that the operator R_λ is small for large λ so that the operator $\mathbb{1} + R_\lambda = A_\lambda B_\lambda$ is invertible.

For large n the operator $(\lambda - A)^{-(n+1)}$ is of trace class and then we conclude that

$$\boldsymbol{Tr}\, e^{-tA} = \frac{(-1)^n n!}{2\pi\boldsymbol{i}t^n}\int_{\gamma_R} e^{0t\lambda}\,\boldsymbol{Tr}(\lambda - A)^{-(n+1)}\,d\lambda, \quad n \in \mathbb{Z}_{\geq 0}. \tag{4.1.4}$$

In fact, the Schwarz kernel of $(\lambda - A)^{-(n+1)}$ is given by a *continuous* section $K_\lambda(x, y)$ of the bundle $\boldsymbol{E} \boxtimes E^\vee \to M \times M$, and we have

$$\boldsymbol{Tr}(\lambda - A)^{-(n+1)} = \int_M \operatorname{tr} K_\lambda(x, x)\,|dV_g(x)| =: f_A(\lambda).$$

[1]To understand the equality (4.1.1) think that A is a positive real number and then use the residue theorem.

We obtain a smooth kernel

$$\boldsymbol{Tr}\, e^{-tA} = \frac{(-1)^n n!}{2\pi \boldsymbol{i} t^n} \int_{\gamma_R} e^{-t\lambda} f_A(\lambda)\, d\lambda. \tag{4.1.5}$$

From here we proceed using two clever tricks of classical real analysis. The first will allow us to convert an asymptotic expansion of $f_A(\lambda)$ for λ near ∞ to an asymptotic expansion of $\boldsymbol{Tr}\, e^{-tA}$ as $t \searrow 0$. Next using a Tauberian theorem we convert the latter asymptotic expansion into an information about the asymptotic behavior of λ_n as $n \to \infty$.

The key moment in the proof is the construction of the operator B_λ satisfying (4.1.2) and (4.1.3). This is based on the concept of ψdo with parameters.

4.2. Pseudo-differential operators with parameters

We have to redo most of Chapter 2 working with symbols depending in a rather constrained way on a complex parameter. We follow the approach in [**Shu**, Chap. II] which suffices for the application we have in mind but has some limitations. For more general classes of symbols depending on parameters we refer to [**GrSe95, GrH**].

Fix $\varepsilon > 0$ very small and denote by Λ the open cone (It is the complement of the shaded area Figure 4.2.)

$$\Lambda := \big\{ re^{\boldsymbol{i}\theta} \in \mathbb{C};\ \ r > 0,\ \ |\theta| > \varepsilon \big\}.$$

Let $\boldsymbol{U}, \boldsymbol{V}$ be real Euclidean spaces of dimensions N an respectively m, Ω an open subset in $\boldsymbol{V}$, and

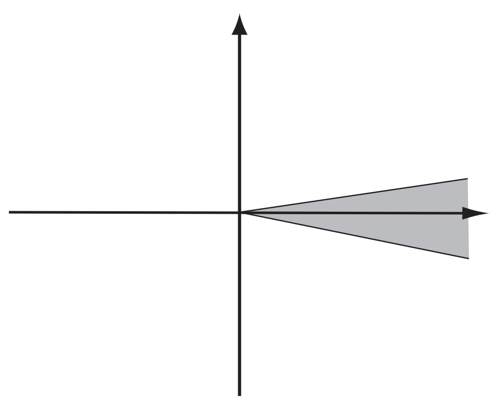

Figure 4.2. *The cone Λ is the complement of the narrow shaded angle.*

$\mathcal{O}$ an open subset of $\boldsymbol{U}$. For any numbers $s \in \mathbb{R}$ and $d > 0$ we define $\boldsymbol{A}^{s,d}_\Lambda(\mathcal{O} \times \boldsymbol{V})$ to be the space of smooth functions

$$a : \mathcal{O} \times \boldsymbol{V} \times \Lambda \to \mathbb{C},\ \ \mathcal{O} \times \boldsymbol{V} \times \Lambda \ni (x, \xi, \lambda) \mapsto a_\lambda(x, \xi),$$

such that the following hold.

- For any $(x, \xi) \in \mathcal{O} \times \boldsymbol{V}$, the map $\lambda \mapsto a_\lambda(x, \xi)$ is holomorphic.
- For any compact $K \subset \mathcal{O}$, any multi-indices $\alpha, \beta \in \mathbb{Z}^m_{\geq 0}$, and any $j \in \mathbb{Z}_{\geq 0}$, there exists a constant $C = C(\alpha, \beta, K)$ such that

$$|D^\beta_x \partial^j_\lambda \partial^\alpha_\xi a_\lambda(x, \xi)| \leq C\big(1 + |\xi| + |\lambda|^{1/d}\big)^{s - |\alpha| - jd},\ \ \forall x \in K,\ \ \lambda \in \Lambda. \tag{4.2.1}$$

We set

$$\boldsymbol{\mathcal{A}}^{\infty,d} := \bigcup_{s\in\mathbb{R}} \boldsymbol{\mathcal{A}}^{s,d}_{\Lambda},\ \ \boldsymbol{\mathcal{A}}^{-\infty,d} := \bigcap_{s\in\mathbb{R}} \boldsymbol{\mathcal{A}}^{s,d}_{\Lambda}.$$

The quantity $\bigl(1+|\xi|^2+|\lambda|^{2/d}\bigr)^{1/2}$ will appear very frequently in the sequel and for this reason we introduce the notation

$$\boldsymbol{\varrho}(\xi,\lambda)=\boldsymbol{\varrho}_d(\xi,\lambda)=\bigl(1+|\xi|+|\lambda|^{1/d}\bigr),\ \ \forall \xi\in\mathbb{V},\ \ \lambda\in\mathbb{C}.$$

Observe that $\boldsymbol{\varrho}(\xi,0)=\langle\xi\rangle$. Note that (4.2.1) is equivalent to

$$|D^\gamma_x\partial^\beta_\lambda\partial^\alpha_\xi a_\lambda(x,\xi)|\leq C\boldsymbol{\varrho}_d(\xi,\lambda)^{s-|\alpha|-d\beta},\ \ \forall x\in K,\ \ \lambda\in\Lambda. \tag{4.2.2}$$

Example 4.2.1. (a) Suppose that for any $x\in \boldsymbol{V}$ the function $a(x,\xi)$ is a *polynomial* in ξ of degree ℓ. Equivalently, $a(x,\xi)$ is a polynomial in ξ with smooth coefficients. Then

$$a_\lambda(x,\xi)=a(x,\xi)-\lambda\in\boldsymbol{\mathcal{A}}^{\ell,\ell}(\boldsymbol{V}).$$

This follows from the fact that in this case we need to check the inequalities (4.2.1) involving only derivatives ∂^α_ξ and ∂^j_λ with $|\alpha|\leq\ell$ and $j\leq 1$ so that $\boldsymbol{\varrho}(\xi,\lambda)^{\ell-|\alpha|}\geq\langle\xi\rangle^{\ell-|\alpha|}$.

(b) The function $(\xi,\lambda)\mapsto b_\lambda(\xi)=(1+|\xi|^2)^{1/2}-\lambda$ *is not* a symbol with parameters, though the function $\xi\mapsto b_\lambda(\xi)$ is a symbol of order 1 for every $\lambda\in\Lambda$. □

Given $a_\lambda\in\boldsymbol{\mathcal{A}}^{s,d}_\Lambda(\Omega\times\Omega\times\boldsymbol{V})$ we can define a continuous operator

$$\mathbf{Op}(a_\lambda): C^\infty_0(\Omega)\to C^{-\infty}(\Omega)$$

whose Schwartz kernel K_{a_λ} is given by the oscillatory integral

$$K_{a_\lambda}(x,y)=(2\pi)^{-m/2}\int^\sim_{\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)}a_\lambda(x,y,\xi)\,|d\xi|_*.$$

We denote by $\boldsymbol{\Psi}^{s,d}(\Omega,\Lambda)$ this class of pseudo-differential operators.

We say that $\mathbf{Op}(a_\lambda)$ is *properly supported* if there exists a proper subset $C\subset\Omega\times\Omega$ (see Definition 2.3.5) such that

$$\operatorname{supp} K_{a_\lambda}\subset C,\ \ \forall\lambda\in\Lambda.$$

We denote by $\boldsymbol{\Psi}^{s,d}_0(\Omega,\Lambda)$ the subclass of $\boldsymbol{\Psi}^{s,d}(\Omega,\Lambda)$ consisting of properly supported operators. Again, we have a decomposition

$$\boldsymbol{\Psi}^{s,d}(\Omega,\Lambda)=\boldsymbol{\Psi}^{s,d}_0(\Omega,\Lambda)+\boldsymbol{\Psi}^{-\infty,d}(\Omega,\Lambda).$$

If $A_\lambda\in\boldsymbol{\Psi}^{s,d}_0(\Omega,\Lambda)$ then we can define

$$\sigma_{A_\lambda}(x,\xi)=e_{-\xi}(\xi)\bigl(A_\lambda e_\xi\bigr)(x).$$

Then

$$\sigma_{A_\lambda}\in\boldsymbol{\mathcal{A}}^{s,d}_\Lambda(\Omega\times\boldsymbol{V})=:\boldsymbol{\mathcal{S}}^{s,d}_\Lambda(\Omega),$$

and for any $u\in C^\infty_0(\Omega)$ we have

$$A_\lambda u(x)=\int_{\boldsymbol{V}} e^{\boldsymbol{i}(\xi,x)}\sigma_{A_\lambda}(x,\xi)\widehat{u}(\xi)\,|d\xi|_*.$$

The space $\boldsymbol{\mathcal{S}}^{s,d}_\Lambda(\Omega)$ is called the space of *symbols with parameters* of bi-order (s,d). The theory of asymptotic expansions extends almost word for word to the parametric case. In particular, we have the following parametric version of Theorem 2.4.6.

Theorem 4.2.2. *Suppose* $A_\lambda \in \boldsymbol{\Psi}_0^{k,d}(\Omega)$ *is a properly supported* ψdo*,*

$$A_\lambda = \mathbf{Op}(a_\lambda),\ \ a \in \boldsymbol{\mathcal{A}}_\Lambda^{k,d}(\Omega\times\Omega\times\boldsymbol{V}).$$

Then its symbol $\sigma_{A_\lambda}(x,\xi) = e_{-\xi}A_\lambda e_\xi$ *admits the asymptotic expansion*

$$\sigma_{A_\lambda}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha a_\lambda(x,y,\xi)|_{x=y},. \tag{4.2.3}$$

□

Similarly, Theorems 2.5.1 and 2.5.2 have a parametric counterpart whose formulations can be left to the reader.

A symbol with parameters $a_\lambda \in \mathcal{S}_\Lambda^{s,d}(\Omega)$ is said to be *polyhomogeneous* or *classical* if it admits an asymptotic expansion of the form

$$a_\lambda \sim \sum_{j=0}^\infty a_{s-j}(x,\xi,\lambda),$$

where $a_{s-j}(x,\xi,\lambda) \in \boldsymbol{\mathcal{A}}_\Lambda^{s-j,d}(\Omega\times\boldsymbol{V})$ is *quasi-homogeneous* of degree $(s-j)$ for $|\xi|+|\lambda|^{1/d} \geq 1$, i.e.,

$$a_{s-j}(x,t\xi,t^d\lambda) = t^{s-j}a(x,\xi,\lambda),\ \ \forall t\geq 1,\ \ |\xi|+|\lambda|^{1/d}\geq 1.$$

The symbol in Example 4.2.1 is an example of classical symbol with parameter. We will denote by $\mathcal{S}_{\Lambda,\mathrm{phg}}^{k,d}(\Omega)$ the subclass of $\mathcal{S}_\Lambda^{k,d}(\Omega)$ consisting of classical symbols.

We want to spend a bit more time investigating the functional properties of the pseudo-differential operators with parameters. Clearly, the pseudo-differential operators with parameters do define continuous linear maps between appropriate Sobolev spaces. More precisely, if $a_\lambda \in \mathcal{S}_\Lambda^{k,d}(\Omega)$, then for any $\varphi \in C_0^\infty(\Omega)$ we have $\varphi a_\lambda \in \mathcal{S}_\Lambda^{k,d}(\boldsymbol{V})$, and for any $s\in\mathbb{R}$ we obtain a bounded linear operator

$$\mathbf{Op}(\varphi a_\lambda) : H^s(\boldsymbol{V}) \to H^{s-k}(\boldsymbol{V}).$$

The resulting family of bounded operators $\lambda \mapsto \mathbf{Op}(\varphi a_\lambda)$ depends holomorphically on $\lambda\in\Lambda$, i.e.,

$$\frac{\partial}{\partial\bar\lambda}\mathbf{Op}(\varphi a_\lambda) = 0,$$

where the above derivative is computed using the norm topology on the space of bounded linear operators $H^s(\boldsymbol{V}) \to H^{s-k}(\boldsymbol{V})$. Moreover

$$\frac{\partial}{\partial\lambda}\mathbf{Op}(\varphi a_\lambda) = \mathbf{Op}\left(\varphi\frac{\partial a_\lambda}{\partial\lambda}\right) \tag{4.2.4}$$

Observe that (4.2.1) implies that

$$\frac{\partial a_\lambda}{\partial\lambda} \in \mathcal{S}_\Lambda^{k-d,d}(\Omega)$$

so that

$$\frac{\partial}{\partial\lambda}\mathbf{Op}(\varphi a_\lambda) \in \boldsymbol{\Psi}^{k-d,d}(\Omega,\Lambda) \tag{4.2.5}$$

The dependence of the norms of these operators on the parameters will play a crucial role in this chapter, and for this reason we want to prove the following more refined version of Theorem 2.8.1.

Theorem 4.2.3. *Suppose $a \in \mathbf{S}_\Lambda^{-k,d}(\Omega)$, $k \geq 0$. Then for every $\varphi \in C_0^\infty(\Omega)$, any $0 \leq \ell \leq k$ and any $s \in \mathbb{R}$ there exists a constant $C = C(s, \ell, \varphi, a)$ such that for any $f \in C_0^\infty(\Omega)$ we have*

$$\|\varphi \, \mathbf{Op}(a) f\|_{s+\ell} \leq C\big(1 + |\lambda|^{1/d}\big)^{-(k-\ell)} \|f\|_s.$$

In particular, if we choose $s = 0$, $\ell = 0$, we deduce

$$\|\varphi \, \mathbf{Op}(a) f\|_{L^2} \leq C\big(1 + |\lambda|^{1/d}\big)^{-k} \|f\|_{L^2}. \tag{4.2.6}$$

Proof. Observe that $\varphi \, \mathbf{Op}(a_\lambda) f = \mathbf{Op}(\varphi a_\lambda) f$. Set

$$\sigma_\lambda(x, \xi) = \varphi(x) a_\lambda(x, \xi) \in \mathbf{S}^\ell(\Omega).$$

Observe that σ_λ has compact x-support, i.e., there exists a compact set $S \subset \Omega$ such that

$$\sigma_\lambda(x, \xi) = 0, \;\; \forall (x, \xi, \lambda) \in (\Omega \setminus S) \times \boldsymbol{V} \times \Lambda.$$

In particular, extending σ_λ by 0 for $x \in \boldsymbol{V} \setminus \Omega$ we can regard it as a symbol $\sigma_\lambda \in \mathbf{S}^\ell(\boldsymbol{V})$.

