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THE MASLOV INDEX, THE SPECTRAL FLOW, AND DECOMPOSITIONS OF MANIFOLDS

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0. Introduction. Consider a closed, compact-oriented Riemann manifold (M, g) and a Clifford bundle $\mathscr{E} \to M$ over M. The spectral flow of a smooth path of selfadjoint Dirac operators D^t: $C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ is the integer obtained by counting, with sign, the number of eigenvalues of D^t that cross 0 as t varies; it is a homotopy invariant of the path (cf. [AS]). The aim of this paper is to describe the spectral flow in terms of a decomposition of the manifold.

More precisely, suppose that M is divided into two manifolds-with-boundary M_1 and M_2 by an oriented hypersurface $\Sigma \subset M$. Assume that in a tubular neighborhood N of Σ, the metric is a product and the operators D^t have the "cylindrical" form

$$D^t = c(ds)(\partial/\partial s + D_0^t), \tag{0.1}$$

where s is the longitudinal coordinate in N, $c(ds)$ is the Clifford multiplication by ds, and D_0^t is independent of s. Set $\mathscr{E}_0 = \mathscr{E}|_\Sigma$ and denote by D_1^t and D_2^t the restriction of D^t to M_1 and M_2.

The kernels of D_j^t are infinite-dimensional spaces of solutions of $D_j^t \psi = 0$ on M_j. Restriction to Σ gives the *Cauchy-data spaces* (CD spaces)

$$\Lambda_1(t) = \operatorname{Ker} D_1^t|_\Sigma, \qquad \Lambda_2(t) = \operatorname{Ker} D_2^t|_\Sigma$$

in $L^2(\mathscr{E}_0)$. Note that the intersection $\Lambda_1(t) \cap \Lambda_2(t)$ is the finite-dimensional space of solutions of $D^t\psi = 0$ on M.

This setup has a rich symplectic structure. Multiplication by $c(ds)$ introduces a complex structure in $L^2(\mathscr{E}_0)$ and hence a symplectic structure in this space. The CD spaces $\Lambda_j(t)$ are then infinite-dimensional Lagrangian subspaces of $L^2(\mathscr{E}_0)$ that vary smoothly with t, and the pair $(\Lambda_1(t), \Lambda_2(t))$ is a Fredholm pair (as defined in Section 1). As in the finite-dimensional case, one can associate to a path of Fredholm pairs of Lagrangians an integer called the Maslov index. The main result of this paper is Theorem 3.14, which states that this Maslov index equals the spectral flow of the family D^t.

The Lagrangians defined by the CD spaces are infinite-dimensional, but the setup can be reduced to finite-dimensional symplectic geometry by "stretching the neck." This is done by changing the metric on M to one in which the neck is

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isometric to a long cylinder $(-r, r) \times \Sigma$. We study the Cauchy-data spaces in the adiabatic limits $r \to \infty$. These limits exist if we assume that D is "neck-compatible," i.e., is cylindrical, and that the operator D_0 in (1.1) is selfadjoint. If, moreover, certain nondegeneracy conditions are satisfied, these limits have a nice description, and our Cauchy-data spaces $\Lambda_j(t)$ stabilize to asymptotic Cauchy-data spaces Λ_j^∞. The limiting spaces arise naturally in the Atiyah-Patodi-Singer index problem [APS1]–[APS3]. A related adiabatic analysis was considered in [CLM2]. After performing this adiabatic deformation, we can reduce the Maslov index computation to a finite-dimensional situation by passing to a symplectic quotient. This generalizes a recent result of Yoshida [Y] in the context of Floer's instanton homology.

The paper consists of four sections. In Section 1, we translate some basic facts of finite-dimensional symplectic topology into infinite dimensions. We prove that the space of Fredholm pairs of Lagrangians has the homotopy type of the classifying space of KO^1. Next, we deal with the Maslov index in infinite dimensions. Using Arnold's definition [Ar] as a model, we define it as an intersection number and then derive some computational formulae which play a crucial part later.

Section 2 contains the main analytical technicalities of this paper. Many of these results are known, but we have reformulated them in a symplectic context (see [BW4] for an extended presentation of this subject).

Section 3 contains our main result: the Maslov index equals the spectral flow. The idea of the proof is to reduce the general problem via successive homotopies to a simple situation. For this we rely on a genericity result first used by Floer [F] in the context of symplectic homology (we give a complete proof in the appendix). After reducing to the case of piecewise affine homotopies, the theorem follows by an integration by parts formula. Again, this has an elegant symplectic interpretation.

Finally, in Section 4 we take up the problem of stretching the neck. This entails studying the behavior of the Cauchy-data spaces of a neck-compatible Dirac on a manifold M as the length of the neck tends to infinity. We begin by studying a related finite-dimensional problem. Namely, suppose that A is a $2n \times 2n$ symmetric matrix that anticommutes with the canonical complex structure J on $\mathbf{R}^{2n}$. We then get a 1-parameter group of symplectic transformations $r \to e^{-rA}$, and hence a flow on the Lagrangian Grassmanian $\Lambda(n)$ of $\mathbf{R}^{2n}$. In Corollary 4.4, we show that each trajectory in $\Lambda(n)$ has a unique limit point as $r \to \infty$; this limit is an A-invariant Lagrangian in $\mathbf{R}^{2n}$. This follows from a simple trick we learned from Tom Parker. We then return to the infinite-dimensional problem, where we can regard the CD spaces as infinite-dimensional Lagrangians evolving by the "flow" $r \to e^{-rD_0}$ as the neck length $r \to \infty$. By passing to a carefully defined symplectic quotient, we relate this to the above finite-dimensional situation. This yields Theorem 4.9, which shows that as the neck length $r \to \infty$, the Cauchy-data spaces stabilize to limiting infinite dimensional Lagrangians that can be explicitly described. We can then obtain the Maslov index from a computation in the finite-dimensional symplectic quotient (Corollary 4.14).

The families of Dirac operators for which we proved the splitting formula have constant symbol. In [N2] we deal with higher-dimensional families of Dirac operators and prove higher-dimensional splitting formulae using an entirely different approach. The techniques there can be used to successfully discuss the nonconstant symbol case as well.

Tom Mrowka informed the author that he proved these results using a similar approach. After this work was completed, the author learned that Ulrich Bunke independently obtained a splitting formula for the spectral flow (see [Bu2]) as consequence of a glueing result for the eta function of a neck-compatible Dirac (see [Bu1]). The results of this paper were announced in [N1].

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1. Infinite dimensional symplectic geometry. In this section, we study Lagrangian subspaces in an infinite-dimensional symplectic space. In contrast to the finite-dimensional situation, the Grassmanian of Lagrangian subspaces is contractible. A related, but topologically more interesting, space is the space of Fredholm pairs of Lagrangians. We will show this is a classifying space for KO^1 and then we will explicitly describe an isomorphism, called *the Maslov index*, between its fundamental group and $\mathbf{Z}$.

Let H be a separable real Hilbert space with inner product $(\ ,\ )$. We will denote the $*$-algebra of bounded linear operators on H by $B(H)$. Let $GL(H)$ be the group of invertible elements in $B(H)$, and let $O(H)$ be the subgroup of bounded orthogonal operators. For A, $B \in B(H)$, define the commutator and the anticommutator as usual:

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA.$$

Fix once and for all a complex structure on H, that is, an operator $J \in O(H)$ with $J^2 = -I$. Thus, H becomes a symplectic space with symplectic form

$$\omega(x, y) = (Jx, y) \quad \forall x, y \in H.$$

We can then introduce the basic notations of symplectic geometry. Let W be a subspace of H ("subspace" will always mean closed subspace). Its *annihilator* is the subspace

$$W^0 = \{y \in H; \omega(w, y) = 0\ \forall w \in W\}.$$

It is easily seen that $W^0 = JW^\perp$ where $W^\perp$ is the orthogonal complement of W in H.

Definition. A subspace W of H is called *isotropic* if $W \subset W^0$, *coisotropic* if $W^0 \subset W$, and *Lagrangian* if $W^0 = W$. Equivalently, W is Lagrangian if and only if $W^\perp = JW$.

Let $\mathscr{L} = \mathscr{L}_J$ be the set of *Lagrangian* subspaces of H. To topologize $\mathscr{L}$, we identify it with a space of operators using the following construction. Associated to each Lagrangian are three operators: the orthogonal projection P_L onto L, the complementary projection $Q_L = I - P_L$ onto the orthogonal complement of L, and the conjugation operator (reflection through L)

$$C_L = P_L - Q_L = 2P_L - I.$$

Note that $C = C_L$ satisfies

$$C = C^*, \qquad C^2 = I, \qquad \{C, J\} = 0. \tag{1.1}$$

It is easy to see that if C satisfies (1.1), then $\mathrm{Ker}(I - C)$ is a Lagrangian subspace with projection $P_L = 1/2(I + C)$. Thus, we can identify $\mathscr{L}_J$ with

$$\mathscr{C}_J = \{C;\, C \text{ satisfies (1.1)}\} \tag{1.2}$$

and topologize it using the operator norm. We will use this identification $\mathscr{L}_J = \mathscr{C}_{\mathscr{J}}$ frequently below.

The unitary group $\mathscr{U}_J(H) = \{U \in O(H);\, [U, J] = 0\}$ is a topological group that is contractible by Kuiper's theorem [Ku] and acts on $\mathscr{L}$ by $C \mapsto UCU^{-1}$. This action is transitive (just as in the finite-dimensional case, cf. [GS]). The stabilizer of L is $O(L) = \{U \in \mathscr{U}_J;\, [U, C_L] = 0\}$. Using standard arguments ([BW2] or [AS]), we get a fibration

$$O(L) \to \mathscr{U}_J \to \mathscr{L},$$

where, again by Kuiper's theorem, $O(L)$ is contractible. The long exact sequence in homotopy implies the following result.

PROPOSITION 1.1. *$\mathscr{L}$ is contractible.*

Thus, in infinite dimensions, $\mathscr{L}$ has no interesting topology. To get something interesting, we will consider

$$\mathscr{L}^{(2)} = \{(\Lambda_1, \Lambda_2) \in \mathscr{L}^2;\, (\Lambda_1, \Lambda_2) \text{ Fredholm pair}\}.$$

Recall that a pair of (V, W) of infinite-dimensional subspaces of H is called *Fredholm* if both subspaces have infinite codimension, $V + W$ is closed, and both

$\dim(V \cap W)$ and $\operatorname{codim}(V + W)$ are finite. The *Fredholm index* of this pair is defined as

$$i(V, W) = \dim(V \cap W) - \operatorname{codim}(V + W).$$

(For basic facts about Fredholm pairs, we refer to [C] or [K].) Note that Fredholm pairs of Lagrangians automatically have index 0 since

$$\begin{aligned} i(\Lambda_1, \Lambda_2) &= \dim(\Lambda_1 \cap \Lambda_2) - \dim(\Lambda_1^\perp \cap \Lambda_2^\perp) \\ &= \dim(\Lambda_1 \cap \Lambda_2) - \dim J(\Lambda_1 \cap \Lambda_2) = 0. \end{aligned} \tag{1.3}$$

We can also describe $\mathscr{L}^{(2)}$ in terms of conjugation operators. By Lemma 2.6 of [BW2], $(\Lambda_1, \Lambda_2) \in \mathscr{L}^{(2)}$ if and only if the corresponding conjugations satisfy $C_1 + C_2 \in \mathscr{K}$, where $\mathscr{K}$ is the space of compact operators on H. Thus,

$$\mathscr{L}^{(2)} = \{(C_1, C_2) \in \mathscr{C}^2; C_1 + C_2 \in \mathscr{K}\}.$$

Now fix $C_0 \in \mathscr{C}$. We have a fibration

$$\mathscr{L}_C \hookrightarrow \mathscr{L}^{(2)} \xrightarrow{p} \mathscr{L},$$

where $p(C_1, C_2) = C_1$ and $\mathscr{L}_0 = p^{-1}(C_0) = (-C_0 + \mathscr{K}) \cap \mathscr{C}$. Since $\mathscr{L}$ is contractible, we get a weak homotopy equivalence $\mathscr{L}^{(2)} \cong \mathscr{L}_C$. Set $\mathscr{U}_{\mathscr{K}} = \mathscr{U}_J \cap (I + \mathscr{K})$, and for $C \in \mathscr{C}$ set $O_{\mathscr{K},C} = (I + \mathscr{K}) \cap O_C$.

THEOREM 1.2. *There exists a weak homotopy equivalence*

$$\mathscr{L}_0 \cong U(\infty)/O(\infty)$$

where

$$U(\infty) = \varinjlim U(n) \qquad O(\infty) = \varinjlim O(n).$$

Proof. The proof will be carried out in several steps, with some intervening lemmas.

Step 1. $\mathscr{L}_0$ is path-connected. We associate to each finite-dimensional subspace $V \subset \Lambda_0$ the set

$$\mathscr{L}_0(V) = \{\Lambda \in \mathscr{L}_0; \Lambda \cap \Lambda_0 \subset V\} \subset \mathscr{L}_0.$$

These define a filtration of $\mathscr{L}_0$. To show that $\mathscr{L}_0$ is connected, it suffices to show that each $\mathscr{L}_0(V)$ is connected. Now in finite dimensions, the space of Lagrangians in $V + JV$ is connected (see [GS]). Hence, any Lagrangian in $\mathscr{L}_0(V)$ can be connected in $\mathscr{L}_0(V)$ to a Lagrangian in $\mathscr{L}_0(0)$. Thus, it suffices to show that $\mathscr{L}_0(0)$ is connected. This follows immediately from the next lemma, which gives an alternate description of $\mathscr{L}_0(0)$. The idea, which is standard in the finite-dimensional case, is to regard Lagrangian subspaces as the graphs of symmetric operators (cf. [Ar], [GS]).

LEMMA 1.3. *There is an identification*

$$\mathscr{L}_0(0) \cong \{\text{selfadjoint operators } J\Lambda_0 \to J\Lambda_0\}$$

and hence $\mathscr{L}_0(0)$ *is contractible.*

Proof. Suppose that (Λ, Λ_0) is a Fredholm pair of transversal Lagrangians. Let $P = P_{\Lambda_0}$ and $Q = I - P$. We deduce that $H = \Lambda + \Lambda_0$. In particular, this implies that $Q(\Lambda) = J\Lambda_0$ (see Figure 1). Using the fact that $\Lambda \cap \Lambda_0 = 0$, we see that $Q\colon \Lambda \to J\Lambda_0$ is also injective. The open mapping theorem implies that Q is an isomorphism. Construct the operator $A\colon J\Lambda_0 \to J\Lambda_0$ by

$$l^{\perp} \mapsto Q^{-1} l_{\perp} \mapsto PQ^{-1} l_{\perp} \mapsto JPQ^{-1} l_{\perp}.$$

Clearly, A is a bounded operator (by the closed graph theorem). Note that

(i) each $u \in \Lambda$ can be uniquely written as $u = l_{\perp} - JAl_{\perp}$, where $l_{\perp} = Qu \in J\Lambda_0$;

(ii) the condition that L is Lagrangian is equivalent to A being selfadjoint.

Conversely, given a selfadjoint operator $A\colon J\Lambda_0 \to J\Lambda_0$, its "graph"

$$\Lambda_A = \{l_{\perp} - JAl_{\perp};\, l_{\perp} \in J\Lambda_0\}$$

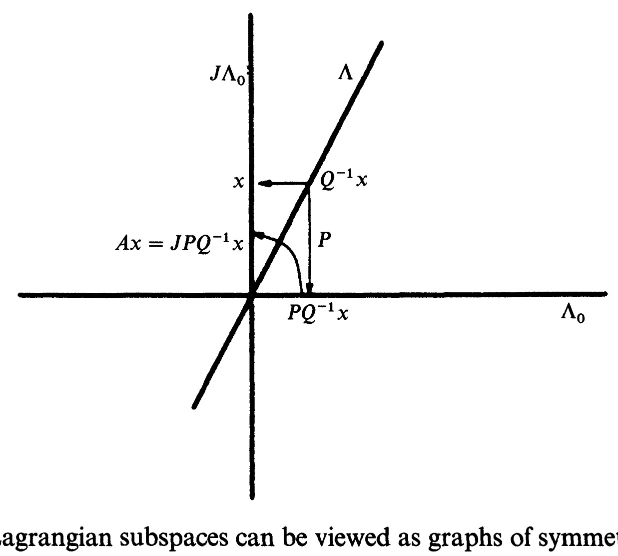

FIGURE 1. Lagrangian subspaces can be viewed as graphs of symmetric operators.

is a Lagrangian. Note that $\Lambda_A \cap \Lambda_0 = 0$. Now consider the operator

$$\bar{A}: H \to H, \qquad \bar{A}(l, l_\perp) = JAl_\perp, \qquad (l, l_\perp) \in \Lambda_0 \oplus J\Lambda_0.$$

One sees that Range$(I - \bar{A}) = \Lambda_0 + \Lambda_A$, so that the transversality of the pair (Λ_A, Λ_0) is equivalent to the surjectivity of $I - \bar{A}$. Since $\bar{A}^2 = 0$ for any selfadjoint $A: J\Lambda_0 \to J\Lambda_0$, we deduce that for any such A, $I - \bar{A}$ is invertible. Hence, (Λ_A, Λ_0) is a transversal Fredholm pair. □

Step 2. If $C_1, C_2 \in \mathscr{L}_0$ satisfy $\|C_1 - C_2\| < 2$, then there is a T in

$$GL_{\mathscr{K}} = \{T \in GL(H) \cap (I + \mathscr{K}); [T, J] = 0\}$$

such that

$$C_2 = TC_1T^{-1}. \tag{1.4}$$

Following [W], we set $T = I + 1/2(C_1 - C_2)C_1$. Then T is invertible, since $\|(C_1 - C_2)C_1\| < 2$, and T commutes with J because C_1 and C_2 anticommute with J. On the other hand, C_1 and C_2 lie in $-C + \mathscr{K}$, so $T \in GL_{\mathscr{K}}$. A simple computation shows that (1.4) holds.

Step 3. For each pair $C_1, C_2 \in \mathscr{L}_0$ there is a $T \in GL_{\mathscr{K}}$ such that

$$C_2 = TC_1T^{-1}. \tag{1.5}$$

This follows from Step 2 and the path-connectedness of $\mathscr{L}_0$; the details are left to the reader.

To proceed further we need the following technical result.