We set $\Lambda_s = \mathbf{Op}(\langle \xi \rangle^s) \in \mathbf{\Psi}^s(\boldsymbol{V})$ so that Λ_s defines isometries $\Lambda_s : H^t(\boldsymbol{V}) \to H^{t-s}(\boldsymbol{V})$. We observe that

$$\| \mathbf{Op}(\varphi a_\lambda) f\|_{s+\ell} = \|\Lambda_{s+\ell} \, \mathbf{Op}(\varphi a_\lambda) f\|_{L^2}$$

If we write $g = \Lambda_s f$ then

$$f = \Lambda_{-s} g \;\; \text{and} \;\; \|f\|_s = \|g\|_{L^2},$$

and thus we have to estimate $\|\Lambda_{s+\ell} \, \mathbf{Op}(\varphi a_\lambda) \Lambda_{-s} g\|_{L^2}$ in terms of $\|g\|_{L^2}$. In other words, we need to estimate the norm of the bounded operator

$$A_s : \Lambda_{s+\ell} \, \mathbf{Op}(\varphi a_\lambda) \Lambda_{-s} : L^2(\boldsymbol{V}) \to L^2(\boldsymbol{V}).$$

Define

$$\widehat{\sigma}_\lambda(\eta, \xi) := \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\eta)} \sigma_\lambda(x, \xi) \, |dx|_*.$$

Using the support condition on σ_λ we deduce

$$\eta^\alpha \widehat{\sigma}(\eta, \xi) = \int_{\boldsymbol{V}} D_x^\alpha \sigma_\lambda(x, \xi) e^{-\boldsymbol{i}(x,\eta)} \, |dx|_*, \;\; \forall \alpha, \eta.$$

This implies that for every $N > 0$, there exists $C_N > 0$, independent of ξ such that

$$|\widehat{\sigma}_\lambda(\eta, \xi)| \leq C_N \boldsymbol{\varrho}(\xi, \lambda)^{-k} \langle \eta \rangle^{-N}, \;\; \forall \xi, \eta \in \boldsymbol{V}. \tag{4.2.7}$$

For $f \in C_0^\infty(\boldsymbol{V})$ we have

$$\widehat{A_s f}(\eta) = \langle \eta \rangle^{s+\ell} \boldsymbol{\mathfrak{F}}\big(\mathbf{Op}(\sigma_\lambda) \Lambda_{-s} f \,\big)(\eta),$$

and

$$\boldsymbol{\mathfrak{F}}\big(\mathbf{Op}(\sigma_\lambda) \Lambda_s f \,\big)(\eta) = \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\eta)} \left(\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi)} \sigma_\lambda(x, \xi) \langle \xi \rangle^{-s} \widehat{f}(\xi) \, |d\xi|_* \right) |dx|_*$$

$$= \int_{\boldsymbol{V}} \left(\int_{\boldsymbol{V}} e^{\boldsymbol{i}(x,\xi-\eta)} \sigma_\lambda(x, \xi) \langle \xi \rangle^{-s} \widehat{f}(\xi) \, |dx|_* \right) |d\xi|_* = \int_{\boldsymbol{V}} \widehat{\sigma}_\lambda(\eta - \xi, \xi) \langle \xi \rangle^{-s} \widehat{f}(\xi) \, |d\xi|_*.$$

Hence

$$\widehat{A_s f}(\eta) = \int_{\boldsymbol{V}} \underbrace{\widehat{\sigma}(\eta - \xi, \xi) \langle \eta \rangle^{s-\ell} \langle \xi \rangle^{-s}}_{=: K_s(\eta, \xi)} \widehat{f}(\xi) \, |d\xi|_*.$$

Using (4.2.7) we deduce that for any $N > 0$ there exists $C = C(a, N) > 0$ such that

$$|K_s(\eta,\xi)| \leq C\langle \eta-\xi\rangle^{-N}\langle\eta\rangle^{s+\ell}\langle\xi\rangle^{-s}\boldsymbol{\varrho}(\xi,\lambda)^{-k}$$

Observe that

$$\boldsymbol{\varrho}(\xi,\lambda)^{-k} = \boldsymbol{\varrho}(\xi,\lambda)^{-\ell}\boldsymbol{\varrho}(\xi,\lambda)^{-(k-\ell)} \leq \boldsymbol{\varrho}(\xi,0)^{-\ell}\boldsymbol{\varrho}(0,\lambda)^{-(k-\ell)} \leq C(1+|\lambda|^{1/d})^{-(k-\ell)}\langle\xi\rangle^{-\ell}.$$

Hence

$$|K_s(\eta,\xi)| \leq C(1+|\lambda|^{1/d})^{-(k-\ell)}\langle\eta-\xi\rangle^{-N}\langle\eta\rangle^{s+\ell}\langle\xi\rangle^{-s-\ell}.$$

Using Peetre's inequality we deduce

$$\langle\xi\rangle^{-s-\ell} \leq 2^{|s+\ell|}\langle\eta\rangle^{-s-\ell}\langle\eta-\xi\rangle^{|s+\ell|}$$

so that

$$|K_s(\eta,\xi)| \leq 2^{|\ell+s|}C(1+|\lambda|^{1/d})^{-(k-\ell)}\langle\eta-\xi\rangle^{|\ell+s|-N}.$$

Choosing $N := m+1+|\ell+s|$ we deduce

$$|K(\eta,\xi)| \leq 2^{|\ell+s|}C(1+|\lambda|^{1/d})^{-(k-\ell)}\langle\eta-\xi\rangle^{-(m+1)}.$$

If we set

$$C_{m,s} := 2^{|\ell+s|}C\int_{\boldsymbol{V}}\langle\xi\rangle^{-(m+1)}\,|d\xi|_*$$

we deduce from Schur's Lemma 2.8.2 that

$$\|\widehat{Af}\|_{L^2} \leq C_{m,s}(1+|\lambda|^{1/d})^{-(k-\ell)}\|\widehat{f}\|_{L^2}$$

□

The extension to vectorial ψdo's is immediate we leave it to the reader. For two Hermitian vector spaces with get parametric versions $\mathbf{S}^{\infty,d}_{\Lambda}(\Omega, E_0, E_1)$, $\boldsymbol{\Psi}^{\infty,d}(\Omega,\Lambda,E_0,E_1)$ of the spaces of vectorial symbols and ψdo's. When $E_0 = E_1 = E$ we use the simpler notations $\mathbf{S}^{\infty,d}_{\Lambda}(\Omega, E)$ and $\boldsymbol{\Psi}^{\infty,d}(\Omega,\Lambda,E)$. The following result will play an important part in our investigation of the heat kernel.

Proposition 4.2.4. *Suppose $A_\lambda \in \boldsymbol{\Psi}_0^{-k,d}(\Omega,\Lambda,E)$ and let $K_{A_\lambda} \in C^{-\infty}(\Omega\times\Omega, E\otimes E^*)$ be the Schwartz kernel of the operator A_λ. Assume $k > m = \dim\boldsymbol{V}$. Then the following hold.*

(a) The Schwartz kernel is a continuous function $\Omega\times\Omega\to E\otimes E^* = \mathrm{End}(E)$.

(b) For any compact $K\subset\Omega$ there exists a constant $C>0$, independent of λ such that

$$\sup_{x,y\in K}|K_{A_\lambda}(x,y)| \leq C(1+|\lambda|^{1/d})^{-(k-m)},\ \ \forall\lambda\in\Lambda. \tag{4.2.8}$$

Proof. Let $a_\lambda(x,\xi)$ denote the symbol of A_λ. Then the Schwartz kernel K_{A_λ} is given by the oscillatory integral (see (2.4.4))

$$K_{A_\lambda} = (2\pi)^{-m/2}\int_{\boldsymbol{V}} e^{\boldsymbol{i}(\xi,x-y)}a_\lambda(x,\xi)|d\xi|_*.$$

The estimate (4.2.1) implies that for every compact $K\subset\Omega$ there exists a constant $C>0$ independent of λ such that

$$\sup_{x\in K}\big|\,a_\lambda(x,\xi)\,\big| \leq C(1+|\xi|^2+|\lambda|^{2/d})^{-k/2}.$$

Since $k > m$ we deduce that the function $\xi \mapsto a_\lambda(x,\xi)$ is integrable over $\boldsymbol{V}$. Thus the above oscillatory integral is a classical Lebesgue integral depending continuously on the parameters x, y. This proves that the kernel is continuous.

To prove part (b) notice first that there exists a constant κ depending only on the dimension r of E such that

$$\big|\operatorname{tr} a_\lambda(x,y,\xi)\big| \leq \big|a_\lambda(x,\xi)\big|, \quad \forall x,y,\xi,\lambda.$$

Thus for any compact $K \subset \Omega$ there exists a constant $C > 0$ independent of λ such that

$$\sup_{x\in K} |\operatorname{tr} K_{A_\lambda}(x,x)| \leq (2\pi)^{-m} \sup_{x\in K} \int_{\boldsymbol{V}} \big|\operatorname{tr} a_\lambda(x,\xi)\big| |d\xi|$$

$$\leq C\int_{\boldsymbol{V}} \big(1 + |\lambda|^{2/d} + |\xi|^2\big)^{-k/2} |d\xi|.$$

We set $u^2 := 1 + |\lambda|^{2/d}$ and we deduce

$$\int_{\boldsymbol{V}} \big(u^2 + |\xi|^2\big)^{-k/2} |d\xi| \stackrel{(1.1.2)}{=} u^{m-k} \frac{\boldsymbol{\sigma}_{m-1}\Gamma(p)\Gamma(k/2-p)}{2\Gamma(k/2)}, \quad p = \frac{m-2}{2}.$$

□

Let us say a few words about elliptic operators with parameters.

Definition 4.2.5. Let $a_\lambda \in \boldsymbol{S}^{k,d}_{\Lambda,\mathrm{phg}}(\Omega, E_0, E_1)$ be a classical symbol with parameters

$$a_\lambda \sim \sum_{j=0}^{\infty} a_{k-j}(x,\xi,\lambda).$$

Then a_λ is said to be an *elliptic symbol with parameters* if $a_k(x,\xi,\lambda) \in \operatorname{Hom}(E_0,E_1)$ is invertible for any $(\xi,\lambda) \in (\boldsymbol{V}\setminus\{0\}) \times \Lambda$, $|\xi| + |\lambda|^{1/d} > 1$. A properly supported classical ψdo with parameters is called *elliptic with parameters* if its symbol is elliptic with parameters. □

Example 4.2.6. Suppose E is a Hermitian vector space and $A : C^\infty(\underline{E}_\Omega) \to C^\infty(\underline{E}_\Omega)$ is a formally selfadjoint differential operator of order k such that, for any $x \in \Omega$ and any $\xi \in \boldsymbol{V}\setminus\{0\}$ the principal symbol $[\sigma_A](x,\xi)$ is a *positive definite* symmetric endomorphism of E. Then the pseudo-differential operator with parameters $\lambda - A$ is elliptic with parameters. □

Arguing exactly as in the proof of Theorem 2.9.4 we obtain the following parametric version.

Theorem 4.2.7. *Let* $A_\lambda \in \boldsymbol{\Psi}^{k,d}_0(\Omega,\Lambda,E_0,E_1)$ *and set* $a_\lambda = \sigma_A$. *Then the following statements are equivalent.*

(a) *The operator* A_λ *is elliptic with parameters.*

(b) *There exists a* ψdo *with parameters* $B_\lambda \in \boldsymbol{\Psi}^{-k,d}_0(\Omega,\Lambda,E_1,E_0)$ *such that*

$$A_\lambda B_\lambda - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty,d}(\Omega,\Lambda,E_1,E_1), \quad B_\lambda A_\lambda - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty,d}(\Omega,\Lambda,E_0,E_0).$$

(c) *There exists a* ψdo *with parameters* $B_\lambda \in \boldsymbol{\Psi}^{-k}_0(\Omega,\Lambda,E_1,E_0)$ *such that*

$$B_\lambda A_\lambda - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty,d}(\Omega,\Lambda,E_0,E_0).$$

(d) *There exists a* ψdo *with parameters* $B_\lambda \in \boldsymbol{\Psi}_0^{-k}(E_1, E_0)$ *such that*

$$A_\lambda B_\lambda - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty,d}(\Omega, \Lambda, E_1, E_1).$$

An operator B_λ *satisfying one of the equivalent properties (b),(c), (d) is called a* parametrix with parameters. □

Example 4.2.8. Let us explain how to find a parametrix (with) parameters of the operator in Example 4.2.6. The symbol of A has the form

$$\sigma_A(x,\xi) = \sum_{j=0}^{k} a_j(x,\xi)$$

where $a_j(x,\xi)$ is a homogeneous polynomial of degree j in ξ with coefficients $\operatorname{End}(E)$-valued smooth functions on Ω. Then

$$\lambda - A \in \boldsymbol{\Psi}^{k,k}(\Omega, \Lambda, E).$$

We seek $B_\lambda \in \boldsymbol{\Psi}_0^{-k,k}(\Omega, \Lambda, E))$ such that

$$(\lambda - A)B_\lambda - \mathbb{1} \in \boldsymbol{\Psi}^{-\infty,k}(\Omega, \Lambda, E).$$

The symbol b_λ of B_λ has an asymptotic expansion

$$b_\lambda \sim \sum_{\nu=0}^{\infty} b_{-k-\nu}(x,\xi,\lambda),$$

where $b_{-k-\nu}(x,\xi,\lambda)$ satisfying the quasi-homogeneity condition

$$b_{-k-\nu}(x, t\xi, t^k\lambda) = t^{-k-\nu} b_{-k-\nu}(x,\xi,\lambda),\ \ \forall t \geq 1,\ \ |\xi| + |\lambda|^{1/k} \geq 1,\ \ (\xi,\lambda) \in \boldsymbol{V} \times \Lambda. \tag{4.2.9}$$

The function $b_{-k-\nu}(x,\xi,\lambda)$ determines a unique function $\beta_{-k-\nu}(x,\xi,\lambda)$ satisfying the above quasi-homogeneity condition for any $(\xi,\lambda) \in V \times \Lambda \setminus \{(0,0)\}$. We set

$$a^h_{j,\lambda}(x,\xi) = \begin{cases} \lambda - a_k(x,\xi), & j = k \\ -a_j(x,\xi), & j < k. \end{cases}$$

Arguing as in the second proof of Theorem 2.9.4 we deduce that the sequence $(b_{-k-\nu})$ satisfies the following system of linear equations.

$$\mathbb{1} = a^h_{k,\lambda}\beta_{-k}, \tag{4.2.10a}$$

$$\beta_{-k-\nu}a^h_k + \sum_{\substack{\ell+|\alpha|+j=\nu \\ \ell<\nu}} \frac{1}{\alpha!}\partial^\alpha_\xi a^h_{k-j,\lambda} D^\alpha_x \beta_{-k-\ell} = 0, \quad \nu > 0, \tag{4.2.10b}$$

We deduce

$$\beta_{-k}(x,\xi,\lambda) = (\lambda - a_k)^{-1},$$

$$\beta_{-k-\nu} = -\Big(\sum_{\substack{\ell+|\alpha|+j=\nu \\ \ell<\nu}} \frac{1}{\alpha!}\partial^\alpha_\xi a^h_{k-j} D^\alpha_x \beta_{-k-\ell} \Big)(\lambda - a_k)^{-1},\ \ \nu \geq 1.$$

For example, for $\nu = 1$ we deduce

$$\beta_{-k-1} = -\Big(\sum_{|\alpha|+j=1} \partial^\alpha_\xi a^h_{k-j} D^\alpha_x (\lambda - a_k)^{-1} \Big)(\lambda - a_k)^{-1}$$

$$= a_{k-1}(\lambda - a_k)^{-2} + \Big(\sum_{|\alpha|=1} \partial_\xi^\alpha a_k D_x^\alpha (\lambda - a_k)^{-1} \Big) (\lambda - a_k)^{-1}.$$

For many of the applications we have in mind the operator A is a generalized Laplacian. Thus A has order 2 and its principal symbol is of the form

$$a_2(x,\xi) = |\xi|^2_{g(x)} \mathbb{1}_E,$$

where $|\xi|_{g(x)}$ denotes the norm of a covector $\xi \in T_x^*\Omega$ with respect to some Riemann metric g on Ω. In this case we deduce

$$\beta_{-2} = (\lambda - a_2)^{-1},$$

$$\beta_{-3} = (\lambda - a_2)^{-2} a_1 + (\lambda - a_2)^{-3} \sum_{|\alpha|=1} (\partial_\xi^\alpha a_2)(D_x^\alpha a_2),$$

$$\beta_{-4} = -(\lambda - a_2)^{-1} \Big(\sum_{\substack{\ell+|\alpha|+j=2 \\ \ell<\nu}} \frac{1}{\alpha!} \partial_\xi^\alpha a_{2-j}^h D_x^\alpha \beta_{-2-\ell} \Big)$$

$$= (\lambda - a_2)^{-1} \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_\xi^\alpha a_2 D_x^\alpha (\lambda - a_2)^{-1} + (\lambda - a_2)^{-1} \sum_{|\alpha|=1} \sum_{\ell=0}^{1} \partial_\xi^\alpha a_{1-\ell} D_x^\alpha \beta_{-2-\ell}$$

$$-(\lambda - a_2)^{-1} \big(a_0 \beta_{-2} + a_1 \beta_{-3} \big).$$

Now choose a smooth function

$$\varphi : \mathbb{R} \to [0,\infty), \quad \varphi(t) = \begin{cases} 0, & |t| \le \frac{1}{2}, \\ \\ 1, & |t| \ge 1, \end{cases}$$

and define

$$b_{-k-\nu} := \varphi\big(\boldsymbol{\varrho}_d(\xi,\lambda) \big) \beta_{-k-\nu}(x,\lambda).$$

Then the operator with B_λ such that

$$\sigma_{B_\lambda}(x,\xi) \sim \sum_{\nu=0}^{\infty} b_{-k-\nu}(x,\xi,\lambda)$$

will be a parametric ψdo with parameters. If we define $B_\nu(\lambda) \in \mathbf{\Psi}^{-k,k}(\Omega,\Lambda,E)$ to be the operator with symbol

$$\sigma_{B_\nu(\lambda)}(x,\xi) = \sum_{\ell=0}^{\nu} b_{-k-\ell}(x,\xi)$$

then we deduce

$$(\lambda - A) B_\nu(\lambda) - \mathbb{1} \in \mathbf{\Psi}^{-\nu-1,k}(\Omega,\Lambda,E), \quad \forall \nu \ge 0. \qquad \square$$

The change in variables formula (2.7.4b) extends to ψdo's with parameters. As in Chapter 3 we can use this fact to define ψdo's with parameters on manifolds.

Theorem 4.2.9. *Suppose (M, g) is a smooth Riemannian manifold of dimension m. Let $\boldsymbol{E} \to M$ be a smooth complex vector bundle equipped with a hermitian metric h and suppose that*

$$A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$$

is a formally selfadjoint partial differential operator of order k such that for any $x \in M$ and $\xi \in T_x^* M \setminus \{0\}$ the principal symbol $[\sigma_A](x, \xi) : \boldsymbol{E}_x \to \boldsymbol{E}_x$ is a positive definite hermitian endomorphisms, i.e.,

$$h\big([\sigma_A](x, \xi)u, u \big) > 0, \;\; \forall u \in \boldsymbol{E}_x \setminus \{0\}.$$

Then the operator $\lambda - A \in \boldsymbol{\Psi}^{k,k}(M, \Lambda, \boldsymbol{E})$ is elliptic with parameters and there exists $R > 0$ such that for any $|\lambda| > R$ the operator $(\lambda - A) : L^{k,2}(\boldsymbol{E}) \to L^2(\boldsymbol{E})$ is invertible and there exists a constant $C > 0$ independent of $\lambda \in \Lambda$, $|\lambda| > R$ such that

$$\|(\lambda - A)^{-1} u\|_{L^2} \leq C\big(1 + |\lambda|^{1/k} \big)^{-k} \|u\|_{L^2}, \;\; \forall u \in L^2(\boldsymbol{E}). \tag{4.2.11}$$

Proof. From Example 4.2.6 we deduce that the operator $(\lambda - A)$ is elliptic with parameters. Using the computations in Example 4.2.8 and arguing exactly as in the proof of Theorem 3.2.2 we can find for every $\nu > 0$ and operator $B_\nu(\lambda) \in \boldsymbol{\Psi}^{-k,k}(M, \Lambda, \boldsymbol{E})$ such that

$$S_\nu(\lambda) = \big(\lambda - A \big) B_\nu(\lambda) - \mathbb{1} \in \boldsymbol{\Psi}^{-\nu-1,k}(M, \Lambda, \boldsymbol{E}).$$

Theorem 4.2.3 implies that there exists a constant $C > 0$, independent of λ such that

$$\|S_\nu(\lambda f\|_{L^2} \leq C\big(1 + \lambda|^{1/k} \big)^{-\nu-1} \|f\|_{L^2}, \;\; \forall f \in L^2(\boldsymbol{E}).$$

If we choose $R > 0$ such that

$$C\big(1 + R^{1/k} \big)^{-\nu-1} < \frac{1}{2},$$

then we deduce that for $|\lambda| > R$ the operator

$$(\lambda - A) B_\nu(\lambda) = \mathbb{1} + S_\nu(\lambda) : L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E})$$

is invertible with inverse

$$\big(\mathbb{1} + S_\nu(\lambda) \big)^{-1} = \sum_{n=0}^\infty (-1)^n S_\nu(\lambda)^n.$$

As inverse of $(\lambda - A)$ we can take the operator

$$B_\nu(\lambda) \big(\mathbb{1} + S_\nu(\lambda) \big)^{-1}.$$

Since the norm of $(\mathbb{1} + S\nu(\lambda))$ as a bounded operator $L^2 \to L^2$ is bounded from above by

$$\sum_{n \geq 0} \frac{1}{2^n} = 2,$$

we deduce that for any $u \in L^2(\boldsymbol{E})$ we have

$$\|(\lambda - A)^{-1} u\|_{L^2} \leq 2 \|B_\nu(\lambda) u\|_{L^2}.$$

We observe that $B_\nu \in \boldsymbol{\Psi}^{-k,k}(M, \Lambda, \boldsymbol{E})$. Invoking Theorem 4.2.3 we deduce that there exists $C > 0$ independent of $\lambda \in \Lambda$ such that

$$\|B_\nu(\lambda) u\|_{L^2} \leq C(1 + |\lambda|^{1/k})^{-k} \|u\|_{L^2}, \;\; \forall u \in L^2(\boldsymbol{E}).$$

This proves (4.2.11). □

4.3. Trace class and Hilbert-Schmidt operators

We want to collect here a few basic facts about two important classes of bounded operators that will be needed for our further developments. For proofs and more information we refer to our main sources, [**DS2**, XI], [**ReSi**, VI.6], [**RSz**, §66,97,98] and [**Si**].

Suppose H is a separable, complex Hilbert space. and is a Hilbert basis. We denote by $(-,-)$ the inner product on H. It is linear in the first variable, and *conjugate* linear in the second variable. We denote by $\mathcal{B}(H)$ the collection of bounded linear operators $H \to H$.

A bounded operator $A : H \to H$ is called *nonnegative* if

- it is self-adjoint, $A = A^*$, and
- $(Ax, x) \geq 0, \forall x \in H$.

A non-negative operator is said to be *trace class* if for some Hilbert basis $(\boldsymbol{e}_n)_{n\geq 0}$ of H we have

$$\boldsymbol{Tr}(A) := \sum_{n\geq 0}(A\boldsymbol{e}_n, e_n) < \infty.$$

In fact this condition is independent of the Hilbert basis, so that, for any pair of Hilbert bases $(\boldsymbol{e}_n)_{n\geq 0}$ and $(\boldsymbol{f}_n)_{n\geq 0}$ we have

$$\sum_{n\geq 0}(A\boldsymbol{e}_n, \boldsymbol{e}_n) = \sum_{n\geq 0}(A\boldsymbol{f}_n, \boldsymbol{f}_n).$$

For any bounded operator $T : H \to H$ we set

$$|T| := (T^*T)^{1/2}.$$

The operator T is said to be *trace class* if $|T|$ is trace class. We denote by $\mathcal{I}_1$ the collection of trace class operators. For $T \in \mathcal{I}_1$ we set

$$\|T\|_1 := \boldsymbol{Tr}\,|T|.$$

Theorem 4.3.1. *(a) The function $\mathcal{I}_1 \ni T \mapsto \|T\|_1 \in [0,\infty)$ is a norm on $\mathcal{I}_1$, and $\mathcal{I}_1$ equipped with this norm is a Banach space. Moreover*

$$\|T\| \leq \|T\|_1, \quad \forall T \in \mathcal{I}_1.$$

(b) The collection $\mathcal{I}_1$ is a $*$-ideal of $\mathcal{B}(H)$, i.e., it is an ideal of the ring $\mathcal{B}(H)$ such that $T \in \mathcal{I}_1 \Longleftrightarrow T^* \in \mathcal{I}_1$. Moreover,

$$\|TS\|_1, \|ST\|_1 \leq \|S\| \cdot \|T\|_1 \;\; \forall T \in \mathcal{I}_1, \quad S \in \mathcal{B}(H).$$

(c) If $T \in \mathcal{I}_1$ then for any Hilbert basis $(\boldsymbol{e}_n)_{n\geq 0}$ the series $\sum_{n\geq 0}(T\boldsymbol{e}_n, \boldsymbol{e}_n)$ converges absolutely. Its sum is independent of the choice of the basis $(\boldsymbol{e}_n)_{n\geq 0}$. It is called the trace *of T and it is denoted by $\boldsymbol{Tr}\,T$. It defines a continuous linear map*

$$\boldsymbol{Tr} : (\mathcal{I}_1, \|-\|_1) \to \mathbb{C}.$$

Moreover

$$\boldsymbol{Tr}(AB) = \boldsymbol{Tr}(BA), \;\; \boldsymbol{Tr}(A^*) = \overline{\boldsymbol{Tr}\,A}, \;\; \forall A \in \mathcal{I}_1, \quad B \in B \in \mathcal{B}(H).$$

(d) Any trace class operator is compact.

(e) If T is compact and self-adjoint, and $(\lambda_n)_{n\geq 0}$ are its eigenvalues, counted with multiplicities then

$$T \in \mathcal{I}_1 \Longleftrightarrow \sum_{n\geq 0}|\lambda_n| < \infty.$$

Moreover, if $T \in \mathcal{I}_1$ then

$$\boldsymbol{Tr}\, T = \sum_n \lambda_n. \qquad \square$$

An operator $T \in \mathcal{B}(H)$ is called *Hilbert-Schmidt* if $T^*T \in \mathcal{I}_1$. We denote by $\mathcal{I}_2$ the space of Hilbert-Schmidt operators. Note that $\mathcal{I}_1 \subset \mathcal{I}_2$.

Theorem 4.3.2. *(a) The space $\mathcal{I}_2$ is an $*$-ideal of $\mathcal{B}(H)$.*
(b) $A \in \mathcal{I}_1$ if and only if $A = BC$, for $B, C \in \mathcal{I}_2$.
(c) If we define

$$(-,-)_2 : \mathcal{I}_2 \times \mathcal{I}_2 \to \mathbb{C}, \;\; (A,B)_2 := \boldsymbol{Tr}(AB^*),$$

then $(-,-)_2$ defines a Hilbert space structure on $\mathcal{I}_2$. For $T \in \mathcal{I}_2$ we set

$$\|T\|_2 = \sqrt{(T,T)_2}.$$

Then

$$\|T\| \leq \|T\|_2 \leq \|T\|_1, \;\; \|ST\|_1 \leq \|S\|_2 \cdot \|T\|_2, \;\; \forall S, T \in \mathcal{I}_2.$$

(d) Any Hilbert-Schmidt operator is compact. Moreover if $T \in \mathcal{B}(H)$ is self-adjoint, then $T \in \mathcal{I}_2$ if and only if

$$\sum_{n \geq 0} \lambda_n^2 < \infty,$$

where as in Theorem 4.3.1 the summation is carried over all the eigenvalues of T counted with their multiplicities. $\square$

Example 4.3.3. Suppose (X,μ) is a measure space. Then a bounded operator $T : L^2(X,\mu) \to L^2(X,\mu)$ is Hilbert-Schmidt if and only if there exists $K \in L^2(X \times X, \mu \times \mu)$. such that

$$Tf(x) = T_K f(x) := \int_X K(x,y) f(y)\, d\mu(y), \;\; \forall f \in L^2(X,\mu).$$

In this case we have

$$\|T_K\|_2 = \|K\|_{L^2}.$$

Observe that $(T_K)^* = T_{K^\dagger}$, where

$$K^\dagger(x,y) := \overline{K(y,x)}.$$

If $K_1, K_2 \in L^2(X \times X, \mu \times \mu)$ then $T_{K_1} \circ T_{K_2} = T_{K_1 * K_2}$, where

$$K_1 * K_2(x,y) := \int_X K_1(x,z) K_2(z,y) d\mu(z).$$

In this case $T_{K_1 * K_2} \in \mathcal{I}_1$ and

$$\begin{aligned} \boldsymbol{Tr}\, T_{K_1 * K_2} = (K_1, K_2^\dagger)_{L^2} &= \int_{X \times X} K_1(x,y) K_2(y,x) d\mu(x) d\mu(y) \\ &\stackrel{?}{=} \int_X K_1 * K_2(x,x) d\mu(x). \end{aligned} \tag{4.3.1}$$

We left a question mark over the last equality since $K_1 * K_2$ is a measurable function, defined only almost everywhere and thus we may be able to assign a meaning to its restriction to the diagonal on $X \times X$ that has null measure. If both K_1 and K_2 are continuous then the last equality is valid.

This result has an obvious extension to operator $T : L^2(X, E, \mu) \to L^2(X, E, \mu)$ where E is a finite dimensional complex hermitian space and $L^2(XE, \mu)$ denotes the space of L^2-functions $f : X \to E$. In this case the kernel is a function $K : X \times X \to \operatorname{End}(E)$ and

$$(T_K)^* = T_{K^\dagger}, \;\; K^\dagger(x, y) := K(y, x)^*. \qquad \square$$

Proposition 4.3.4. *Consider the real Euclidean space $\boldsymbol{V}$ of dimension m, and suppose $A \in \boldsymbol{\Psi}_0^{-\ell}(\boldsymbol{V})$ is a properly supported ψdo of order $-\ell$ with symbol $\sigma(x, \xi)$ such that $\sigma_A(x, \xi) = 0$ for $|x| \gg 0$. Then the operator $A : C_0^\infty(\boldsymbol{V}) \to C_0^\infty(\boldsymbol{V})$ induces a Hilbert-Schmidt operator $L^2(\boldsymbol{V}) \to L^2(\boldsymbol{V})$ if $\ell > m/2$.*

Proof. We set

$$\widehat{\sigma}(\eta, \xi) := \int_{\boldsymbol{V}} e^{-\boldsymbol{i}(x,\eta)} \sigma(x, \xi)\, |dx|_*.$$

Let $f \in C_0^\infty(\boldsymbol{V})$. Arguing as in the proof of (2.8.4) we deduce

$$\boldsymbol{\mathfrak{F}}(Af) = \int_{\boldsymbol{V}} \underbrace{\widehat{\sigma}(\eta - \xi, \xi)}_{K(\eta,\xi)} \widehat{f}(\xi)\, |d\xi|_*.$$

Using the notations in Example 4.3.3 we can rewrite the above equality $\boldsymbol{\mathfrak{F}} \circ A = T_K \circ \boldsymbol{\mathfrak{F}}$ so that $A = \boldsymbol{\mathfrak{F}}^{-1} T_K \boldsymbol{\mathfrak{F}}$.