LEMMA 1.4. *If $T \in GL_{\mathscr{K}}$, then $(T^*T)^{1/2} \in GL_{\mathscr{K}}$.*

Proof. Set $S = T^*T$. Clearly, $S^{1/2} \in GL(H)$ and $S^{1/2}$ commutes with J. We have to show that $S^{1/2} \in I + \mathscr{K}$. Then $S \in I + \mathscr{K}$. To find $S^{1/2}$, we use Newton's iteration, as in [Ku]:

$$S_0 = I, \qquad S_{n+1} = 1/2(S_n + S_n^{-1}S).$$

Note that this iteration is well defined, since all S_n's are invertible (they are positive selfadjoint operators with their spectra bounded away from 0). One sees inductively that the right-hand side of the iteration is an affine combination of terms in $I + \mathscr{K}$. Thus, $S^{1/2} = \lim S_n \in I + \mathscr{K}$. □

Step 4. For each pair $C_1, C_2 \in \mathscr{L}_0$, there is a $U \in \mathscr{U}_{\mathscr{K}}$ such that

$$C_2 = UC_1U^*. \tag{1.6}$$

We follow the idea in [B, Proposition 4.6.5]. Consider $T \in GL_{\mathcal{K}}$ as in (1.5). Then $TC_1 = C_2 T$ and $C_1 T^* = T^* C_2$. It follows that C_1 commutes with $S = S = T^*T$, and hence with $S^{1/2}$. Setting $U = T(TT^*)^{-1/2}$, we clearly have $U^*U = I$, and $U \in GL_{\mathcal{K}}$ by Lemma 1.4. Therefore, U is in $\mathcal{U}_{\mathcal{K}}$ and satisfies (1.6).

Step 4 shows that $\mathcal{U}_{\mathcal{K}}$ acts transitively on $\mathcal{L}$. For $C \in \mathcal{L}_0$, the stabilizer of this action is $O_{\mathcal{K},C}$. Thus,

$$\mathcal{L}_0 \cong \mathcal{U}_{\mathcal{K}}/O_{\mathcal{K},C}, \qquad C \in \mathcal{L}_0. \tag{1.7}$$

Step 5. There exist homotopy equivalences

$$\mathcal{U}_{\mathcal{K}} \cong GL_{\mathcal{K}} \cong GL(\infty, \mathbf{C}), \qquad O_{\mathcal{K},C} \cong GL(\infty, \mathbf{R}). \tag{1.8}$$

The proof of $\mathcal{U}_{\mathcal{K}} \cong GL_{\mathcal{K}}$ is identical to the proof of Lemma 2.9 of [BW2]. It essentially uses the polar decomposition which by Lemma 1.4 is an internal decomposition in $GL_{\mathcal{K}}$, followed by an affine deformation of the positive symmetric term of the polarization. $I + \mathcal{K}$ is an affine space, so this deformation stays within $GL_{\mathcal{K}}$. Then by the results of Palais [P1], we have a homotopy equivalence

$$GL_{\mathcal{K}} = GL(\infty, \mathbf{C}).$$

The second part is completely analogous. Classically,

$$U(\infty) \cong GL(\infty, \mathbf{C}), \qquad 0(\infty) \cong GL(\infty, \mathbf{R}) \quad \text{homotopically}. \tag{1.9}$$

Theorem 1.2 follows from (1.7), (1.8), and (1.9). □

Remark 1.5. A related result was proved in [W], [BW3]. In that context, $\mathcal{K}$ represents compact *pseudodifferential* operators in some *complex* L^2 space.

The above arguments apply in finite dimensions to show that the Grassmanian $\Lambda(n)$ of Lagrangians in $\mathbf{C}^n$ is diffeomorphic to $U(n)/O(n)$. Taking the direct limits over the embeddings $\Lambda(n) \hookrightarrow \Lambda(n+1)$ then gives

$$\Lambda(\infty) = \varinjlim \Lambda(n) \cong U(\infty)/O(\infty).$$

Hence, we get the following corollary.

Corollary 1.6. *We have*

$$\mathcal{L}^{(2)} \cong \mathcal{L}_0 \cong U(\infty)/O(\infty) \cong \Lambda(\infty).$$

It is known that U/O is a classifying space for KO^1 (cf. [Kar]). On the other hand, Atiyah-Singer [AS] have shown that this classifying space can also be identified (up to homotopy) with the space of selfadjoint Fredholm operators on a real Hilbert space. Its fundamental group is isomorphic to **Z**. The isomorphism is given by the spectral flow (of a loop of selfadjoint Fredholm operators). Obviously,

$$\pi_1(U/O) \cong \mathbf{Z},$$

and the isomorphism is given by the Maslov index. Thus, Corollary 1.6 displays the double nature of $\mathscr{L}^{(2)}$: the operator theoretic nature and the symplectic nature. In the sequel, we will further analyze this duality.

It will be very convenient to have a computational description of the infinite-dimensional Maslov index. In the finite-dimensional situation, there are many excellent presentations of the Maslov index (see, e.g., [Ar], [CLM1], [D1], [D2], [GS], [RS]). However, all these assume the finite-dimensionality, especially when dealing with orientability questions. For a Banach manifold, orientability is a delicate question. To avoid this issue, we will give a meaning to a local intersection number without any elaborate considerations of orientability. Our approach is inspired from Arnold's description of the finite-dimensional index [Ar].

Consider a Lagrangian $\Lambda_0 = \mathrm{Ker}(I - C_0)$ specified by the conjugation C_0. The next several lemmas describe the geometry of the space

$$\mathscr{L}_0 = \{\Lambda \in \mathscr{L}/(\Lambda_0, \Lambda) \text{ is a Fredholm pair}\}.$$

LEMMA 1.7. *$\mathscr{L}_0$ is a smooth Banach manifold modelled on the space* $\mathrm{Sym}(J\Lambda_0)$ *of bounded symmetric operators on $J\Lambda_0$.*

Proof. To each finite-dimensional subspace V of Λ_0, we associate an orthogonal operator I_V commuting with J by

$$I_V(v) = \begin{cases} Jv & \text{for } v \in V \\ v & \text{for } v \in \Lambda \cap V^\perp \end{cases}$$

and the open subset

$$\mathscr{D}_V = \{\Lambda \in \mathscr{L}/\Lambda \cap I_V\Lambda_0 = 0\}.$$

Thus, $\mathscr{D}_V = I_V\mathscr{L}_0^*$, where

$$\mathscr{L}_0^* = \{\Lambda \in \mathscr{L}_0/\Lambda \cap \Lambda_0 = 0\}$$

is the dense open set of transverse pairs (in particular, $\mathscr{D}_0 = \mathscr{L}_0^*$). Notice that $I_V \in \mathscr{U}_{\mathscr{K}}$, so $(I_V\Lambda, \Lambda_0)$ is a Fredholm pair and thus $\mathscr{D}_V \subset \mathscr{L}_0$. The sets $\mathscr{D}_V$ cover $\mathscr{L}_0$: if $\Lambda \in \mathscr{L}_0$, then $\Lambda \in \mathscr{D}_V$ for $V = \Lambda \cap \Lambda_0$.

The isomorphism of Lemma 1.3 is a map $\Psi_0: \mathscr{D}_0 = \mathscr{L}_0^* \to \mathrm{Sym}(J\Lambda_0)$. For other V, set

$$\Psi_V = \Psi_0 \circ I_V^{-1}: \mathscr{D}_V \to \mathrm{Sym}(J\Lambda_0).$$

Then the collection

$$\{(\mathscr{D}_V, \Psi_V);\ V \in \mathscr{V},\ \Psi_V\colon \mathscr{D}_V \to \mathrm{Sym}(J\Lambda_0)\}$$

forms an atlas of $\mathscr{L}_0$. The verification that the transition functions are smooth is accomplished by writing the conjugation operator C associated to a Lagrangian in terms of these coordinates. The details are left to the reader. □

The manifold $\mathscr{L}_0$ is filtered by the subspaces

$$\mathscr{L}_0^m = \{\Lambda \in \mathscr{L}_0;\ \dim(\Lambda \cap \Lambda_0) = m\}.$$

In fact, these are subvarieties. Indeed, note that $\mathscr{L}_0^m$ is covered by charts of the form $\mathscr{D}_V$ with $\dim V = m$. Fix one such chart and write $S = \Psi_V$. By an elementary argument of Arnold [Ar, Lemma 3.3.3], one sees that $\Lambda \in \mathscr{D}_V$ lies in $\mathscr{L}_0^m$ if and only if

$$(SJu, Jv) = 0 \qquad \text{for all } u, v \in V \quad (S = \Psi_V(\Lambda)). \tag{1.10}$$

Since S is symmetric and $\dim V = m$, this describes $\mathscr{L}_0^m$ in this chart as the solution set of $m(m+1)/2$ algebraic equations. In particular, if $\Lambda \in \mathscr{L}_0^1$, then $\Lambda \cap \Lambda_0$ is a 1-dimensional space $V_0 = \mathbf{R}e$, $\Lambda \in \mathscr{D}_{V_0}$, and $S = \Psi_{V_0}(\Lambda)$. Then

$$(SJe, Je) = 0. \tag{1.11}$$

COROLLARY 1.8. *The closure $\overline{\mathscr{L}_0^1}$ is a codimension-1 subvariety of $\mathscr{L}_0$ called the "resonance divisor." It is stratified by subvarieties $\mathscr{L}_0^m$ of codimension $m(m+1)/2$.*

We may think of $\overline{\mathscr{L}_0^1}$ as a divisor in $\mathscr{L}_0$ defining an element in $H^1(\mathscr{L}_0, \mathbf{Z}) \cong \mathbf{Z}$ dual to the generator of $H_1(\mathscr{L}_0, \mathbf{Z})$. Dually, given a loop γ in $\mathscr{L}_0$, we may think of its Maslov index $\mu(\gamma)$ as being the intersection number $\gamma \cap \mathscr{L}_0$. Most of the rest of this section is devoted to making this intuition rigorous. We will first show that if a path γ intersects $\mathscr{L}_0^1$ transversally, one can associate a sign to each intersection point. The sum of the intersection numbers is a homotopy invariant of the path. As a byproduct, we will get several formulae for the local intersection number.