Since the Fourier transform is an isometry $L^2(\boldsymbol{V}) \to L^2(\boldsymbol{V})$ it suffices to show that the kernel K is in $L^2(\boldsymbol{V} \times \boldsymbol{V})$. Since σ has compact support in the x-variable we deduce that for any $N > 0$ there exists a constant $C > 0$ such that

$$|\widehat{\sigma}(\eta - \xi, \xi)| \leq C \langle \eta - \xi \rangle^{-N} \langle \xi \rangle^{-\ell}.$$

We deduce that if $N > m/2$ and then for any $\xi \in \boldsymbol{V}$ we have

$$\int_{\boldsymbol{V}} |K(\eta, \xi)|^2\, |d\eta| \leq C \langle \xi \rangle^{-2\ell} \int_{\boldsymbol{V}} \langle \eta - \xi \rangle^{-2N}\, |d\eta|$$

$(\zeta := \eta - \xi)$

$$= C \langle \xi \rangle^{-2\ell} \int_{\boldsymbol{V}} \langle \zeta \rangle^{-2N}\, |d\zeta| \stackrel{(1.1.2)}{=} C(m, N) \langle \xi \rangle^{-2\ell},$$

for some constant $C(m, N)$ depending only on m and N. Since $\ell > m/2$ we deduce that the function $\xi \mapsto \langle \xi \rangle^{-2\ell}$ is integrable. The Fubini-Tonnelli theorem now implies that $K \in L^2(\boldsymbol{V} \times \boldsymbol{V})$. $\square$

Corollary 4.3.5. *Suppose (M, g) is a compact Riemann manifold of dimension m, $\boldsymbol{E} \to M$ is a smooth, complex hermitian vector bundle of rank r and $A \in \boldsymbol{\Psi}^{-\ell}(\boldsymbol{E})$ is a ψdo of order $-\ell < -m/2$. Then A induces a Hilbert-Schmidt operator $A : L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E})$.*

Proof. We follow closely the approach in the proof of Theorem 3.2.2. Choose a finite open cover $(\mathcal{O}_i)_{i \in I}$ of M by coordinate domains, and let $(\eta_i)_{i \in I}$, $\eta_i \in C_0^\infty(\mathcal{O}_i)$ be a partition of unity subordinated to the cover $(\mathcal{O}_i)_{i \in I}$. Next, choose $\varphi_i \in C_0^\infty(\mathcal{O}_i)$ such that $\varphi_i \equiv 1$ on an open neighborhood $\mathcal{N}_i$ of $\operatorname{supp} \eta_i$ in $\mathcal{O}_i$. We define

$$A' = \sum_i \eta_i A \varphi_i.$$

Arguing as in the proof of Theorem 3.2.2 we deduce that A' is a ψdo and $A' - A$ is a smoothing operator. In particular, we deduce that $A' - A$ is Hilbert-Schmidt since its Schwartz kernel is smooth thus L^2. Proposition 4.3.4 implies that each of the operators $\eta_i A\varphi_i$ is Hilbert-Schmidt. Hence A is Hilbert-Schmidt and so is A. □

Corollary 4.3.6. *Suppose (M,g) is a compact Riemann manifold of dimension m, $\boldsymbol{E}\to M$ is a smooth, complex hermitian vector bundle of rank r and $A\in\boldsymbol{\Psi}^{-\ell}(\boldsymbol{E})$ is a ψdo of order $-\ell<-m$. Then A induces a trace class operator $L^2(\boldsymbol{E})\to L^2(\boldsymbol{E})$.*

Proof. Let observe that for any $k>0$ there exists a selfadjoint, positive definite elliptic operator $\Lambda_k\in\boldsymbol{\Psi}^k(\boldsymbol{E})$. Indeed, we can find an operator $S\in\boldsymbol{\Psi}^{k/2}(\boldsymbol{E})$ such that

$$[\sigma_S](x,\xi)=|\xi|_g^{k/2}\mathbb{1}_{\boldsymbol{E}_x},\;\;\forall x\in M,\;\;\xi\in T_x^*M\setminus 0.$$

Then the operator $S^*S\in\boldsymbol{\Psi}^k(\boldsymbol{E})$ is self-adjoint, elliptic and nonnegative definite. Thus, for some constant $C_k>0$ the operator $\Lambda_k=S^*S+C_k$ is elliptic, self-adjoint and positive definite. In particular, Λ_k defines a continuous bijective operator $\Lambda_k: C^\infty(\boldsymbol{E})\to C^\infty(\boldsymbol{E})$. Its inverse is continuous[2] and it is a ψdo of order $-k$.

Observe now that $T=\Lambda_{\ell/2}\Lambda_{\ell/2}A$ is a ψdo of order 0 and thus defines a bounded operator $L^2(\boldsymbol{E})\to L^2(\boldsymbol{E})$. Next we observe that $A=(\Lambda_{\ell/2}^{-1})^2T$. By Corollary 4.3.5 the induced operator $\Lambda_{-\ell/2}^{-1}: L^2(\boldsymbol{E})\to L^2(\boldsymbol{E})$ is Hilbert-Schmidt so that $(\Lambda_{\ell/2}^{-1})^2$ is trace class. Since $\mathcal{J}_1$ is an ideal, we conclude that A is trace class. □

From (2.4.4) we deduce that the Schwartz kernel K_A of an operator $A\in\boldsymbol{\Psi}^{-\ell}(\boldsymbol{E})$, $\ell>m$ is continuous, and we would like to conclude that

$$\boldsymbol{Tr}\,A=\int_M \operatorname{tr} K_A(x,x)\,|dV_g(x)|.$$

This is however not necessarily true (see [**GGL**, §5.3]). Still, using the discussion in Example 4.3.3 we salvage something.

Corollary 4.3.7. *Suppose (M,g) is a compact Riemann manifold of dimension m, $\boldsymbol{E}\to M$ is a smooth, complex hermitian vector bundle of rank r and $A\in\boldsymbol{\Psi}^{-\ell}(\boldsymbol{E})$ is a ψdo of order $-\ell<-2m$. Then A induces a trace class operator $L^2(\boldsymbol{E})\to L^2(\boldsymbol{E})$ and if $K_A\in C^{-\infty}\big(\operatorname{End}(\boldsymbol{E})\big)$ is its Schwartz kernel then*

$$\boldsymbol{Tr}\,A=\int_M \operatorname{tr} K_A(x,x)\,|dV_g(x)|. \tag{4.3.2}$$

Moreover, there exists a constant $C>0$ that depends only on the geometry of M and E *such that*

$$\|A\|_1\leq C\Big(\int_{M\times M}|K_A(x,y)|^2|dV_{g\times g}(x,y)|\Big)^{1/2} \tag{4.3.3}$$

Proof. Consider again the operators Λ_k used in the proof of Corollary 4.3.6. We have $A=\Lambda_{\ell/2}^{-1}(\Lambda_{\ell/2}A)$. Both ψdo's $\Lambda_{\ell/2}^{-1}$ and $(\Lambda_{\ell/2}A)$ have order $-\ell/2<-m$. Hence they are Hilbert-Schmidt and (4.3.2) conclusion follows from the discussion in Example 4.3.3 . To prove (4.3.3) we observe that

$$\|A\|_1\leq\|\Lambda_{\ell/2}^{-1}\|\cdot\|\Lambda_{\ell/2}A\|_1\leq\|\Lambda_{\ell/2}^{-1}\|\cdot\|\Lambda_{\ell/2}\|_2\cdot\|A\|_2.$$

□

[2] We can see this in two ways, either invoking the open mapping theorem for Frèchet spaces, or using elliptic estimates.

Remark 4.3.8 (***Another word of warning!***). At this point we need to interrupt our line of thought and comment on an ambiguity built in the above equality. As explained in (1.4.2), the inclusion of

$$U : C^{\infty}(\boldsymbol{E} \boxtimes \boldsymbol{E}^{\vee}) \hookrightarrow C^{-\infty}(E \boxtimes \boldsymbol{E}^{\vee})$$

depends on the choice of metric. This affects all the local computations. We want to explain how. To keep the notation at bay, let us assume that $\boldsymbol{E}$ is the trivial complex line bundle, so we are dealing with operators acting on functions.

The Schwartz kernel of a ψdo determines an operator

$$T_K : C_0^{\infty}(M) \to C^{-\infty}(M),$$

but throughout this chapter we consistently regarded as an operator $C_0^{\infty}(M) \to C^{\infty}(M)$. When doing so we have implicitly used the map $C^{\infty}(M) \hookrightarrow C^{-\infty}(M)$ which is metric dependent. This is not the only tacit identification that we used. More precisely, we have identified the Schwartz kernel with a continuous function, so that we have implicitly used the embedding

$$C^0(M \times M) \hookrightarrow C^{-\infty}(M \times M)$$

which is also metric dependent. Suppose g_0, g_1 are two metrics on M. There exists a positive function ρ such that

$$|dV_{g_1}(x)| = \rho(x)|dV_{g_0}(x)|.$$

Informally, we can write

$$\rho(x) = |dV_{g_1}(x)|/|dV_{g_0}(x)|.$$

Suppose are given a Schwartz kernel $K \in C^{-\infty}(M \times M)$ that is smooth. This means that there exist two smooth functions $K_0, K_1 \in C^{\infty}(M \times M)$ such that for any $w \in C_0^{\infty}(M \times M)$ we have

$$\langle K, w\rangle = \int_{M\times M} K_0(x,y)w(x,y)|dV_{g_0 \boxtimes g_0}(x,y)| = \int_{M\times M} K_1(x)w(x)|dV_{g_1\boxtimes g_1}(x)|$$

$$= \int_{M\times M} K_1(x,y)w(x)\rho(x)\rho(y)|dV_{g_1\boxtimes g_1}(x,y)|.$$

Hence

$$K_0(x,y) = K_1(x,y)\rho(x)\rho(y).$$

This implies that

$$\int_M K_1(x,x)\,|dV_{g_1}(x)| = \int_M \frac{1}{\rho^2} K_0(x,x)\rho\,|dV_{g_0}(x)|,$$

i.e.,

$$\int_M K_1(x,x)\,|dV_{g_1}(x)| = \int_M \frac{1}{\rho(x)} K_0(x,x)\rho\,|dV_{g_0}(x)|, \quad \rho = |dV_{g_1}(x)|/|dV_{g_0}(x)|. \tag{4.3.4}$$

The distribution K also determines a continuous linear operator

$$T_K : C_0^{\infty}(M) \to C^{-\infty}(M),$$

such that, for any $u \in C_0^{\infty}(M)$ we can identify $T_K u$ with a smooth function on M. We can do this in two ways: using the identification given by the metric g_0, or using that given by g_1. In any case we obtain two smooth functions $v_0 = T_{K,g_0}u$ and $v_1 = T_{K,g_1}u$ related by the equality

$$\int_M v_0(x)v(x)\,|dV_{g_0}(x)| = \langle K, vu\rangle = \int_M v_1(x)v(x)\,|dV_{g_1}(x)|, \quad \forall v \in C_0\infty(M).$$

We deduce that $v_0 = \rho v_1$.

4.4. The heat kernel

Suppose (M, g) is a smooth, compact Riemann manifold of dimension m, $\boldsymbol{E} \to M$ is a smooth complex vector bundle over M of rank r and h is a hermitian metric on $\boldsymbol{E}$.

A partial differential operator of order k $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ is called *admissible* if the following conditions are satisfied.

- It is elliptic and formally self-adjoint.
- Its principal symbol is positive definite, i.e., for any $x \in M$ and any $\xi \in T_x^*M \setminus \{0\}$ the operator
$$[\sigma_A](x, \xi) : \boldsymbol{E}_x \to \boldsymbol{E}_x$$
is self-adjoint and positive definite.

The spectral decomposition theorem implies that the spectrum of A is real, discrete and consists only of eigenvalues of finite multiplicity. Theorem 4.2.9 implies that there exists $R > 0$ such that

$$\operatorname{spec}(A) \subset (-R, \infty). \tag{4.4.1}$$

We can thus label the eigenvalues of A

$$-R < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots \nearrow \infty$$

such that in the sequence $(\lambda_n)_{n\geq 0}$ each eigenvalue of A appears as many times as its multiplicity.

We fix a Hilbert basis $(\boldsymbol{\phi}_n)_{n\geq 0}$ of $L^2(\boldsymbol{E})$ such that

$$A\boldsymbol{\phi}_n = \lambda_n \boldsymbol{\phi}_n, \quad \forall n \geq 0.$$

For any $t > 0$ we define a bounded operator

$$e^{-tA} : L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E}),$$

$$e^{-tA}\Big(\sum_{n\geq 0} u_n \boldsymbol{\phi}_n\Big) = \sum_{n\geq 0} e^{-t\lambda_n} u_n \boldsymbol{\phi}_n, \quad \forall u = \sum_{n\geq 0} u_n \boldsymbol{\phi}_n \in L^2(\boldsymbol{E}).$$

The series $\sum_{n\geq 0} |e^{-t\lambda_n} u_n|^2$ is convergent since

$$|e^{-\lambda_n t} u_n|^2 \leq e^{-2t\lambda_0} |u_n|^2, \quad \forall n \geq 0,$$

and the series $\sum_{n\geq 0} |u_n|^2$ is convergent.

We want to prove that e^{-tA} is a trace class operator, i.e.,

$$\boldsymbol{Tr}\big(\, e^{-tA}\,\big) := \sum_{n\geq 0} e^{-t\lambda_n} < \infty, \quad \forall t > 0$$

and then investigate the behavior of $\boldsymbol{Tr}\big(\, e^{-tA}\,\big)$ as $t \searrow 0$. The next result will play a key role in this investigation.

Proposition 4.4.1. *Suppose $S_\lambda \in \boldsymbol{\Psi}^{-\nu,d}(M, \Lambda, \boldsymbol{E})$, $\nu > 0$. Then for any $t > 0$ the integral*

$$\mathcal{L}_S := \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma_R} e^{-\lambda t} S_\lambda d\lambda \tag{4.4.2}$$

is absolutely convergent with respect to the norm on the space bounded operator on $L^2(\boldsymbol{E})$, and it is independent of the parameter R defining the path γ_R. Moreover, the operator $\mathcal{L}_S$ is smoothing, and for $j > 0$ sufficiently large we have

$$\boldsymbol{Tr}\,\mathcal{L}_S = \frac{1}{2\pi \boldsymbol{i} t^j}\int_{\gamma_R} e^{-\lambda t}\,\boldsymbol{Tr}\,\partial_\lambda^j S_\lambda d\lambda. \tag{4.4.3}$$

Proof. Denote by $\|S\|_{L^2,L^2}$ the norm of a bounded operator $S : L^2(\boldsymbol{E}) \to L^2(\boldsymbol{E})$. To prove the convergence we use (4.2.6) to conclude that there exists a constant $C > 0$ independent of $\lambda \in \Gamma_R$ such that

$$\|S_\lambda\|_{L^2,L^2} \leq C(1 + |\lambda|^{1/d})^{-\nu}.$$

Since $\mathbf{Re}\,\lambda \to \infty$ as $|\lambda| \to \infty$ on γ_R we deduce that this (operator valued) integral is absolutely convergent to a bounded operator. Since

$$|\lambda|^k e^{-\lambda t}\|S_\lambda\|_{L^2,L^2} \to 0$$

as $|\lambda| \to \infty$ along γ_R we deduce from an integration by parts that

$$\mathcal{L}_S = \frac{1}{2\pi \boldsymbol{i} t^m}\int_{\gamma_R} e^{-\lambda t}\partial_\lambda^j S_\lambda d\lambda,\ \ \forall j \geq 0.$$

From (4.2.5) we deduce that $\partial_\lambda^j S_\lambda$ is a ψdo of order $-\nu - jd$. We deduce that for any $k > 0$ we can find $j = j(k)$ such that the Schwartz kernel $\mathcal{K}_{\partial_\lambda^j S_\lambda}$ of $\partial_\lambda^j S_\lambda$ is of class C^k. Moreover, (4.2.8) shows that the integral

$$\frac{1}{2\pi \boldsymbol{i} t^j}\int_{\gamma_R} e^{-\lambda t}\mathcal{K}_{\partial_\lambda^j S_\lambda} d\lambda.$$

is convergent and defines a section of $\boldsymbol{E} \boxtimes \boldsymbol{E}^\vee$ of class C^k representing the Schwartz kernel of $\mathcal{L}_S$. This shows that $\mathcal{L}_S$ is smoothing. The fact that it is independent of R follows from the fact that $\lambda \mapsto S_\Lambda$ so that the integral of $e^{-\lambda t}S_\lambda$ along any closed path contained in Λ is trivial. We we denote by γ_R^n the portion of the path γ_R in the region $\mathbf{Re}\,\lambda < n$ then we deduce that for any $R_1 < R_2$ and any $n > 0$ we have (see Figure 4.3)

$$\int_{\gamma_{R_1}^n - \gamma_{R_2}^n} e^{-\lambda t}S_\lambda d\lambda = 0.$$

We then let $n \to \infty$ in the above equality.

To prove (4.4.3) we first need prove that if j is sufficiently large

$$\int_{\gamma_R}\left\| e^{-t\lambda}\partial_\lambda^j S_\lambda\right\|_1 d\lambda < \infty. \tag{4.4.4}$$

Recall that $\partial_\lambda^j S_\lambda$ is an operator of order $-\nu - jd$. If we choose j such that $\nu + jd > 2m$, then Corollary 4.3.7 implies that $\partial_\lambda^j S_\lambda$ is trace class. Using (4.4.3) and (4.2.8) we deduce

$$\|\partial_\lambda^j S_\lambda\|_1 \leq C\big(1 + |\lambda|^{1/d}\big)^{-(\nu + jd - m)},$$

for some constant $C > 0$ depending only on the symbol of R and the geometry of M. This proves the convergence of (4.4.4) . To prove (4.4.3) it suffices to take the traces of both sides of (4.4.2).

□

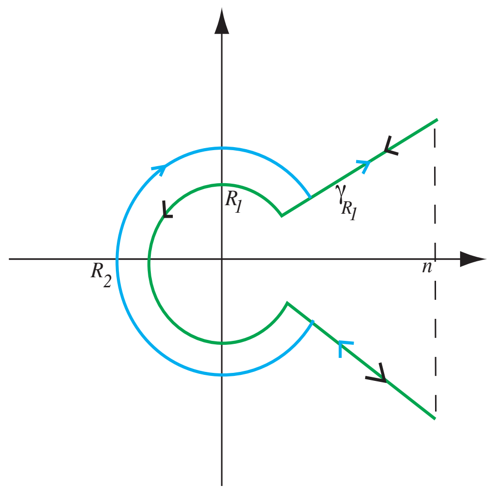

Figure 4.3. *The contours* $\gamma^n_{R_1}$ and $-\gamma^n_{R_2}$.

Corollary 4.4.2. *Fix $R > 0$ sufficiently large such that (4.4.1) holds and consider the path γ_R depicted in Figure 4.1. Then the following hold.*

(a) *$(\lambda - A)$ is invertible for any $\lambda \in \gamma_R$.*

(b) *For any $\ell \geq 0$ and any $t > 0$ we have*

$$e^{-tA} = \frac{(-1)^\ell \ell!}{2\pi \boldsymbol{i} t^\ell} \int_{\gamma_R} e^{-t\lambda} (\lambda - A)^{-(\ell+1)}\, d\lambda, \tag{4.4.5}$$

where the integral in the right hand side is absolutely convergent.

Proof. Part (a) follows from (4.4.1). The convergence follows from Proposition 4.4.1. To prove that $S_A(t) = e^{-tA}$ for $t > 0$ it suffices to show that

$$S_A(t)\boldsymbol{\phi}_n = e^{-t\lambda_n}\boldsymbol{\phi}_n, \quad \forall n \geq 0. \tag{4.4.6}$$

Fix $n \geq 0$, a real number $L > \lambda_n$ and form the path γ_R^L as in Figure 4.4. Then for any $\lambda \in \gamma_R^L \cup \gamma_R$ we have

$$e^{-t\lambda}(\lambda - A)^{-(\ell+1)}\boldsymbol{\phi}_n = \underbrace{e^{-t\lambda}(\lambda - \lambda_n)^{-(\ell+1)}}_{f_n(\lambda)} \boldsymbol{\phi}_n.$$

Hence

$$S_A(t)\boldsymbol{\phi}_n = \frac{(-1)^\ell \ell!}{2\pi \boldsymbol{i} t^\ell} \left(\int_{\gamma_R} f_n(\lambda) d\lambda \right) \boldsymbol{\phi}_n.$$

The function $f_n(\lambda)$ has a single pole inside the contour γ_R^L located at λ_n. The residue at this pole is

$$\frac{d^\ell f}{d\lambda^\ell}|_{\lambda = \lambda_n} = \frac{(-t)^\ell}{\ell!} e^{-t\lambda_n}.$$

The residue theorem implies that

$$\frac{(-1)^\ell \ell!}{2\pi \boldsymbol{i} t^\ell} \int_{\gamma_R^L} e^{-t\lambda}(\lambda - A)^{-(\ell+1)} \boldsymbol{\phi}_n\, d\lambda = \frac{(-1)^\ell \ell!}{2\pi \boldsymbol{i} t^\ell} \left(\int_{\gamma_R^L} f_n(\lambda) d\lambda \right) \boldsymbol{\phi}_n = e^{-t\lambda_n}\boldsymbol{\phi}_n.$$

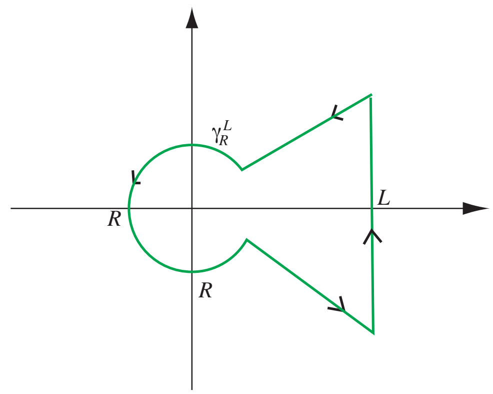

Figure 4.4. *The contour γ_R^L.*

The equality (4.4.6) now follows from the following elementary equality whose proof is left to the reader as an exercise.

$$\int_{\gamma_R} f_n(\lambda)d\lambda = \lim_{L\to\infty}\int_{\gamma_R^L} f_n(\lambda)d\lambda. \tag{4.4.7}$$

□

Corollary 4.4.3. *For any $t > 0$ the operator e^{-tA} is smoothing, trace class and*

$$\boldsymbol{Tr}\, e^{-tA} = \frac{(-1)^\ell \ell!}{2\pi \boldsymbol{i} t^\ell}\int_{\gamma_R} e^{-t\lambda}\,\boldsymbol{Tr}(\lambda - A)^{-(\ell+1)}\,d\lambda,\ \ \forall \ell + 1 > \frac{2m}{d}. \tag{4.4.8}$$

Definition 4.4.4. Let A be an admissible operator. Then the Schwartz kernel of e^{-tA} is called the *heat kernel* of A. □

Definition 4.4.5. Suppose $f : (0,\infty) \to \mathbb{C}$ is a smooth function, $(s_j)_{j\geq 0}$ is strictly increasing sequence of real numbers such that $s_j \nearrow \infty$, and $(c_j)_{j\geq 0}$ is a sequence of complex numbers. We say that formal series $\sum_{j\geq 0} c_j t^{s_j}$ is an asymptotic expansion of $f(t)$ as $t \searrow 0$, and we write this

$$f(t) \sim_0 \sum_{j\geq 0} c_j t^{s_j},$$

if for any $k > 0$ we have

$$\Big|\, f(t) - \sum_{j=0}^{k} c_j t^{s_j}\,\Big| = O(t^{s_{k+1}})\ \text{ as } t \searrow 0.$$ □

Theorem 4.4.6 (Heat kernel expansion)**.** *Let (M, g) be a smooth, compact, Riemann manifold of dimension m, and $\boldsymbol{E} \to M$ is a smooth, complex vector bundle of rank r equipped with a Hermitian metric h. Suppose $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ is an admissible elliptic partial differential operator of order k which is also nonnegative definite, i.e.,*

$$\int_M (Au(x), u(x))_h\, |dV_g(x)| \geq 0,\ \ \forall u \in C^\infty(\boldsymbol{E}).$$

Then as $t \to 0$ we have the asymptotic expansion

$$\boldsymbol{Tr}\, e^{-tA} \sim_0 t^{-\frac{m}{k}} \sum_{p\geq 0} c_p t^{\frac{p}{k}}, \tag{4.4.9}$$

where the coefficients $c_p = c_p(A)$ can be expressed as integrals

$$c_p = \int_M \boldsymbol{e}_p(x)\,|dV_g|,$$

where for each $x \in M$ the quantity $\boldsymbol{e}_p(x)$ is a universal (but horrendous) expression in the symbol of A and its partial derivatives at x.

In particular, when A is a generalized Laplacian we have

$$c_0(A) = (4\pi)^{-m/2} r \operatorname{vol}_g(M). \tag{4.4.10}$$

Proof. The key trick is contained in the following technical result.

Lemma 4.4.7. *Let Ω be an open subset of the Euclidean space $\boldsymbol{V}$, and let E be a complex Hermitian vector space of dimension r. Suppose we are given the following data.*

- *A compactly supported function $\eta \in C_0^\infty(\Omega)$.*
- *A bounded continuous function $\rho : \Omega \to (0,\infty)$.*
- *A polyhomogeneous symbol with parameters $b = b(x,\lambda,\xi) \in \mathbf{S}^{-\nu,d}_{\Lambda,\mathrm{phg}}(\Omega, E)$.*

For every $j \geq 0$ we denote by $K^{(j)}_{b,\lambda}$ the Schwartz kernel of the operators $\mathbf{Op}(\eta\partial_b^{(j)})$, where

$$b^{(j)} := \partial_\lambda^j b.$$

Then the following hold.

(a) If $\nu + jd > m$, then $K^{(j)}_{b,\lambda}$ is continuous, the integral

$$\frac{1}{t^j}\int_{\gamma_R} e^{-t\lambda} K^{(j)}_{b,\lambda}(x,y)\,d\lambda$$

converges absolutely and uniformly in $x, y \in \Omega$. It is independent of R and j, and determines for every $t > 0$ a continuous, bounded map

$$\mathcal{L}_t[\eta b] : \Omega \times \Omega \to E \otimes E^*.$$

(b) There exists a constant $C > 0$ such that

$$\int_\Omega \big|\operatorname{tr}\mathcal{L}_t[\eta b](x,x)\,\big|\,\rho(x)|dx| \leq C t^{-1+\frac{\nu-m}{d}} \quad \forall t \in (0,1).$$

(c) If

$$b \sim \sum_{k\geq 0} b_{-\nu-k}(x,\lambda,\xi)$$

Then

$$\int_\Omega \operatorname{tr}\mathcal{L}_t[\eta b](x,x)\,\rho(x)|dx| \sim_0 t^{-1+\frac{\nu-m}{d}} \sum_{k\geq 0} c_k t^{k/d},$$

where

$$c_k = (2\pi)^{-m/2}\int_\Omega \operatorname{tr}\mathcal{L}_{t=1}[\eta b^{(j)}_{-\nu-k}](x,x)\,\rho(x)|dx|, \tag{4.4.11}$$

for any $j > 0$ such that $\nu + k + jd > m$.

Proof of Lemma 4.4.7. (a) Assume $\nu + jd > m$. Then

$$K^{(j)}_{b,\lambda}(x,y) = (2\pi)^{-m/2} \int_{\boldsymbol{V}} e^{\boldsymbol{i}(x-y,\xi)} \eta(x) b^{(j)}(x,\lambda,\xi)\,|d\xi|_*. \tag{4.4.12}$$

This integral is absolutely convergent since

$$|b^{(j)}(x,\lambda,\xi)| = O(\boldsymbol{\varrho}_d(\lambda,\xi)^{-\nu-jd})$$

We deduce that $K^{(j)}_{b,\lambda}$ depends holomorphically on λ and

$$\lim_{\mathbf{Re}\,\lambda\to\infty} e^{-t\lambda} \sup_{x,y\in\Omega} |K^{(j)}_{b,\lambda}(x,y)| = 0, \quad \forall t > 0.$$

From the equality

$$K^{(j+1)}_{b,\lambda} = \partial_\lambda K^{(j)}_{b,\lambda}$$

we deduce by an integration by parts that

$$\int_{\gamma_R} e^{-t\lambda} K^{(j)}_{b,\lambda} d\lambda = \frac{1}{t} \int_{\gamma_R} e^{-t\lambda} K_{b,j+1,\lambda} d\lambda, \quad \forall t > 0.$$

The independence on R is proved exactly as in Proposition 4.4.1. This proves (a).

To prove (b) observe that for $\nu + jd > m$ we have

$$\mathcal{L}_t[\eta b](x,y) = t^{-j} \int_{\gamma_{1/t}} e^{-t\lambda} \eta(x) K^{(j)}_{b,\lambda}(x,y) d\lambda = t^{-j-1} \int_{\gamma_1} e^{-\mu} K^{(j)}_{b,t^{-1}\mu}(x,y) d\mu. \tag{4.4.13}$$

Now observe that for any $z \in \Lambda$, and any $x, y \in \Omega$ we have

$$\big|\, K^{(j)}_{b,z}(x,y) \,\big| \leq C \int_{\boldsymbol{V}} \big| b^{(j)}(x,z,\xi) \big|\, |d\xi| \leq C \int_{\boldsymbol{V}} (1 + |z|^{2/d} + |\xi|^2)^{-(\nu+jd)/2} |d\xi|.$$

In the last integral we make the substitutions $z = t^{-1}\mu$, $\xi = t^{-1/d}\eta$ and we deduce

$$\big|\, K^{(j)}_{b,t^{-1}\mu}(x,y) \,\big| \leq C t^{j+\frac{(\nu-m)}{d}} \int_{\boldsymbol{V}} (t^{2/d} + |\mu|^2 + |\eta|^2)^{-(\nu+jd)/2}\, |d\eta|$$

$$\overset{(1.1.2)}{=} C t^{j+\frac{(\nu-m)}{d}} (t^{2/d} + |\mu|^{2/d})^{\frac{m}{2}-\frac{(\nu+jd)}{2}}$$

For any $\mu \in \gamma_1$ we have $|\mu| > 1$ and we conclude

$$\big| K^{(j)}_{b,t^{-1}\mu}(x,y) \,\big| \leq C t^{j+\frac{(\nu-m)}{d}}, \quad \forall \mu \in \gamma_1.$$

Using this last inequality in (4.4.13) we obtain the estimate (b).