Consider the vector field χ over $\mathscr{L}_0$ defined by

$$\chi(\Lambda) = \left.\frac{d}{dt}\right|_{t=0} (e^{Jt}\Lambda).$$

PROPOSITION 1.9. *χ defines a transversal orientation on $\mathscr{L}_0^1$.*

Proof. This follows easily from a computation of Arnold [Ar, Lemma 3.5.3]. Consider $\Lambda \in \mathscr{L}_0$ and assume Λ lies in a coordinate chart $\mathscr{D}_V$, $V = \mathrm{span}(e)$, $|e| = 1$. If $S = \Psi_V(\Lambda)$ are the coordinates of Λ, then the coordinates of χ are given by the formula

$$\chi(\Lambda) = -(I + S^2).$$

Hence, $(\chi(\Lambda)Je, Je) < 0$ and thus, in view of (1.11), χ defines a transversal orientation along $\mathscr{L}_0^1$. □

Consider a path $\Lambda(t)$, which for $|t|$ small, lies in a single chart $\mathscr{D}_v = \mathscr{D}_{\mathbf{R}v}$ and such that $\Lambda(0) \cap \Lambda_0 = \mathbf{R}e_0$, $|e_0| = 1$. Let $S_t^v = \Psi_v(\Lambda(t))$. Assume $\Lambda(t)$ intersects $\mathscr{L}_0^1$ transversally at $t = 0$. The transversality can be rewritten as

$$(\dot{S}_t^v Je_0, Je_0) \neq 0,$$

where—here and below—the dot denotes d/dt at $t = 0$. Let $\mathscr{M} = \{v \in \Lambda_0;\, |v| = 1,\, \Lambda(0) \in \mathscr{D}_v\}$, and define a map $\sigma = \sigma_{\Lambda(\cdot)}\colon \mathscr{M} \to \{\pm 1\}$ by $\sigma(v) = \operatorname{sign}(\dot{S}_t^v Je_0, Je_0)$.

LEMMA 1.10. *For a path $\Lambda(t)$ as above, the map $\sigma_{\Lambda(\cdot)}$ is constant.*

Proof. One can alternatively characterize $\mathscr{M}$ as $\{v \in \Lambda_0;\, |v| = 1,\, (v, e_0) \neq 0\}$. Hence, $\mathscr{M}$ has two components:

$$\mathscr{M}_\pm = \{v \in \mathscr{M};\, \pm(v, e_0) > 0\}.$$

Now $\dot{S}_t^v$ varies continuously with v, and obviously $\sigma(v) = \sigma(-v)$. Thus, $\sigma\colon \mathscr{M} \to \{\pm 1\}$ a continuous even map, so is constant. □

Definition. For a path $\Lambda(t)$ as in Lemma 1.10, we define the local Maslov index by

$$\mu(\Lambda_0, \Lambda(t)) = \sigma_{L(\cdot)}(v), \qquad v \in \mathscr{M}. \tag{1.12}$$

By Lemma 1.10, this definition is independent of coordinates.

We will next give several more concrete versions of formula (1.12). To begin, note that in (1.12) $\sigma_\Lambda(v)$ is independent of v, so we are free to choose v as we please. Choose $v = e_0$. Set $f_0 = Je_0$ and $R_t = \Psi_0(I_0^{-1}\Lambda(t))$, where $I_0 = I_{\mathbf{R}e_0}$. Thus, (1.12) becomes

$$\mu(\Lambda_0, \Lambda(t)) = \operatorname{sign}(\dot{R}_t f_0, f_0). \tag{1.13}$$

Now consider the path $x_t = f_0 - JR_t f_0 \in I_0^{-1}\Lambda(t)$. Then $x_0 = f_0$ (since $e_0 \in \Lambda(0)$ so that $f_0 = -I_0^{-1}e_0 \in I_0^{-1}\Lambda(0)$), and hence

$$(x_t - x_0, e_0) = -(JR_t f_0, e_0) = (R_t f_0, f_0).$$

Differentiating at $t = 0$, we get

$$(\dot{x}_t, e_0) = (\dot{R}_t f_0, f_0). \tag{1.14}$$

Now introduce the conjugation $D = D(t)$ associated to $I_0^{-1}\Lambda(t)$. Since $x_t \in I_0^{-1}\Lambda(t)$, we have $x_t = D(t)x_t$. Differentiating this at $t = 0$, taking the inner product with e_0, and noting that $D(0)e_0 = -e_0$, we get

$$(\dot{x}_0, e_0) = (\dot{D}f_0, e_0) - (\dot{x}, e_0)$$

and therefore

$$2(\dot{x}_0, e_0) = (\dot{D}f_0, e_0) = (J\dot{C}f_0, f_0). \tag{1.15}$$

The conjugation associated with $\Lambda(t)$ is $C(t) = I_0^{-1}D(t)I_0$. Using this in (1.15), we deduce

$$2(\dot{x}_0, e_0) = (JI_0\dot{C}I_0^{-1}f_0, f_0) = (J\dot{C}e_0, e_0). \tag{1.16}$$

Combining (1.13), (1.14), and (1.16), we get the following corollary.

COROLLARY 1.11. *We have*

$$\mu(\Lambda_0, \Lambda(t)) = \operatorname{sign}(J\dot{C}e_0, e_0) = \operatorname{sign}\omega(\dot{C}e_0, e_0)$$

where $\Lambda(0) \cap \Lambda_0 = \mathbf{R}e_0$ *and* $\omega(x, y) = (Jx, y)$ *is the symplectic form.*

Note that the above formula is independent of coordinates. For the application we have in mind, we will need another variant of this formula. Consider a family $U(t) \in \mathcal{U}_J$ with

$$U(0) = I, \qquad C(t) = U(t)C(0)U(t)^*.$$

If we write $\dot{U} = JA$, where A commutes with J and A is selfadjoint, then

$$\dot{C} = JAC(0) - C(0)JA = JAC(0) + JC(0)A$$

$$\dot{C}e_0 = JAC(0)e_0 + JC(0)Ae_0 = J(Ae_0 + C(0)Ae_0) = J(I + C(0))Ae_0.$$

But $P(0) = 1/2(I + C(0))$ is the orthogonal projection onto $\Lambda(0)$, so

$$(J\dot{C}e_0, e_0) = -2(P(0)Ae_0, e_0) = -2(Ae_0, e_0).$$

Hence we have the following result.

COROLLARY 1.12. *If* $\Lambda(t) = U(t)\Lambda(0)$ *with* $U(t) = I + tJA + O(t^2)$, *then*

$$\mu(\Lambda_0, \Lambda(t))|_{t=0} = -\operatorname{sign}(Ae_0, e_0) = \operatorname{sign}\omega(\dot{U}e_0, e_0). \tag{1.17}$$

Remark 1.13. There is an ambiguity in the definition of the Maslov index, and, without a proper normalization, the Maslov index is well defined up to a sign. This is easily seen in the "mirror symmetry" of the Maslov index (cf. [CLM1, Prop. XI]:

$$\mu(\Lambda_1(t), \Lambda_2(t)) = -\mu(\Lambda_2(t), \Lambda_1(t)).$$

We consider as standard normalization the one in Property VII of [CLM1], and we want to compare it with our definition of the Maslov index. For this, we consider $\mathbf{R}^2$ with the standard symplectic structure

$$\omega(x, y) = -(Jx, y), \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $L_0 = \operatorname{span}(e_0)$, where $e_0 = (1, 0)$, and consider the path $L_t = e^{Jt}L_0$ for t in a small neighborhood of 0. Corollary 1.12 gives $\mu(L_0, L_t) = \operatorname{sign} \omega(Je_0, e_0) = 1$, which agrees with the standard normalization.

Consider

$$\mathscr{P}_0 = \{\gamma\colon (I, \partial I) \to (\mathscr{L}_0, \mathscr{L}_0^*)\}, \qquad I = [a, b] - \text{compact interval}$$

$$\mathscr{P}_0^* = \{\gamma \in \mathscr{P}_0;\ \gamma(t) \text{ intersects } \mathscr{L}_0^1 \text{ transversally}\}.$$

Since $\operatorname{codim} \mathscr{L}_0^k = k(k+1)/2 \geqslant 3$ if $k \geqslant 2$, we see that any path γ in $\mathscr{P}_0$ can be deformed (in $\mathscr{P}_0$) to a path in $\mathscr{P}_0^*$. For $\gamma^* \in \mathscr{P}_0^*$ define

$$\mu(\Lambda_0, \gamma^*) = \sum_{\gamma^*(t_j) \in \mathscr{L}_0^1} \mu(\Lambda_0, \gamma^*(t)|_{|t-t_j|<\varepsilon}).$$

This is the usual definition of an intersection number. In particular, standard arguments show that the above μ can be extended to the whole $\mathscr{P}_0$ as a homotopy invariant function. Now define

$$\mathscr{L}_*^{(2)} = \{(\Lambda_1, \Lambda_2) \in \mathscr{L}^{(2)};\ \Lambda_1 \cap \Lambda_2 = 0\}$$

and

$$\mathscr{P}^{(2)} = \{\gamma\colon (I, \partial I) \to (\mathscr{L}^{(2)}, \mathscr{L}_*^{(2)}\}.$$

Any $\gamma \in \mathscr{P}^{(2)}$ looks like $\gamma(t) = (\Lambda_1(t), \Lambda_2(t))$. Without any loss of generality, we may assume that $\Lambda_1(0) = \Lambda_0$. We can find a smooth family of unitary operators $U(t) \in \mathscr{U}_{\mathscr{X}}$ such that

$$\Lambda_1(t) = U(t)\Lambda_0, \qquad U(0) = I$$

and define

$$\mu(\gamma) = \mu(\Lambda_1(t), \Lambda_2(t)) = \mu(\Lambda_0, U(t)^{-1}\Lambda_2(t)).$$

Then one can easily check the following:
(A) $\mu(\gamma)$ is independent of the family $U(t)$;
(B) $\mu(\gamma)$ is a homotopy invariant.