To prove (c) observe that for j sufficiently large we have

$$\mathcal{L}_t[\eta b_{-\nu-k}](x,x) = t^{-j} \int_{\gamma_{1/t}} e^{-t\lambda} K^{(j)}_{b_{-\nu-k},\lambda}(x,x) d\lambda = t^{-1} \int_{\gamma_1} e^{-\mu} K^{(j)}_{b_{-\nu-k},t^{-1}\mu}(x,x) d\mu$$

$$= (2\pi)^{-m/2} t^{-1-j} \int_{\gamma_1} e^{-\mu} \left(\int_{\boldsymbol{V}} \eta(x) b^{(j)}_{-\nu-k}(x, t^{-1}\mu, \xi) |d\xi|_* \right) d\mu$$

$(\xi = t^{-1/d}\eta)$

$$= (2\pi)^{-m/2} t^{-1-j-\frac{m}{d}} \int_{\gamma_1} e^{-\mu} \left(\int_{\boldsymbol{V}} \eta(x) b^{(j)}_{-\nu-k}(x, t^{-1}\mu, t^{-1/d}\eta) |d\eta|_* \right) d\mu$$

Now use the fact that

$$b^{(j)}_{-\nu-k}(x, t^{-1}\mu, t^{-1/d}\eta) = t^{\frac{\nu+k}{d} j} b^{(j)}_{-\nu-k}(x,\mu,\eta),$$

for $|\mu|^{1/d}+|\xi|\geq 1$, and $t\in(0,1)$, and the fact that $|\mu|\geq 1$ on γ_1 to deduce

$$\mathcal{L}_t[\eta(x)b_{-\nu-k}](x,x)=(2\pi)^{-m/2}t^{-1+\frac{\nu+k-m}{d}}\int_{\gamma_1}e^{-\mu}\left(\int_{\boldsymbol{V}}\eta(x)b^{(j)}_{-\nu-k}(x,\mu,\eta)\,|d\eta|_*\right)d\mu$$

For any $k>0$ we set

$$r_k=b-\underbrace{\sum_{0\leq\ell<k}b_{-\nu-\ell}}_{\beta_k}.$$

Then $r_k\in\mathbf{S}^{-\nu-k,d}_{\Lambda,\mathrm{phg}}(\Omega)$, and from (b) we deduce

$$\int_\Omega\bigl|\operatorname{tr}\mathcal{L}_t[\eta r_k](x,x)\bigr|\,\rho(x)|dx|\leq Ct^{-1+\frac{\nu+k-m}{d}}\;\;\forall t\in(0,1).$$

Using (c) we deduce that

$$\int_\Omega\bigl|\operatorname{tr}\mathcal{L}_t[\eta\beta_k](x,x)\bigr|\,\rho(x)|dx|=t^{-1+\frac{\nu-m}{d}}\sum_{0\leq\ell<k}c_\ell t^{\ell/d},$$

where c_ℓ are defined as in (4.4.11). This concludes the proof of Lemma 4.4.7. □

We want to work in local coordinates using the set-up in the proof of Theorem 3.2.2.

Choose a finite open cover $(\mathcal{O}_\alpha)_{\alpha\in\mathcal{A}}$ of M by pre-compact coordinate neighborhoods, and let $(\eta_\alpha)_{\alpha\in\mathcal{A}}$, $\eta_\alpha\in C_0^\infty(\mathcal{O}_\alpha)$ be a partition of unity subordinated to the cover $(\mathcal{O}_\alpha)_{\alpha\in\mathcal{A}}$. Next, choose $\varphi_\alpha\in C_0^\infty(\mathcal{O}_\alpha)$ such that $\varphi_\alpha\equiv 1$ on an open neighborhood $\mathcal{N}_\alpha$ of $\operatorname{supp}\eta_\alpha$ in $\mathcal{O}_\alpha$. We construct a parametrix $B_\alpha(\lambda)$ of A on $\mathcal{O}_\alpha$. Then the operator

$$B(\lambda)=\sum_{\alpha\in\mathcal{A}}\eta_\alpha B_\alpha(\lambda)\varphi_\alpha\in\boldsymbol{\Psi}^{-k,k}(\Lambda,\boldsymbol{E}),$$

is a parametrix (with parameters) of $(\lambda-A)$. Since A is self-adjoint and non-negative definite we deduce that $(\lambda-A)$ is invertible for any $\lambda\in\Lambda$ so that $S(\lambda)=(\lambda-A)^{-1}-B(\lambda)$ is a smoothing operator with parameters. Observing that

$$\int_{\gamma_R}e^{-t\lambda}\operatorname{tr}\partial^j_\lambda S(\lambda)d\lambda\sim_0 0,\;\;\forall j\geq 0,$$

we deduce that

$$\boldsymbol{Tr}\,e^{-tA}-\frac{1}{2\pi\boldsymbol{i}t^j}\int_{\gamma_R}e^{-t\lambda}\operatorname{tr}\partial^j_\lambda B(\lambda)d\lambda\sim_0 0,\;\;\forall j\gg 0.$$

We have

$$\frac{1}{2\pi\boldsymbol{i}t^j}\int_{\gamma_R}e^{-t\lambda}\operatorname{tr}\partial^j_\lambda B(\lambda)d\lambda=\frac{1}{2\pi\boldsymbol{i}t^j}\int_{\gamma_R}e^{-t\lambda}\left(\int_M\operatorname{tr}K^{(j)}_{B(\lambda)}(x,x)\,|dV_g(x)|\right)d\lambda,$$

where $K^{(j)}_{B(\lambda)}$ denotes the Schwartz kernel of $\partial^j_\lambda B(\lambda)$. Hence

$$\boldsymbol{Tr}\,e^{-tA}\sim_0\frac{1}{2\pi\boldsymbol{i}t^j}\sum_{\alpha\in\mathcal{A}}\int_{\gamma_R}e^{-t\lambda}\left(\int_M\operatorname{tr}K^{(j)}_{\eta_\alpha B_\alpha(\lambda)\varphi_\alpha}(x,x)\,|dV_g(x)|\right)d\lambda$$

Set $\mathcal{B}_\alpha(\lambda)=\eta_\alpha B_\alpha(\lambda)\varphi_\alpha$ so that,

$$\boldsymbol{Tr}\,e^{-tA}\sim_0\frac{1}{2\pi\boldsymbol{i}t^j}\sum_{\alpha\in\mathcal{A}}\int_{\gamma_R}e^{-t\lambda}\left(\int_{\mathcal{O}_\alpha}\operatorname{tr}K^{(j)}_{\mathcal{B}_\alpha(\lambda)}(x,x)\,|dV_g(x)|\right)d\lambda.$$

Fox $\alpha \in A$ and denote by $a_\alpha(x,\xi)$ the symbol of A defined by a choice of local coordinates on $\mathcal{O}_\alpha$ and a choice of trivialization of $\boldsymbol{E}|_{\mathcal{O}_\alpha}$. Denote by $b_\alpha(\lambda) \in \mathcal{S}^{-k,k}(\mathcal{O}_\alpha, \Lambda, \boldsymbol{E})$ the symbol of $B_\alpha(\lambda)$ computed using the recursive procedure detailed in Example 4.2.8. Then, for large j and any $x \in \mathcal{N}_\alpha$ we have

$$K^{(j)}_{\mathcal{B}_\alpha(\lambda)}(x,x) = K^{(j)}_{\eta_\alpha B_\alpha(\lambda)}(x,x).$$

The Schwartz kernel of $\eta_\alpha \partial^j_\lambda B_\alpha(\lambda)$ can be identified with a function of $\mathcal{O}_\alpha \times \mathcal{O}_\alpha$ using the metric on $\mathcal{O}_\alpha$ that is Euclidean in the local coordinates on $\mathcal{O}_\alpha$. More precisely, we identify it with the function decribed in (4.4.12). Using the terminology in Lemma 4.4.7 and (4.3.4) we deduce

$$\boldsymbol{Tr}\, e^{-tA} \sim_0 \frac{1}{2\pi \boldsymbol{i}} \sum_{\alpha \in \mathcal{A}} \int_{\mathcal{O}_\alpha} \frac{1}{\rho_\alpha(x)} \operatorname{tr} \mathcal{L}_t[\eta_\alpha b_\alpha](x,x)\, |dV_g(x)|, \tag{4.4.14}$$

where $\rho_\alpha(x)\,|dx|$ is the description of the metric density $|dV_g(x)|$ in the local coordinates on $\mathcal{O}_\alpha$,

$$|dV_g(x)| = \rho_\alpha(x)\,|dx| \ \text{ on } \mathcal{O}_\alpha. \tag{4.4.15}$$

We can now invoke Lemma 4.4.7(c) in the case $\nu = d = k$ for the symbols $b_\alpha \in \mathcal{S}^{-k,k}(\mathcal{O}_\alpha, \Lambda, \boldsymbol{E})$ to obtain an asymptotic expansion

$$\boldsymbol{Tr}\, e^{-tA} \sim_0 t^{-m/k} \sum_{p \geq 0} c_p t^{\frac{p}{k}}.$$

The coefficients c_ℓ are described by integrals

$$c_p = \int_M \boldsymbol{e}_p(x)\, |dV_g(x)|,$$

where the functions $\boldsymbol{e}_p$ are obtained as follows.

On $\mathcal{O}_\alpha$ the symbols a_α and $b_\alpha(\lambda)$ has asymptotic expansions

$$a_\alpha \sim \sum_{\ell \geq 0} a_\alpha^{k-p}(x,\xi),$$

$$b_\alpha(\lambda) \sim \sum_{p \geq 0} b_\alpha^{-k-p}(x,\lambda,\xi),$$

where a_α^{k-p} is homogeneous of degree $k-p$ in $|\xi| > 0$, and $b_\alpha^{-k-p}(x,\lambda,\xi)$ is quasi-homogeneous of degree $-k-p$ for $|\xi| + |\lambda|^{1/k} \geq 1$.

Then

$$c_p = (2\pi)^{-m/2} \sum_\alpha \int_{\mathcal{O}_\alpha} \frac{\eta_\alpha(x)}{\rho_\alpha(x)} \left(\int_{T^*_x M} \left(\frac{1}{2\pi \boldsymbol{i}} \int_{\gamma_1} e^{-\mu} \operatorname{tr} \partial^j_\mu b_\alpha^{-k-p}(x,\mu,\xi) d\mu \right) |d\xi|_* \right) |dV_g(x)|. \tag{4.4.16}$$

The computations in Example 4.2.8 show that each b_α^{-N} is a linear combination of terms of the form

$$T_0(\lambda - a_\alpha^k)^{-n_1} T_1 \cdots T_{r-1} (\lambda - a_\alpha)^{-n_r} T_r,$$

where T_j is of the form $D_x^\beta D_\xi^\gamma a^{\ell_j}(x,\xi)$ for some multi-indices β and γ. The integral over γ_1 can be computed[3] using the residue formula. This proves the claim about the general structure of $\boldsymbol{e}_p$.

If A is a generalized Laplacian we have $k = 2$ and

$$a_2(x,\xi) = |\xi|^2_x \mathbb{1}_{\boldsymbol{E}_x}, \ \ b_{-2}(x,\xi) = (\lambda - |\xi|^2_x)^{-1} \mathbb{1}_{\boldsymbol{E}_x},$$

[3] This leads to some horrible expressions that can be simplified somewhat using the orthogonal invariance of those expressions.

where $|\xi|_x$ denotes the length of the covector $\xi \in T^*_x M$ computed using the metric g. We fix a point $p_0 \in M$. We have

$$\frac{1}{2\pi \boldsymbol{i}}\int_{\gamma_1} e^{-\mu}\partial^j_\mu(\mu - |\xi|^2_{p_0})^{-2}\,\mathrm{tr}\,\mathbb{1}_{\boldsymbol{E}_x} d\mu = \frac{r}{2\pi \boldsymbol{i}}\int_{\gamma_1} e^{-\mu}(\mu - |\xi|^2_{p_0})^{-2} d\mu = e^{-|\xi|^2_{p_0}}, \;\; r = \dim \boldsymbol{E}_x.$$

To compute the integral

$$\int_{T^*_{p_0}M} e^{-|\xi|^2_{p_0}} |d\xi|_*$$

we identify a neighborhood of p_0 in M with a neighborhood of 0 in the Euclidean space $\boldsymbol{V}$. For p near p_0, the metric g_p on T_pM is then described by symmetric positive definite map $G_p : \boldsymbol{V} \to \boldsymbol{V}$, while the induced metric on $T^*_{p_0}M$ is described by its inverse, i.e.,

$$|\xi|^2_p = (G_p^{-1}\xi, \xi),$$

where $(-,-)$ denotes the inner product on $\boldsymbol{V}$.

Let $\lambda_1(p), \dots, \lambda_m(p) > 0$ the eigenvalues of G_p. Let us observe that

$$|dV_g| = \sqrt{\det G_p}|dx| = \sqrt{\lambda_1(p)\cdots\lambda_m(p)}|dx|.$$

Using (4.4.15) we can rewrite the above equality as

$$\rho_\alpha(p) = \sqrt{\det G_p}. \tag{4.4.17}$$

We can now choose Euclidean coordinates $\xi_1, \dots, \xi_m$ on $\boldsymbol{V}$ that diagonalize G_0. We then have

$$\int_{T^*_{p_0}M} e^{-|\xi|^2_{p_0}} |d\xi|_* = \prod_{j=1}^m \int_{\mathbb{R}} e^{-r^2/\lambda_j(p_0)}|dr|_*$$

$$= \sqrt{\lambda_1(p_0)\cdots\lambda_m(p_0)}\prod_{j=1}^m \int_{\mathbb{R}} e^{-s^2}|ds|_* = 2^{-m/2}\sqrt{\lambda_1(p_0)\cdots\lambda_m(p_0)}.$$

Using the last equality and (4.4.17) in (4.4.16) where $p = 0$ we deduce

$$c_0 = (4\pi)^{-m/2} r \mathrm{vol}_g(M).$$

□

Remark 4.4.8. The same arguments used in the proof of Theorem 4.4.6 imply a slightly stronger result. To formulate it let us introduce the cones

$$C_\varphi := \{ z = r^{\boldsymbol{i}\theta} \in \mathbb{C};\;\; r > 0,\;\; |\theta| \leq \varphi \},\;\; \varphi \in [0, \infty). \tag{4.4.18}$$

Fix $|\varphi| < \frac{\pi}{4}$ so that the cone C_φ is surrounded by the contour γ_R. Then one can show that e^{-tA} defined as in (4.4.5) makes sense for any $t \in C_\varphi$. The resulting operator is smoothing and we have an asymptotic expansion

$$\boldsymbol{Tr}\, e^{-tA} \sim t^{-\frac{m}{k}} \sum_{p \geq 0} c_p t^{\frac{p}{k}} \;\; \text{as} \;\; t \to 0, \;\; t \in C_\varphi, \tag{4.4.19}$$

where the coefficients c_p are the ones in (4.4.9). □

Example 4.4.9. We want to investigate a very simple example and confirm (4.4.10) in this simple case by an alternate method. Consider the scalar Laplacian on the unit circle

$$\Delta := -\frac{d^2}{d\theta^2} : C^\infty(S^1) \to C^\infty(S^1).$$

Above, we identify $C^\infty(S^1)$ with the space of smooth 2π-periodic functions $\mathbb{R} \to \mathbb{C}$. For any $n \in \mathbb{Z}$ we set

$$e_n(\theta) := (2\pi)^{-1/2} e^{\boldsymbol{i}n\theta}.$$

The collection $\{e_n(\theta)\}_{n\in\mathbb{Z}}$ is a unitary Hilbert basis of $L^2(S^1)$. Moreover

$$\operatorname{spec}(\Delta) = \{ n^2;\ \ n \in \mathbb{Z} \} \ \text{ and } \ \ker(n^2 - \Delta) = \operatorname{span}_{\mathbb{C}}\{e_{\pm n}(\theta)\}.$$

Hence

$$\boldsymbol{Tr}\, e^{-t\Delta} = \sum_{n\in\mathbb{Z}} e^{-tn^2} =: f(t).$$

The equality (4.4.10) predicts that

$$\lim_{t\searrow 0} t^{1/2} f(t) = \pi^{1/2}. \tag{4.4.20}$$

We want to confirm this by independent means.

The function $f(t)$ is closely related to the classical theta function

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi\boldsymbol{i}n^2\tau + 2\pi\boldsymbol{i}nz), \quad \mathbf{Im}\,\tau > 0.$$

More precisely $f(t) = \vartheta(0, \boldsymbol{i}t)$. The asymptotic behavior of $f(t)$ is a simple consequence of the modularity of the function $\tau \mapsto \vartheta(z,\tau)$. In more concrete terms, we will prove a very surprising identity involving $f(t)$ which will imply immediately the equality (4.4.20). We follow the approach in [**Be**, §9] based on the so called *Poisson formula*.

For every $t > 0$ we consider the function $g_t \in \mathcal{S}(\mathbb{R})$, $g_t(x) = e^{-tx^2}$. Note that its Fourier transform is

$$\widehat{g}_t(\xi) = \int_{\mathbb{R}} e^{-tx^2} e^{-\boldsymbol{i}x\xi} |dx|_* = (2t)^{-1/2} \int_{\mathbb{R}} e^{-y^2/2} e^{\boldsymbol{i}y\xi/\sqrt{2t}}\, |dy|_* \overset{(1.1.10)}{=} (2t)^{-1/2} e^{-\xi^2/4t}. \tag{4.4.21}$$

We form the 2π-periodic function

$$G_t(x) = \sum_{n\in\mathbb{Z}} g_t(x + 2\pi n).$$

The above series is uniform;ly convergent since the function $g_t(x)$ decays very fast as $|x| \to \infty$. We regard G_t as a function on S^1. As such, it has a Fourier series decomposition

$$G_t(x) = \sum_{n\in\mathbb{Z}} c_n(t) e_n(x), \tag{4.4.22}$$

where the Fourier coefficient $c_n(t)$ is given by

$$c_n(t) = \int_0^{2\pi} G_t(x)\overline{e_n(x)}\, |dx|.$$

Observe that

$$c_n(t) = \sum_{k\in\mathbb{Z}} \int_0^{2\pi} g_t(x + 2\pi k) e^{-\boldsymbol{i}nx} |dx|_* = \int_{\mathbb{R}} g_t(x) e^{-\boldsymbol{i}nx} |dx|_* = \widehat{g}_t(n) = (2t)^{-1/2} e^{-n^2/4t}.$$

This shows that the series (4.4.22) is uniformly convergent for $0 \leq x \leq 2\pi$. We obtain in this fashion the *Poisson formula*

$$\sum_{n\in\mathbb{Z}} g_t(x+2\pi n) = G_t(x) = \sum_{n\in\mathbb{Z}} \widehat{g}_t(n)e_n(x) = (4\pi t)^{-1/2}\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{4t}}e^{-\boldsymbol{i}nx}, \;\; \forall x\in[0,2\pi], \;\; \forall t>0 \tag{4.4.23}$$

If we let $x=0$ in the above equality we deduce

$$\sum_{n\in\mathbb{Z}} e^{-t(2\pi n)^2} = G_t(0) = (4\pi t)^{-1/2}\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{4t}}.$$

so that if we use the substitution $t = t/(4\pi^2)$ we obtain

$$\sum_{n\in\mathbb{Z}} e^{-tn^2} = \left(\frac{\pi}{t}\right)^{1/2}\sum_{n\in\mathbb{Z}} e^{-\frac{(2\pi n)^2}{4t}} \tag{4.4.24}$$

This proves that

$$\lim_{t\searrow 0} t^{1/2}\sum_{n\in\mathbb{Z}} e^{-tn^2} = \pi^{1/2}\lim_{t\searrow 0}\sum_{n\in\mathbb{Z}} e^{-\frac{(2\pi n)^2}{4t}} = \pi^{1/2}.$$ □

The asymptotic expansion (4.4.9) has the following remarkable consequence.

Theorem 4.4.10 (Weyl asymptotic formula)**.** *Let (M,g) be a smooth, compact, Riemann manifold of dimension m, and $\boldsymbol{E}\to M$ is a smooth, complex vector bundle of rank r equipped with a Hermitian metric h. Suppose $A: C^\infty(\boldsymbol{E})\to C^\infty(\boldsymbol{E})$ is an admissible partial differential operator of order k which is also nonnegative definite. We collect the eigenvalues of A in a nondecreasing sequence*

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$$

such that each eigenvalue λ appears in this sequence as many times as its multiplicity $m(\lambda) = \dim\ker(\lambda - A)$. For every $\lambda>0$ we set

$$n_A(\lambda) := \#\big\{n;\;\; \lambda_n\leq\lambda\,\big\}.$$

Then

$$n_A(\lambda) \sim \frac{c_0(A)}{\Gamma(1+m/k)}\lambda^{m/k} \;\; \textit{as } \lambda\to\infty, \tag{4.4.25}$$

where $c_0(A)$ is given by the asymptotic expansion (4.4.9), i.e.,

$$c_0(A) = \lim_{t\searrow 0} t^{m/k}\,\boldsymbol{Tr}\, e^{-tA},$$

and Γ denotes Euler's Gamma function. In particular, if A is a generalized Laplacian, then

$$n_A(\lambda) \sim \frac{r\mathrm{vol}_g(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2}$$

Proof. The equality (4.4.25) is a consequence of the following Tauberian theorem..

Theorem 4.4.11 (Karamata)**.** *Suppose $(\lambda_j)_{j\geq 0}$ is a non-increasing sequence of non-negative real numbers such that*

$$f(t) = \sum_{j\geq 0} e^{-t\lambda_j} < \infty,$$

and there exist $\alpha, A>0$ such that

$$\lim_{t\searrow 0} t^\alpha f(t) = A. \tag{4.4.26}$$

We set

$$N(\lambda) := \#\{\, n;\ \ \lambda_n \leq \lambda \,\}.$$

Then

$$N(\lambda) \sim \frac{A\lambda^{\alpha}}{\Gamma(\alpha+1)},\ \ \text{as } \lambda \to \infty.$$

Proof of Karamata's theorem For any continuous function $g : [0,1] \to \mathbb{R}$ we set

$$w_g(t) := \sum_{j\geq 0} g(e^{-t\lambda_j})e^{-t\lambda_j}.$$

We first want to prove that for any such g we have

$$\lim_{t\searrow 0} t^{\alpha} w_g(t) = \frac{A}{\Gamma(\alpha)} \int_0^{\infty} g(e^{-s})s^{\alpha-1}e^{-s}ds =: I(g). \tag{4.4.27}$$

Denote by $\mathcal{X}$ the set of $g \in C^0([0,1])$ for which (4.4.27) holds. We will prove that $\mathcal{X} = C^0([0,1])$.

Clearly $\mathcal{X}$ is nonempty vector space since $0 \in \mathcal{X}$. Let us show that $\mathcal{X}$ contains all the monomials $g(x) = x^n$, $n \geq 0$. Indeed, we have

$$w_{x^n}(t) = \sum_{j\geq 0} e^{-t(n+1)\lambda_j} = f\big(\,(n+1)t\,\big) \overset{(4.4.26)}{\sim} \frac{A}{(n+1)^{\alpha}} t^{-\alpha},\ \ \text{as } t \searrow 0.$$

On the other hand, in this case we have

$$\frac{A}{\Gamma(\alpha)} \int_0^{\infty} g(e^{-s})s^{\alpha-1}e^{-s}ds = \frac{A}{\Gamma(\alpha)} \int_0^{\infty} e^{-(n+1)s}s^{\alpha-1}ds = \frac{A}{(n+1)^{\alpha}\Gamma(\alpha)} \int_0^{\infty} e^{-y}y^{\alpha-1}dy$$
$$= \frac{A}{(n+1)^{\alpha}}.$$

This shows that $\mathcal{X}$ contains all the polynomials.

Now observe that if $g_0, g_1 : [0,1] \to \mathbb{R}$ are two continuous functions then

$$|w_{g_0}(t) - w_{g_1}(t)| \leq \sum_{j\geq 0} \big|\, g_0(e^{-t\lambda_j}) - g_0(e^{-t\lambda_j})\,\big| e^{-\lambda_j t} \leq \|g_0 - g_1\|_{\infty} f(t),$$

where $\|-\|_{\infty}$ denotes the sup-norm in $C^0([0,1])$. We conclude that

$$|t^{\alpha}w_{g_0}(t) - t^{\alpha}w_{g_1}(t)| \leq \|g_0 - g_1\|_{\infty} t^{\alpha} f(t),\ \ \forall t > 0,\ \ g_0, g_1 \in C^0([0,1]).$$

Similarly,

$$\big|\, I(g_0) - I(g_1)\,\big| \leq A\|g_0 - g_1\|_{\infty}.$$

We deduce that there exists a constant $C > 0$ such that for any continuous function $g_0 : [0,1] \to \mathbb{R}$, any $t \in (0,1]$ and any $g_1 \in \mathcal{X}$ we have

$$|t^{\alpha}w_{g_0}(t) - I(g_0)| \leq |t^{\alpha}w_{g_0}(t) - t^{\alpha}w_{g_1}(t)| + |t^{\alpha}w_{g_1}(t) - I(g_1)| + |I(g_1) - I(g_0)|$$
$$\leq C\|g_0 - g_1\|_{\infty} + |t^{\alpha}w_{g_1}(t) - I(g_1)|.$$

Hence there exists $C > 0$ such that for any continuous $g_0 : [0,1] \to \mathbb{R}$ and any $g_1 \in \mathcal{X}$ we have

$$\limsup_{t\searrow 0} |t^{\alpha}w_{g_0}(t) - I(g_0)| \leq C\|g_0 - g_1\|_{\infty}.$$

This proves that $\mathcal{X}$ is closed with respect to the norm $\|-\|_{\infty}$. The Stone-Weierstrass theorem now implies that $\mathcal{X} = C^0([0,1])$.

For any $0 < r < 1$ we let $g_r \in C^0([0,1])$ be the continuous function such that

$$g_r(x) = \begin{cases} 1/x, & x > 1/e, \\ 0, & x < r/e, \\ \text{linear}, & x \in [r/e, 1/e]. \end{cases}$$

We set

$$I_r(\alpha) = \frac{A}{\Gamma(\alpha)} \int_0^\infty g_r(e^{-s}) s^{\alpha-1} e^{-s} ds.$$

Observe that

$$\lim_{r \nearrow 1} I_r(\alpha) = \frac{A}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1} ds = \frac{A}{\alpha\Gamma(\alpha)} = \frac{A}{\Gamma(\alpha+1)}. \tag{4.4.28}$$

Then

$$w_{g_r}(1/\lambda) = \sum_{j \geq 0} g(e^{-\lambda_j/\lambda}) e^{-\lambda_j/\lambda} = \sum_{\lambda_j \leq \lambda(1-\log r)} g(e^{-\lambda_j/\lambda}) e^{-\lambda_j/\lambda} \leq N\big(\lambda(1-\log r)\big).$$

On the other hand, we have

$$w_{g_r}(1/\lambda) \geq \sum_{\lambda_j \leq \lambda} g(e^{-\lambda_j/\lambda}) e^{-\lambda_j/\lambda} \geq N(\lambda)$$

Thus, if we set $q_r := 1 - \log r$, we deduce

$$w_{g_r}(q_r/\lambda) \leq N(\lambda) \leq w_{g_r}(1/\lambda).$$

Letting $\lambda \to \infty$ we deduce from (4.4.27) that

$$q_r^{-\alpha} I_r(\alpha) = \lim_{\lambda\to\infty} \frac{w_{g_r}(q_r/\lambda)}{\lambda^\alpha} \leq \varliminf_{\lambda\infty} \lambda^{-\alpha} N(\lambda) \leq \varlimsup_{\lambda\to\infty} \lambda^{-\alpha} N(\lambda) \leq \lim_{\lambda\to\infty} \frac{w_{g_r}(1/\lambda)}{\lambda^\alpha} = I_r(\alpha).$$

If we now let $r \nearrow 1$ in the above inequalities and use (4.4.28) we obtain (4.4.26). □

Returning to our we see that Karamata's theorem implies that we deduce that

$$n_A(\lambda) \sim \frac{c_0(A)}{\Gamma(1+m/k)} \lambda^{m/k} \ \text{ as } \lambda \to \infty.$$

□

Remark 4.4.12. The above asymptotic estimate suggests that

$$\frac{c_0(A)}{\Gamma(1+m/k)} \lambda_n^{m/k} \sim n \ \text{ as } n \to \infty.$$

This implies

$$\lambda_n \sim \left(\frac{\Gamma(1+m/k)}{c_0(A)}\right)^{k/m} n^{k/m} \ \text{ as } n \to \infty.$$

In other words the eigenvalues of a positive selfadjoint elliptic operators of order k on an m-dimensional manifold ought to grow like $n^{k/m}$. This is indeed the case. For a proof we refer to [**Shu**, Prop. 13.1]. □

4.5. McKean-Singer formula

Suppose (M, g) is a smooth, compact Riemann manifold of dimension m, and $\boldsymbol{E}_0$, $\boldsymbol{E}_1$ are smooth complex vector bundles over M of the same rank r equipped with hermitian metrics and compatible connections.

Suppose now that $L : C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_1)$ is an elliptic *partial differential operator* of order k. We form the operators

$$A_+ = L^*L : C^\infty(\boldsymbol{E}_0) \to C^\infty(\boldsymbol{E}_0), \;\; A_- = LL^* : C^\infty(\boldsymbol{E}_1) \to C^\infty(\boldsymbol{E}_1),$$

Both operators A_+, A_- are admissible and non-negative definite so Theorem 4.4.6 implies that we have asymptotic estimates

$$\boldsymbol{Tr}\, e^{-tA_\pm} = t^{-\frac{m}{2k}} \sum_{p \geq 0} c_p(A_\pm) t^{\frac{p}{2k}}.$$

The coefficients $c_p(A_\pm)$ are described by integrals

$$c_p(A_\pm) = \int_M e_p(x, A_\pm)\, |dV_g(x)|,$$

where the densities $e_p(x, A_\pm)$ are obtained in an universal way from the coefficients of $A_\pm$. We set

$$\boldsymbol{\rho}_L(x) := e_m(x, A_+) - e_m(x, A_-).$$

We will refer to the function $\boldsymbol{\rho}_L$ as the *index density* of L.

Theorem 4.5.1 (McKean-Singer)**.** *If L is as above, then*

$$\operatorname{ind} L = \dim \ker L - \dim \ker L^* = \int_M \boldsymbol{\rho}_L(x)\, |dV_g(x)|.$$

Proof. The key facts behind the proof are contained in the following lemma.

Lemma 4.5.2. *(a)* $\ker A_+ = \ker L$*,* $\ker A_- = \ker L*$*.*
(b) For any $\lambda > 0$ *we have* $\dim \ker(\lambda - A_+) = \dim \ker(\lambda - A_-)$. □

Assuming temporarily the validity of this lemma we deduce

$$\boldsymbol{Tr}\, e^{-tA_+} - \boldsymbol{Tr}\, e^{-tA_-} = \sum_{\lambda \geq 0} e^{-\lambda t}\Big(\dim \ker(\lambda - A_+) - \dim \ker(\lambda - A_-)\Big)$$

$$= \dim \ker A_+ - \dim \ker A_- = \operatorname{ind} L.$$

From the asymptotic expansion as $t \searrow 0$ of the trace of the heat kernel we deduce that

$$\operatorname{ind} L \sim t^{-\frac{m}{2k}} \sum_{p \geq 0} \big(\, c_p(A_+) - c_p(A_-)\,\big) t^{\frac{p}{2k}}.$$

Since the left-hand side of the above expansion is independent of t, we deduce that the terms the right-hand side involving t^r, $r \neq 0$ must be trivial. This leaves us with the equality

$$\operatorname{ind} L = c_m(A_+) - c_m(A_-) = \int_M \boldsymbol{\rho}_L(x)\, |dV_g(x)|.$$

□

Proof of Lemma 4.5.2. (a) Observe that for any $u \in C^\infty(\boldsymbol{E}_0)$ we have

$$\int_M (L^*Lu, u)_{E_0}\,|dV_g| = \int_M (Lu, Lu)_{E_1}\,|dV_g(x)| = \int_M |Lu(x)|^2_{E_1}\,|dV_g(x)|.$$

This shows that $u \in \ker L^*L$ if and only if $u \in \ker L$, i.e., $\ker A_+ = \ker L$. The equality $\ker A_- = \ker L^*$ is proved in a similar fashion.