Both assertions follow from the fact the inclusion $\mathscr{L}_0 \hookrightarrow \mathscr{L}^{(2)}$ is a homotopy equivalence. The details are left to the reader. An immediate consequence of the above considerations is that μ defines a morphism $\mu\colon \pi_1(\mathscr{L}^{(2)}) \to \mathbf{Z}$.

The finite-dimensional Maslov index behaves nicely with respect to symplectic reductions. So does this infinite-dimensional version of it. Recall first the process of reduction.

LEMMA 1.14. *Consider $\Lambda \subset H$ a Lagrangian of H, an isotropic subspace W and its annihilator W^0. If (Λ, W^0) is a Fredholm pair, then:*
(i) *$\mathscr{H}_0 = W^0/W$ has an induced symplectic structure;*
(ii) *$\Lambda^W = (\Lambda \cap W^0)/W$ is a Lagrangian subspace in W/W^0.*

Proof. First, (i) is straightforward and is left to the reader. We now prove (ii) in a special case, which is precisely the situation we will need. We will assume that Λ is *clean mod* **W**, i.e., $\Lambda \cap W = 0$. We will identify $\mathscr{H}_0$ with the orthogonal complement of W in W^0. Finally, set $U = (W^0)^{\perp}$. Then

$$H = \mathscr{H}_0 \oplus W \oplus U.$$

Denote by P_0 (resp., P_W, P_U) the orthogonal projections onto $\mathscr{H}_0$ (resp., W, U). Obviously, Λ^W is an isotropic subspace of $\mathscr{H}_0$. We show that it is maximal isotropic. Let $h_0 \in \mathscr{H}_0$ such that $(Jh_0, \Lambda^W) = 0$. Then

$$Jh_0 \perp (\Lambda^W + W) \Rightarrow Jh_0 \perp (\Lambda \cap W^0),$$

i.e.,

$$\begin{aligned} h_0 \in J(\Lambda \cap W^0)^{\perp} &= J(\Lambda^{\perp} + U) \quad \text{(since } (\Lambda, W^0) \text{ is Fredholm)} \\ &= J(J\Lambda + U) = \Lambda + W \quad \text{(since } W \text{ is isotropic)}. \end{aligned}$$

Thus, $h_0 \in \mathscr{H}_0 \cap (\Lambda + W)$ i.e., $h_0 \in \Lambda^W$. The lemma is proved. □

For any isotropic subspace W, JW is also isotropic, and we define

$$\mathscr{L}_W^{(2)}(H) = \{(\Lambda_1, \Lambda_2) \in \mathscr{L}^{(2)}/(\Lambda_1, W) \text{ is Fredholm } \Lambda_1 \cap W = \Lambda_2 \cap JW = 0\}.$$

(The pairs of $\mathscr{L}_W^{(2)}$ are called *clean mod W*). Note that if $(\Lambda_1, \Lambda_2) \in \mathscr{L}_W^{(2)}$, then $(\Lambda_2, (JW)^0)$ is Fredholm, and we have a natural identification $W^0/W \equiv$

$(JW)^0/JW$ (given by J). The reduction process described in Lemma 1.14 induces a map

$$\pi_W: \mathscr{L}_W^{(2)}(H) \to \mathscr{L}^{(2)}(\mathscr{H}_0)(\Lambda_1, \Lambda_2) \mapsto (\Lambda_1^W, \Lambda_2^{JW}).$$

Since the reduction is clean mod W, we deduce, as in the finite-dimensional case, that π_W is continuous (see [N2] for details). As in finite dimensions, we have the following result.

PROPOSITION 1.15 (Invariance under clean reductions). *If $\gamma(t) \in \mathscr{P}^{(2)}$ is clean mod W at any time, then*

$$\mu(\gamma) = \mu(\pi_W(\gamma)).$$

Proof. As before, it suffices to consider only the special case $\gamma(t) = (\Lambda_0, \Lambda_1(t))$, where t is very small. We can assume without any loss of generality that

$$\Lambda_1(t) = U(t)\Lambda_1(0), \qquad U(0) = I, \qquad U(t)|_{JW} \equiv I.$$

Let $\dot{U}(0) = JA$. Clearly, $A \equiv 0$ on JW. Using (1.17) to compute the local Maslov index, we see that W has no contribution in the formula and thus nothing changes if we mod W out. □

Remark 1.16. One can show that the map π_W is actually a homotopy equivalence (see [N2]). A similar result holds if we allow W to vary with t. As long as the reductions stay clean, we have the invariance of the Maslov class (see [DP], [V] for a related result). We leave the details to the reader.

Using the homotopy long exact sequence for the pair $(\mathscr{L}^{(2)}, \mathscr{L}_*^{(2)})$ and the results proved so far, we deduce the following theorem.

THEOREM 1.17. *Let $\gamma_0, \gamma_1 \in \mathscr{P}^{(2)}$. Then γ_0 is homotopic to γ_1 if and only if*

$$\mu(\gamma_0) = \mu(\gamma_1).$$

In particular, $\mu: \pi_1(\mathscr{L}^{(2)}) \to \mathbf{Z}$ is an isomorphism.

The details are left to the reader.

We now have a flexible definition of the Maslov index. In the following sections, we will apply it in connection to spectral flow computations.

2. Boundary value problems for Dirac operators. We gather in this section various analytical facts about boundary value problems for Dirac operators. Many of these results are known (see [BW4]), but we reformulate them in a form suitable to our purposes.

Consider an oriented Riemann manifold (M, g) and $\mathscr{E} \to M$ a euclidean vector bundle over M. Denote by $C(M)$ the bundle of Clifford algebras over M whose fiber at $x \in M$ is the Clifford algebra $C(T_x^*M)$. We will assume that $\mathscr{E}$ is a *selfadjoint* $C(M)$*-module*, that is, for each 1-form $\eta \in \Omega^1(M)$ the Clifford multiplication $c(\eta) \in \mathrm{End}(\mathscr{E})$ is skew-adjoint. Thus, we can speak of Dirac operators $D: C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ [BGV, Chapter 3]. In the sequel, all Dirac operators will be assumed formally selfadjoint. The space of selfadjoint Dirac operators compatible with a given Clifford action is an affine space modelled on the space of symmetric endomorphisms of $\mathscr{E}$.

Let M be a compact-oriented manifold with boundary $\Sigma = \partial M$, and suppose it is endowed with a cylindrical metric in a neighborhood of the boundary. More precisely, if $U \subset M$ is a collar neighborhood of Σ in M with an identification $\psi: U \cong \Sigma \times (-1, 0]$, then in these coordinates the Riemannian metric on M satisfies $g|_U = h + ds^2$, where h is a Riemannian metric on Σ (Figure 2). Denote by $\nabla = \nabla^g$ the corresponding Levi-Civita connection. Let $\mathscr{E}$ be a selfadjoint $C(M)$-module and $\hat{\nabla}$ a Clifford connection on $\mathscr{E}$. Set $\mathscr{E}_0 = \mathscr{E}|_\Sigma$ and pick an isomorphism (cylindrical gauge)

$$\Psi: \mathscr{E}|_U \cong \mathscr{E}_0 \times (-1, 0]$$

covering ψ such that over the neck

$$\hat{\nabla} = \hat{\nabla}^0 + ds \otimes \partial/\partial s,$$

where $\hat{\nabla}^0 = \hat{\nabla}|_\Sigma$. Fix once and for all the isomorphisms ψ, Ψ, the connection $\hat{\nabla}$, and the metric g.

Definition 2.1. A Dirac operator D is called *cylindrical* if over U it has the

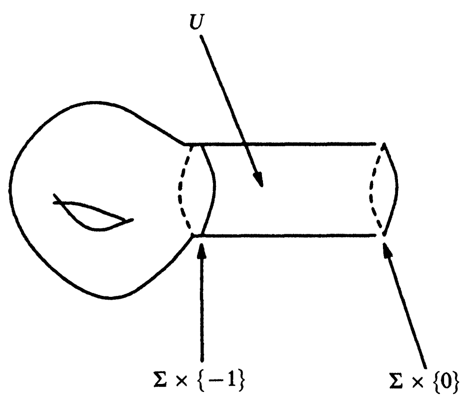

FIGURE 2. The metric is cylindrical in a neighborhood of the boundary.

form

$$D = c(ds)(\partial/\partial s + D_0), \tag{2.1}$$

where $D_0\colon C^\infty(\mathscr{E}_0) \to C^\infty(\mathscr{E}_0)$ is independent of s over U'. In addition, if D_0 is selfadjoint, then D is called *neck-compatible* (n.c.).

In the sequel, all Dirac operators on manifolds with boundary will be assumed cylindrical. The Dirac operator associated to the Clifford connection $\hat{\nabla}$ is neck-compatible. Also note that D_0 is a Dirac operator over the boundary, compatible with the induced Clifford action on $\mathscr{E}_0$.

If A is a *cylindrical* endomorphism of $\mathscr{E}$, i.e., a selfadjoint endomorphism satisfying $\partial/\partial s A = 0$ over U, then $\hat{D} + A$ is cylindrical. We deduce that the space of cylindrical Dirac operators is an affine space modelled on the space of cylindrical endomorphisms.

If A is a neck-compatible endomorphism of $\mathscr{E}$, i.e., a cylindrical endomorphism anticommuting with $d(ds)$

$$\{A, c(ds)\} = 0 \tag{2.2}$$

over U, then $D = \hat{D} + A$ is also an n.c. Dirac operator. In particular, we see that the n.c. operators also form an affine space modelled on the space of n.c. endomorphisms.

The adequate functional framework for all our future considerations is that of Sobolev spaces L^2_σ (distributions "σ-times differentiable" with derivatives in L^2). We will denote the norm of L^2_σ by $|\ |_\sigma$ and the L^2 norm by $|\ |$.

Let D be a Dirac operator. Following Seeley [S], we consider the spaces

$$\mathscr{K}(D) = \{u \in C^\infty(\mathscr{E});\, Du = 0 \text{ in } M\}$$

$$\mathscr{K}_\sigma(D) = \mathscr{K}(D) \cap L^2_\sigma.$$

We are interested in the subspace spanned by the restrictions over Σ of the sections in $\mathscr{K}_\sigma(D)$. For $\sigma > 1/2$, the existence of these restrictions is a consequence of classical trace results for Sobolev spaces (cf. [LiMa]). The case $\sigma = 1/2$ requires a more subtle treatment since the usual trace map is not defined. One uses the fact that $\mathscr{K}_{1/2}(D)$ is a distinguished subspace of sections satisfying an elliptic partial differential equation and a growth condition near the boundary. For $s \in (0, 1)$, consider the restriction map

$$R_s\colon C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E}_0) \qquad u \mapsto u|_{\Sigma \times \{s\}}.$$

For any $u \in \mathscr{K}_{1/2}(D)$, the limit (in $L^2(\mathscr{E}_0)$)

$$R_0 u \stackrel{\text{def}}{=} \lim_{s \to 0} R_s u$$

exists and is uniform in $\{|u|_{1/2} < 1\} \cap \mathscr{K}_{1/2}$ (see [BW4], [S]). This limit map has two important properties.

PROPOSITION 2.2. *R_0: $\mathscr{K}_{1/2} \to L^2(\mathscr{E}_0)$ is a continuous map satisfying*
(a) *unique continuation: If $u \in \mathscr{K}_{1/2}(D)$ and $R_0(u) = 0$, then $u = 0$;*
(b) *boundary estimates: If $u \in \mathscr{K}_{1/2}(D)$, then*

$$|u|_{1/2} \leqslant \text{const.}\, |R_0 u|.$$

For the proof, we refer to [BW4] or [S].

Definition 2.3. The *Cauchy-data space* of D (CD space) is the subspace $\Lambda(D) \subset L^2(\mathscr{E}_0)$ defined as

$$\Lambda(D) = R_0(\mathscr{K}_{1/2}(D)) = \mathscr{K}_{1/2}(D)|_\Sigma .$$

One sees that $\Lambda(D)$ is a closed subspace of $L^2(\mathscr{E}_0)$. It is roughly the subspace consisting of those sections $u \in L^2(\mathscr{E}_0)$ which extend to a solution of $DU = 0$ over M. Proposition 2.2 shows that R_0 is a linear isomorphism between $\mathscr{K}_{1/2}(D)$ and $\Lambda(D)$. The orthogonal projection $P(D)$ onto $\Lambda(D)$ is usually called the *Calderon projector* of D. By the classical results of [S], this projection is induced by a 0th-order pseudodifferential operator whose symbol can be explicitly computed [BW4], [P2], [S].

The dependence of the Calderon projector on the operator is rather nice. The method of constructing the Calderon projectors detailed in [BW4], Theorem 12.4(b) (see also [S]) can be used to prove the following result.

PROPOSITION 2.4. *Let $\{D^t\}$ be a family of cylindrical Dirac operators on M compatible with a fixed Clifford action. Assume D^t is smooth in some Sobolev norm L^2_k, where k is sufficiently large so that $L^2_k \hookrightarrow C^2$, (e.g., $k \geqslant N/2 + 2$, $N = \dim M$). Then the path of orthogonal projections Π_t onto $\Lambda(t) = \Lambda(D(t))$ is C^1 as a path in the Banach space of bounded operators $L^2(\mathscr{E}_0) \to L^2(\mathscr{E}_0)$.*

Proof. We begin by briefly recalling the construction of the Calderon projection. Let $\tilde{M}$ denote the double of M: $\tilde{M} = M \cup_\Sigma (-M)$. Continue to denote by s the longitudinal coordinate along a tubular neighborhood N of Σ in $\tilde{M}$, so that $N \cong \Sigma \times (-1, 1)$ and $s \leqslant 0$ on M.

For each Dirac operator D over M, denote by $\mathscr{D} = \mathscr{D}(D)$ the *invertible double* of D constructed in Theorem 9.1 of [BW4]. This is an invertible Dirac operator on a bundle $\tilde{\mathscr{E}}$ over $\tilde{M}$, extending D. Moreover, $\mathscr{D}$ depends smoothly upon D.

For every $u \in C^\infty(\mathscr{E}_0)$, denote by $\delta \otimes u$ the vector-valued distribution over $\tilde{M}$ defined by

$$\langle \delta \otimes u, V \rangle = \int_\Sigma (u, V|_\Sigma) \qquad V \in C^\infty(\tilde{\mathscr{E}}).$$

Note that supp $\delta \otimes u \subset \Sigma$ and $\delta \otimes u \in L^2_{-1/2-\varepsilon}$ for $0 < \varepsilon < 1/2$. This follows from an equivalent description of the map $u \mapsto \delta \otimes u$ as the adjoint of the trace map

$$\gamma\colon C^\infty(\tilde{\mathscr{E}}) \to C^\infty(\mathscr{E}_0) \qquad V \mapsto V|_\Sigma .$$

This adjoint is a continuous operator $\gamma^*\colon L^2_{-\sigma} \to L^2_{-1/2-\sigma}$ for all $\sigma > 0$.

Given $u \in C^\infty(\mathscr{E}_0)$, denote by $U = U(u)$ the distribution over $\tilde{M}$ defined by $U = \mathscr{D}^{-1}(\delta \otimes u)$. By classical regularity results, sing supp $U \subset \Sigma$ and $U \in L^2_{1/2-\varepsilon}$. In particular, U is smooth over the interior of M; and in [BW4, Theorem 12.4] or [S], it is shown that

$$R_0^- U = \lim_{s \to 0^-} U|\Sigma \times \{s\}$$

exists in any C^k norm. The basic result is that

$$\Pi(D)u = R_0^- U \qquad \forall u \in C^\infty(\mathscr{E}_0).$$

Now let D^t be a smooth path of Dirac operators over M and set $\Pi_t = \Pi(D^t)$, $\mathscr{D}_t = \mathscr{D}(D^t)$, and let $\|\cdot\|$ denote the natural norm in the space of bounded linear operators $L^2(\mathscr{E}_0) \to L^2(\mathscr{E}_0)$. One fact that will be used frequently in the sequel is the following inequality for distributions over a compact manifold:

$$|a\phi|_\sigma \leqslant C(\sigma)\|a\|_{C^2}|\phi|_\sigma \qquad 0 \leqslant \sigma \leqslant 2, \quad a \in C^2, \quad \phi \in L^2_\sigma. \tag{2.3}$$

The proof of (2.3) is immediate. For $\sigma = 0$ or $\sigma = 2$, this is simply the Holder inequality. For the other σ, it follows by interpolation.

The proof that $t \mapsto \Pi_t$ is smooth is carried out in several steps. For every $u \in C^\infty(\mathscr{E}_0)$ and any t set $U_t = \mathscr{D}_t^{-1}(\delta \otimes u)$. Fix $\varepsilon \in (0, 1/2)$. Note that

$$|U_t|_{1/2-\varepsilon} \leqslant C|u| \tag{2.4}$$

where $C > 0$ is independent of t at least for all small t.

Step 1. We will prove that $U_t - U_0 \in L^2_{3/2-\varepsilon}$ and

$$|U_t - U_0|_{3/2-\varepsilon} \leqslant Ct|u| \qquad \forall u \in C^\infty(\mathscr{E}_0). \tag{2.5}$$

To prove (2.5), write $\mathscr{D}_t$ as $\mathscr{D}_t = \mathscr{D}_0 + A(t)$, where $A(t) \in \mathrm{End}(\tilde{\mathscr{E}})$ satisfies

$$\|A(t)\|_{C^2} = O(t) \qquad \text{as } t \to 0. \tag{2.6}$$

U_t satisfies the equation $\mathscr{D}_0 U_t + A(t)U_t = \delta \otimes u$ so that

$$\mathscr{D}_0(U_t - U_0) = -A(t)U_t \in L^2_{1/2-\varepsilon}.$$

By standard elliptic regularity, we deduce $U_t - U_0 \in L^2_{3/2-\varepsilon}$. Using elliptic estimates and the invertibility of $\mathscr{D}_0$, we deduce

$$|U_t - U_0|_{3/2-\varepsilon} \leqslant C|A(t)U_t|_{1/2-\varepsilon}.$$

The estimate (4.7) follows immediately, using (2.3), (2.4), and (2.6).