We will prove (b) by showing that for any $\lambda > 0$ we have

$$\dim\ker(\lambda - A_+) \leq \dim\ker(\lambda - A_-) \ \text{ and } \ \dim\ker(\lambda - A_+) \geq \dim\ker(\lambda - A_-).$$

Observe that $LA_+ = A_-L$. If $u \in \ker(\lambda - A_+)$ then $A_+u = \lambda u$ and

$$A_-Lu = LA_+u = \lambda Lu.$$

Thus L induces a liner map $L : \ker(\lambda - A_+) \to \ker(\lambda - A_-)$. Part (a) shows that this map is injective so that

$$\dim\ker(\lambda - A_+) \leq \dim\ker(\lambda - A_-).$$

Similarly, L^* induces an injection $\ker(\lambda - A_-) \to \ker(\lambda - A_+)$ thus proving the opposite inequality.□

4.6. Zeta functions

Let (M, g) be a smooth, compact, Riemann manifold of dimension m, and $\boldsymbol{E} \to M$ is a smooth, complex vector bundle of rank r equipped with a Hermitian metric h. Suppose $A : C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E})$ is an admissible partial differential operator of order k which is also *positive* definite, i.e., there exists $c > 0$ such that

$$\int_M h(Au(x), u(x))\,|dV_g(x)| \geq c\int_M h(u(x), u(x))\,|dV_g(x)|, \quad \forall u \in C^\infty(\boldsymbol{E}).$$

We collect the eigenvalues of A in a nondecreasing sequence

$$0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$$

such that each eigenvalue λ appears in this sequence as many times as its multiplicity $m(\lambda) = \dim\ker(\lambda - A)$. We set

$$\zeta_A(s) := \sum_{n\geq 0} \lambda_n^{-s}.$$

Lemma 4.6.1. *The series $\zeta_A(s)$ converges absolutely and uniformly on the compacts of the half-plane $\{\mathbf{Re}\, s \geq \frac{m}{k}\}$.*

Proof. We write $n(\lambda) := n_A(\lambda)$, and we set $\mu := \min(\lambda_0, 1)$. Then

$$\sum_{\lambda_\nu \leq (j+1)} \lambda_\nu^{-s} \leq n(1)\lambda_0^{-s} + \big(n(2) - n(1)\big)1^{-s} + \cdots + (\,n(j+1) - n(j)\,)j^{-s}$$

(use Abel's trick)

$$= n(1)\big(\mu^{-s} - 1^{-s}\big) + n(2)\big(1^{-s} - 2^{-s}\big) + \cdots + n(j)\big((j-1)^{-s} - j^{s}\big) + n(j+1)j^{-s}$$

Now observe that

$$(j-1)^{-s} - j^{-s} = O(j^{-s-1}) \ \text{ as } j \to \infty,$$

while

$$n(j) = O(j^{m/k}) \ \text{ as } j \to \infty.$$

□

Proposition 4.6.2. *Let, M, A, $\boldsymbol{E}$ be as above. Then the holomorphic function*

$$\zeta_A : \Big\{ \mathbf{Re}\, s > \frac{m}{k} \Big\} \to \mathbb{C},$$

admits an extension as a meromorphic function $\zeta_A : \mathbb{C} \dashrightarrow \mathbb{C}$ with only simple poles located at

$$s_p := \frac{m-p}{k}, \;\; p = 0, 1, \dots, \;\; s_p \notin \mathbb{Z}_{\leq 0}.$$

The residue of $\zeta_A(s)$ at s_p is

$$\mathrm{Res}_{s=s_p}\big(\zeta_A(s)\big) = \frac{c_p}{\Gamma(s_p)},$$

where $c_p = c_p(A)$ is the coefficient that appears in the asymptotic expansion (4.4.9). This meromorphic extension is called the zeta function of the operator A.

Proof. We follow the approach in [**GrSe96**, Prop. 5.1]. This relies on some basic facts about the Gamma function that can be found in[**La**, §XV.2]. Define

$$e : (0, \infty) \to \mathbb{C}, \;\; e(t) = \sum_{n \geq 0} e^{-t\lambda_n}.$$

The function $e(t)$ decreases exponentially as $t \to \infty$ and we have an asymptotic expansion

$$e(t) \sim \sum_{p \geq 0} c_p t^{a_p} \;\; \text{as} \;\; t \searrow 0, \;\; a_p := -s_p = \frac{p-m}{k}.$$

In particular,

$$|e(t)| = O(|t|^{a_0}), \;\; \text{as} \;\; t \searrow 0.$$

To describe the behavior of $e(t)$ as $t \to \infty$ we argue as in the proof of Lemma 4.6.1. We have

$$\sum_{\lambda_n \leq j+1} \leq e^{-\mu t} n(1) + e^{-t}\big(n(2) - n(1)\big) + \cdots + e^{-jt}\big(n(j+1) - n(j)\big)$$

$$= n(1)(e^{-\mu t} - e^{-t}) + n(2)(e^{-t} - e^{-2t}) + \cdots + n(j)(e^{-(j-1)t} - e^{-jt}) + n(j+1)e^{-jt}$$

$$\leq n(1)e^{-\mu t} + C \sum_{\nu=1}^{j} (\nu+1)^{m/k} e^{-\nu t} + (j+1)^{m/k} e^{-jt}.$$

This shows that $e(t)$ decays exponentially to 0 as $t \to \infty$.

The *Mellin transform* of $e(t)$ is the function

$$f = \mathcal{M}[e] : \big\{ s \in \mathbb{C}; \;\; \mathbf{Re}\, s > -a_0 \big\} \to \mathbb{C}, \;\; f(s) := \int_0^\infty e(t) t^s \frac{dt}{t} = \int_0^\infty e(t) t^{s-1} dt.$$

The function $f(s)$ is holomorphic in the half-plane $\{\mathbf{Re}\, s > -a_0\}$. Moreover

$$f(s) = \sum_{n \geq 0} \int_0^\infty e^{-t\lambda_n} t^{s-1} dt, \;\; \forall\, \mathbf{Re}\, s > -a_0.$$

Observe that

$$\int_0^\infty e^{-t\lambda_n} t^{s-1} dt = \lambda_n^{-s} \int_0^\infty e^{-\tau} \tau^{s-1} ds = \Gamma(s)\lambda_n^{-s},$$

where $\Gamma(s)$ denotes Euler's Gamma function. Hence

$$f(s) = \Gamma(s)\zeta_A(s).$$

We construct a meromorphic extension of $f(s)$ to the entire plane. We have

$$f(s) = \underbrace{\int_0^1 e(t)t^{s-1}dt}_{=:f_0(s)} + \underbrace{\int_1^\infty e(t)t^{s-1}dt}_{=:f_1(s)}.$$

The integral defining $f_1(s)$ is convergent for any $s \in \mathbb{C}$ and thus defines a holomorphic function $f_1 : \mathbb{C} \to \mathbb{C}$. It thus suffices to show that $f_0(s)$ admits a meromorphic extension to the whole plane. We define

$$e_p(t) = e(t) - \sum_{j=0}^{p} c_j t^{a_j}.$$

Then

$$e_p(t) = O(t^{a_{p+1}}) \ \text{ as } \ t \searrow 0, \tag{4.6.1}$$

and for any $\mathbf{Re}\, s > -a_0$ we have

$$f_0(s) = \int_0^1 e(t)t^{s-1}dt = \int_0^1 e_p(t)t^{s-1}dt - \sum_{j=0}^{p} c_j \int_0^1 t^{a_j+s-1}dt$$

$$= \int_0^1 e_p(t)t^{s-1}dt - \sum_{j=0}^{p} \frac{c_j}{s+a_j}.$$

The estimate (4.6.1) implies that the integral $\int_0^1 e_p(t)t^{s-1}dt$ is convergent for any $\mathbf{Re}\, s > -a_{p+1}$ and defines a holomorphic function in this half-plane. The above equality shows that for any $p \geq 0$, the function $f_0(s)$ admits a meromorphic extension to the half-plane $\mathbf{Re}\, s > -a_{p+1}$ with only simple poles located as $s = -a_0, \dots, -a_p$. Moreover, the residues at these points are given by the coefficients $c_0, \dots, c_p$. Letting $p \to \infty$ we deduce that $f(s)$ admits a meromorphic extension to the whole plane, with simple poles located at $s = -a_p$, $p \geq 0$, and residues at these poles given by c_p.

We have

$$\zeta_A(s) = \frac{1}{\Gamma(s)} f(s), \quad \mathbf{Re}\, s > -a_0$$

The function $\frac{1}{\Gamma(s)}$ admits a *holomorphic* extension to the *entire complex plane* given by the Weierstrass product

$$\frac{1}{\Gamma(s)} = ze^{\gamma z} \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) e^{-z/n}, \tag{4.6.2}$$

where γ denotes Euler's constant

$$\gamma = \lim_{n\to\infty} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Big).$$

□

Example 4.6.3. Let $a \in (0,1)$, and consider the first order elliptic operator

$$D_a = -\boldsymbol{i}\frac{d}{d\theta} + a : C^\infty(S^1) \to C^\infty(S^1).$$

Then

$$\operatorname{spec}(D_a) = a + \mathbb{Z}.$$

We form the Laplacian $\Delta_a = D_a^2$. Then its spectrum is

$$\operatorname{spec}(\Delta_a) = \{\, (a+n)^2;\ \ n \in \mathbb{Z}\,\}.$$

If we set $\zeta_a(s) := \zeta_{\Delta_a}(s)$ then we deduce that for any $s > \frac{1}{2}$ we have

$$\zeta_a(s) = \sum_{n\geq 0} \frac{1}{(a+n)^{2s}} + \sum_{n\geq 0} \frac{1}{(1-a+n)^{2s}}.$$

For every z with $\mathbf{Re}\, z > 1$ we define

$$\Xi_a(z) = \sum_{n\geq 0} \frac{1}{(a+n)^z},$$

so that

$$\zeta_a(s) = \Xi_a(2s) + \Xi_{1-a}(2s).$$

We want to show that $\Xi_a(z)$ admits a meromorphic extension to the whole plane with a single simple pole at $z = 1$. We follow the presentation in [**La**, XV§4]. For $\mathbf{Re}\, z > 1$ we have

$$\Gamma(z) := \int_0^\infty e^{-t} t^z \frac{dt}{t} = \int_0^\infty e^{-(n+a)\tau} (n+a)^z \tau^z \frac{d\tau}{\tau},$$

so that

$$\frac{\Gamma(z)}{(n+a)^z} = \int_0^\infty e^{-t} t^z \frac{dt}{t} = \int_0^\infty e^{-(n+a)\tau} (n+a)^z \tau^z \frac{d\tau}{\tau},$$

and thus

$$\Gamma(z)\Xi_a(z) = \sum_{n\geq 0} \int_0^\infty e^{-(n+a)\tau} \tau^z \frac{d\tau}{\tau} = \int_0^\infty \Big(\sum_{n\geq 0} e^{-(n+a)\tau} \Big) \tau^z \frac{d\tau}{\tau} x = \int_0^\infty \frac{e^{-a\tau}}{1-e^{-\tau}} \tau^z \frac{d\tau}{\tau}.$$

Consider the functions

$$F_a(\tau) = \frac{e^{a\tau}}{e^\tau - 1},\ \ G_a(\tau) = \frac{e^{-a\tau}}{1-e^{-\tau}} = -F_a(-\tau),\ \ \tau \in \mathbb{C},$$

so that

$$\Gamma(z)\Xi_a(z) = \int_0^\infty G_a(z) \tau^z \frac{d\tau}{\tau}. \tag{4.6.3}$$

Consider the Hankel contour C_ε depicted in Figure 4.5

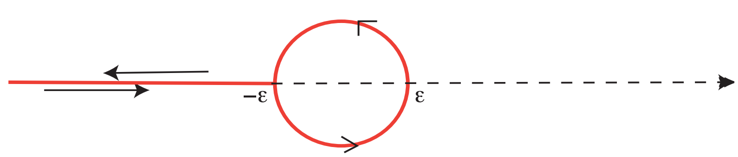

Figure 4.5. *The Hankel contour.*

Consider the function

$$H_a(z) = \int_{C_\varepsilon} F_a(\tau) \tau^z \frac{d\tau}{\tau},$$

where

$$\tau^z = |\tau|^z e^{\boldsymbol{i} z \arg \tau},\ \ \arg \tau \in (-\pi, \pi].$$

This is clearly an entire function. We want to show that

$$H_a(z) = -2\boldsymbol{i} \sin \pi z \Gamma(z) \Xi_a(z),\ \ \forall\, \mathbf{Re}\, z > 1. \tag{4.6.4}$$

Let us show why this equality implies the existence of a meromorphic extension of $\Xi_a(z)$ with a single pole at $z = 1$. We rewrite (4.6.4) as

$$\Xi_a(z) = -\frac{1}{2\boldsymbol{i}\sin\pi z\Gamma(z)}H_a(z),$$

and use the classical identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

to conclude that

$$\Xi_a(z) = -\frac{1}{2\pi\boldsymbol{i}}\Gamma(1-z)H_a(z).$$

This shows that $\Xi_a(z)$ has a meromorphic extension to $\mathbb{C}$. Its poles can be only simple and can be located only at the poles of $\Gamma(1-z)$. From (4.6.2) we deduce these poles are all simple and are located at $z = 1, 2, 3, \dots$. Since $\Xi_a(z)$ is holomorphic for $\mathbf{Re}\, z > 1$, we deduce that it can have at most a simple pole located at $z = 1$.

The proof of (4.6.4) is by direct computation. We have

$$H_a(z) = e^{-\pi\boldsymbol{i}z}\int_{-\infty}^{-\varepsilon} F_a(t)|t|^z\frac{dt}{t} + \int_{|z|=\varepsilon} F_a(\tau)\tau^z\frac{d\tau}{\tau} + e^{\pi\boldsymbol{i}z}\int_{-\varepsilon}^{-\infty} F_a(t)|t|^z\frac{dt}{t}.$$

First we observe that since $\mathbf{Re}\, z > 1$ then

$$\lim_{\varepsilon\searrow 0}\int_{|z|=\varepsilon} F_a(\tau)\tau^z\frac{d\tau}{\tau} = 0.$$

As for the remaining two integrals, we have

$$e^{-\pi\boldsymbol{i}z}\int_{-\infty}^{-\varepsilon} F_a(t)|t|^z\frac{dt}{t} = -e^{-\pi\boldsymbol{i}z}\int_{\varepsilon}^{\infty} F_a(-t)t^z\frac{dt}{t} \to e^{-\pi\boldsymbol{i}z}\int_0^{\infty} G_a(t)\frac{dt}{t},$$

$$e^{\pi\boldsymbol{i}z}\int_{-\varepsilon}^{-\infty} F_a(t)|t|^z\frac{dt}{t} \to= e^{\pi\boldsymbol{i}z}\int_0^{\infty} G_a(t)t^z\frac{dt}{t}.$$

This proves (4.6.4). If we set

$$e_a(t) = \boldsymbol{Tr}\, e^{-t\Delta_a}$$

then we have an asymptotic expansion

$$e_a(t) = t^{-1/2}\sum_{p\geq 0} c_p t^{p/2} \text{ as } t\searrow 0.$$

According to Proposition 4.6.2, the zeta function can only have simple poles located at

$$s = \frac{1}{2}, 0 - \frac{1}{2}, -1, \dots, -\frac{n}{2}, \dots$$

and the residue at $-\frac{n}{2} \leq 0$,

$$r_n = \begin{cases} 0, & n = 2k \\ \frac{(-1)^{k-1}\Gamma(k+\frac{1}{2})c_{2k+1}}{\pi}, & n = 2k-1,\ k > 0. \end{cases}$$

Since we know that $\zeta_a(s)$ has only a simple pole, located at $s = \frac{1}{2}$ we deduce that $c_{2k+1} = 0$, for all $k > 0$. □

4.7. Exercises

Exercise 4.1. Prove the equality (4.4.7). □

Chapter 5

Witten's deformation of the DeRham complex

In this chapter we will describe Witten's analytical proof of the classical Morse inequalities. Our approach follows closely [**Roe**]

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