Step 2.

$$\|\Pi_t - \Pi_0\| = O(t) \qquad \text{as } t \to 0. \tag{2.7}$$

For $u \in C^\infty(\mathscr{E}_0)$, we have

$$\Pi_t u = R_0^- U_t = R_0^- U_0 + R_0^-(U_t - U_0) = \Pi_0 u + R_0^-(U_t - U_0).$$

The existence of $R_0^-(U_t - U_0)$ follows from the regularity established at Step 1 and classical trace results. In particular,

$$|\Pi_t u - \Pi_0 u| = |R_0^-(U_t - U_0)| \leqslant C|U_t - U_0|_{3/2-\varepsilon}.$$

The estimate (2.7) now follows from (2.5). In particular, we proved that Π_t depends continuously upon t.

For any $u \in C^\infty(\mathscr{E}_0)$, let V_0 be defined as the unique solution of the equation

$$\mathscr{D}_0 V_0 + \dot{A}(0)U_0 = 0, \tag{2.8}$$

where as before $\mathscr{D}_t = \mathscr{D}_0 + A(t)$, $U_0 = \mathscr{D}_0^{-1}(\delta \otimes u)$, and the dot denotes the differentiation at $t = 0$.

Step 3.

$$|U_t - U_0 - tV_0|_{3/2-\varepsilon} \leqslant Ct^2|y| \qquad \text{for all } u \in C^\infty(\mathscr{E}_0) \text{ and all } t \text{ small}. \tag{2.9}$$

To prove (2.9), write $A(t) = t\dot{A}(0) + R(t)$, where $R(t) \in \mathrm{End}(\tilde{\mathscr{E}})$ and

$$\|R(t)\|_{C^2} = O(t^2) \qquad \text{as } t \to 0. \tag{2.10}$$

The equation $\mathscr{D}_t U_t = \delta \otimes u$ can be rewritten as

$$\mathscr{D}_0 U_t + t\dot{A}(0)U_t + R(t)U_t = \delta \otimes u.$$

Using (2.8) and $\mathscr{D}_0 U_0 = \delta \otimes u$, we deduce

$$\mathscr{D}_0(U_t - U_0 - tV_0) = -t\dot{A}(0)(U_t - U_0) - R(t)U_t.$$

Hence, by elliptic estimates, we have

$$|U_t - U_0 - tV_0|_{3/2-\varepsilon} \leqslant C(t|\dot{A}(0)(U_t - U_0)|_{1/2-\varepsilon} + |R(t)U_t|_{1/2-\varepsilon}).$$

The estimate (2.9) follows easily, using (2.3), (2.5), and (2.10).

Coupling elliptic estimates in (2.8) with the relations (2.3) and (2.4), we get

$$|V_0|_{3/2-\varepsilon} \leqslant C|u|. \tag{2.11}$$

The reader can now verify immediately that $t \to \Pi_t$ is C^1 and

$$\dot{\Pi}_0 u = R_0^- V_0.$$

Proposition 2.4 is proved. □

We can now relate the Dirac operators and their CD spaces to the infinite-dimensional symplectic topology of the previous sections. All this setup lies over a natural symplectic background. Indeed, $c(ds)$ is a fiberwise isometry, so it defines a unitary operator

$$J: L^2(\mathscr{E}_0) \to L^2(\mathscr{E}_0), \qquad J^2 = -I,$$

i.e., J is a complex structure on $L^2(\mathscr{E}_0)$, thus defining a symplectic structure

$$\omega(u, v) = \int_\Sigma (Ju, v)$$

for all $u, v \in L^2(\mathscr{E}_0)$.

The next result is the key fact which unifies all the topics discussed so far. It is another manifestation of the duality *Selfadjoint operators* ~ *Lagrangian subspaces.*

PROPOSITION 2.5 [BW2, Proposition 3.2]. *$\Lambda(D)$ is a Lagrangian subspace of $L^2(\mathscr{E}_0)$ with respect to the natural symplectic structure induced by the Clifford multiplication with ds.*

Finally, consider the following situation. (M, g) is a compact oriented Riemann manifold and $\mathscr{E}$ a selfadjoint $C(M)$-module over M. Let Σ be an oriented hypersurface in M, which divides it into two manifolds-with-boundary M_1, M_2. Choose N_1, N_2 tubular neighborhoods of M_1, M_2 such that $N_1 \cong \Sigma \times (-1, 0]$, $N_2 \cong \Sigma \times [0, 1)$ (see Figure 3). Set $N = N_1 \cup N_2$. We assume the metric g is a product metric on N, i.e., $g|_N = ds^2 + h$, where h is a metric on Σ and $-1 < s < 1$ is the longitudinal coordinate on N. Let $D: C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ be a Dirac operator on M. Denote by D_1 (*resp.*, D_2) its restrictions to M_1 (resp., M_2). D will be called *cylindrical* if both D_1 and D_2 are cylindrical. As usual, set $\mathscr{E}_0 = \mathscr{E}|_\Sigma$. $L^2(\mathscr{E}_0)$ has a symplectic

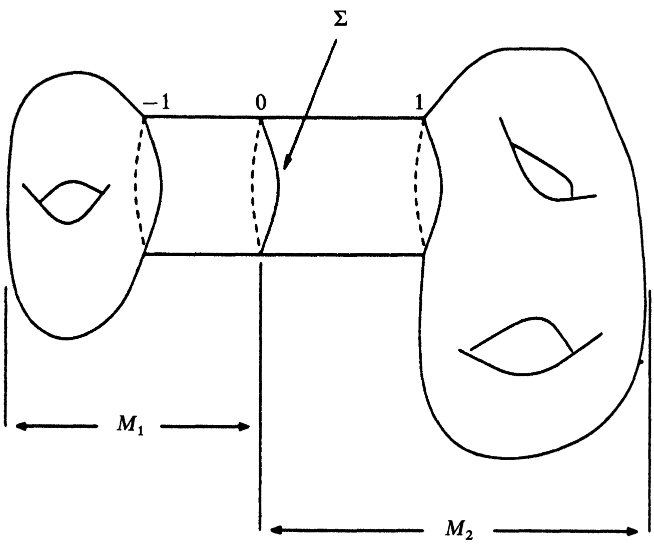

FIGURE 3. A metrically nice splitting

structure induced by the Clifford action. The CD spaces $\Lambda_1(D)$, $\Lambda_2(D)$ of D_1 and D_2 are Lagrangian subspaces in $L^2(\mathscr{E}_0)$. In fact, more can be said.

PROPOSITION 2.6. *$(\Lambda_1(D), \Lambda_2(D))$ is a Fredholm pair. Moreover, (Λ_1, Λ_2) is a transversal pair if and only if D is invertible.*

Proof. Let P_j be the orthogonal projection onto Λ_j, $j = 1, 2$. We have seen that these are 0th-order pseudodifferential operators in $L^2(\mathscr{E}_0)$. In [S] (see also [P2, Chapter XVII]), it is proved that their symbols satisfy

$$\sigma(P_1)(\xi) + \sigma(P_2)(\xi) = \mathrm{Id}.$$

Thus, $P_1 - (I - P_2)$ is a pseudodifferential operator of order $\leqslant -1$ in $L^2(\mathscr{E}_0)$. In particular, $P_1 - (I - P_2)$ is compact, so that (Λ_1, Λ_2) is a Fredholm pair. The second part is intuitively clear (see also [BW2, Corollary 3.4]). □

3. The Maslov index and the spectral flow. The setting of this section is identical to the one at the end of Section 2. We endow the space of cylindrical Dirac operators $\mathscr{D}$ with a Sobolev topology, given by an L_k^2 norm with k sufficiently large so that $L_k^2 \hookrightarrow C^2$. Inside $\mathscr{D}$ sits

$$\mathscr{D}^* = \{D \in \mathscr{D}; D \text{ is invertible}\}.$$

To any continuous path $\gamma = D(t)$ in $\mathscr{D}$ with endpoints in $\mathscr{D}^*$, one can associate an integer, *the spectral flow* $SF(\gamma)$ (see [APS3], [BW1]) defined as the number of eigenvalues of $D(t)$ that change from negative to positive, minus the number of eigenvalues that change from positive to negative. This is a homotopy invariant of γ (cf. [AS], [BW1]) with an obvious additivity property. If $\gamma_1, \gamma_2: (I, \partial I) \to (\mathscr{D}, \mathscr{D}^*)$ with $\gamma_1(1) = \gamma_2(0)$, then

$$SF(\gamma_1 \cdot \gamma_2) = SF(\gamma_1) + SF(\gamma_2),$$

so the spectral flow can be viewed as a homomorphism

$$SF: \pi_1(\mathscr{D}, \mathscr{D}^*) \to \mathbf{Z}.$$

In Section 2, we defined a continuous map

$$\Lambda^{(2)}: (\mathscr{D}, \mathscr{D}^*) \to (\mathscr{L}^{(2)}, \mathscr{L}_*^{(2)}), \qquad D \mapsto (\Lambda_1(D), \Lambda_2(D)).$$

Denote by $\Lambda_*^{(2)}$ the homomorphism between π_1's induced by this map.

We will prove that the following diagram is commutative.

$$\begin{array}{ccc} \pi_1(\mathscr{D}, \mathscr{D}^*) & \xrightarrow{\Lambda_*^{(2)}} & \pi_1(\mathscr{L}^{(2)}, \mathscr{L}_*^{(2)}) \\ & {\scriptstyle SF}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & \mathbf{Z} & \end{array}$$

Here $\mu: \pi_1(\mathscr{L}^{(2)}, \mathscr{L}_*^{(2)}) \to \mathbf{Z}$ is the Maslov index isomorphism constructed in Section 1.

To this end, we will need a localization procedure for the spectral flow. Let $t \mapsto D(t) \in \mathscr{D}$ ($|t| \leqslant \varepsilon$) be a smooth family of cylindrical Dirac operators such that $D(t)$ is invertible for $t \neq 0$. Let $K_0 = \ker D(0)$, and denote by P_0 the orthogonal projection onto K_0. We form the *resonance matrix*:

$$R = R(A): K_0 \to K_0 \qquad R = P_0 \dot{D}(0).$$

We can view R as a symmetric matrix. We have the following result [DRS].

THEOREM 3.1. *Let D and A as above satisfy (1). If the resonance matrix $R(A)$ is nondegenerate, then its signature gives the spectral flow*

$$SF(D(t); |t| \leqslant \varepsilon) = \operatorname{sign} R(A).$$

The above formula follows from an abstract result of Kato (Theorems II.5.4 and II.6.8 of [K]), which we recall now. H is a separable Hilbert space and $A(t)\, t \in \mathbf{R}$ a family of unbounded selfadjoint operators with a fixed dense domain W. W becomes a Hilbert space in its own right using the graph norms. We assume that the embedding $W \hookrightarrow H$ is compact and that the resolvent set of

$A(t)$ is nonempty for every t. Then $A(t)$ has compact resolvent and its spectrum consists entirely of eigenvalues with finite multiplicities. $A(t)$ can also be interpreted as bounded operators $W \to H$. As such, we assume that $A(t)$ depends smoothly upon t. The following result gives precise information about how the eigenvalues of A vary.

THEOREM 3.2 (Kato Selection Theorem). *Let $t_0 \in \mathbf{R}$ and $c_0 > 0$ such that $\pm c_0 \notin \sigma(A(t_0))$. Then there exists a constant $\varepsilon > 0$ and differentiable functions $\lambda_j: (t_0 - \varepsilon, t_0 + \varepsilon) \to (-c_0, c_0)$, $j = 1, 2, \ldots, N$ (N is the dimension of the subspace spanned by the eigenvectors corresponding to eigenvalues in $(-c_0, c_0)$) such that $\lambda_j(t) \in \sigma(A(t))$ and*

$$\dot{\lambda}_j(t) \in \sigma(P_j(t))\dot{A}(t)P_j(t),$$

where $P_j(t): H \to H$ denotes the orthogonal projection onto $\ker(\lambda_j(t)I - A(t))$. Moreover, if $\lambda \in \sigma(A(t)) \cap (c_0, c_0)$ with corresponding spectral projection $P: H \to \ker(\lambda I - A(t))$ and $\theta \in \sigma(P\dot{A}(t)P$ is an eigenvalue of multiplicity m, then there are precisely m indices $j_1, \ldots, j_m$ such that $\lambda_{j_\nu}(t) = \lambda$ and $\dot{\lambda}_{j_\nu}(t) = \theta$ for $\nu = 1, \ldots, m$.

The Kato Selection Theorem has a corollary particularly important for our purposes. To formulate it, introduce the set of *positive* cylindrical endomorphisms

$$\mathrm{Cyl}(\mathscr{E})_+ = \{A \in \mathrm{Cyl}(\mathscr{E})/\exists\lambda > 0\colon \inf \sigma(A(x)) \geqslant \lambda\ \forall x \in M\},$$

where $\sigma(A(x))$ is the spectrum of the selfadjoint endomorphism

$$A(x)\colon \mathscr{E}_x \to \mathscr{E}_x.$$

Set

$$\mathscr{P} = \{\gamma\colon (I, \partial I) \to (\mathscr{D}, \mathscr{D}^*);\ \gamma \in C^1\}.$$

A path $\gamma \in \mathscr{P}$ is called *positive* if $\dot{\gamma} \in \mathrm{Cyl}(\mathscr{E})_+$ and *negative* if $-\dot{\gamma} \in \mathrm{Cyl}(\mathscr{E})_+$. The set of positive (resp., negative) paths is denoted by $\mathscr{P}_+$ (resp., $\mathscr{P}_-$). The *resonance* set $Z = Z(\gamma)$ of a path $\gamma \in \mathscr{P}$ is defined as

$$Z = \{t \in I;\ \ker D(t) \neq 0\}.$$

We can now formulate the following lemma.

LEMMA 3.3. *The resonance set of a positive path is finite.*

Proof. Let $\gamma = D(t) \in \mathscr{P}_+$ and $t_0 \in Z(\gamma)$. Since $\dot{D}(t_0) \in \mathrm{Cyl}(\mathscr{E})_+$, the resonance matrix is positive definite, and by Kato's selection theorem, we deduce that $D(t)$ is invertible when t is in some ε-neighborhood of t_0. Therefore, $Z(\gamma)$ is a discrete set. □

Positive paths have other important properties.

LEMMA 3.4. *Any path $\gamma \in \mathscr{P}$ is homotopic to a product of a positive path with a negative path. (In the sequel, all the homotopies of paths $(I, \partial I) \to (\mathscr{D}, \mathscr{D}^*)$ will be understood as relative homotopies: the endpoints stay invertible during the deformation.)*

Proof. The difference $A = D(1) - D(0) \in \mathrm{Cyl}(\mathscr{E})$ is a bounded, selfadjoint endomorphism of $\mathscr{E}$. Choose $C > 0$ such that

$$C \geqslant 1 + |\sup \sigma(A(x))| \qquad \forall x \in M \tag{3.1}$$

$$D(0) + C \cdot Id_{\mathscr{E}} \in \mathscr{D}^*. \tag{3.2}$$

The choice (3.2) is possible by Lemma 3.3. Now consider

$$\alpha_+ = D(0) + tC \cdot Id_{\mathscr{E}} \qquad t \in I$$

$$\alpha_- = D(0) + C \cdot Id_{\mathscr{E}} + t(A - C \cdot Id_{\mathscr{E}}) \qquad t \in I.$$

By (3.1) and (3.2), $\alpha_\pm \in \mathscr{P}_\pm$. γ is homotopic to $\alpha_+ \cdot \alpha_-$ via an affine homotopy. □

Definition 3.5. A C^1 path $\gamma: (I, \partial I) \to (\mathscr{D}, \mathscr{D}^*)$, $t \mapsto D(t)$ is called
(i) *locally affine* if $\dot{\gamma} = \text{const.}$ in a neighborhood of any $t \in Z(\gamma)$;
(ii) *good* if $Z(\gamma)$ is finite and for all $t \in Z(\gamma)$ $\dim \ker D(t) = 1$.

A key step in our deformation program is a genericity result, which states that almost any path of Dirac operators is good.

PROPOSITION 3.6. *Let D be a cylindrical Dirac operator and assume* rank $\mathscr{E} \geqslant 2$. *Then there exists a Baire set $\mathscr{A}_{\mathrm{reg}} \subset \mathscr{A}$ such that for $\mathscr{A} \in \mathscr{A}_{\mathrm{reg}}$ the path $D(t) = D + A(t)$ is good.*

The proof of this proposition is carried out in the appendix. In particular, since $\mathscr{P}_+$ is open in $\mathscr{P}$, we deduce the following corollary.

COROLLARY 3.7. *Any positive path is homotopic to a positive good path.*

A simple application of Kato's Selection Theorem yields the next lemma.

LEMMA 3.8. *A positive standard path $\gamma \in \mathscr{P}$ is homotopic to a locally affine, positive, good path $\tilde{\gamma}$ such that:*
(i) *$Z(\gamma) = Z(\tilde{\gamma})$;*
(ii) *for all $t \in Z(\gamma)$: $\gamma(t) = \tilde{\gamma}(t)$.*

Proof. The underlying idea is natural: any path is locally homotopic to the tangent line at a point on the path. The only thing we have to prove is that we can find a *relative* homotopy achieving this. Assume $\gamma: [-1, 1] \to \mathscr{D}$ and $Z(\gamma) = \{0\}$. Set $D(t) = D(0) + A(t)$ and $A_0 = \dot{A}(0)$. A_0 is a positive cylindrical endomorphism of $\mathscr{E}$. Consider

$$\tilde{D}_s(t) = (1-s)D(t) + stA_0 \qquad s \in [0,1].$$

By the Kato Selection Theorem, there exists $\varepsilon > 0$ such that for all $0 < |t| < \varepsilon$, $D(t)$ is invertible and its inverse $E(t)$ satisfies

$$\|E(t)\| = O\left(\frac{1}{t\|A_0\|}\right). \tag{3.3}$$

Now

$$\tilde{D}_s(t) = D(t) + R_s(t),$$

where $R_s(t) = s(tA_0 - A(t))$ satisfies

$$\|R_s(t)\| = o(t) \qquad \text{uniformly in } s. \tag{3.4}$$

Thus,

$$E(t)\tilde{D}_s(t) = I + K_s(t) \qquad K_s(t) = E(t)R_s(t), \tag{3.5}$$

where (by (3.3))

$$\|K_s(t)\| = o(1) \qquad \text{uniformly in } s. \tag{3.6}$$

Hence, we can find $t_0 > 0$ such that

$$\|K_s(\pm t_0)\| < 1/2 \qquad \forall s \in [0,1],$$

and from (3.5) we deduce that $\tilde{D}_s(\pm t_0)$ is invertible for all s. Therefore, $\tilde{D}_s(t)$ is an admissible homotopy between γ and a locally affine path satisfying properties (i) and (ii) in the lemma. □

The homotopies constructed so far were between paths close to each other in the C^1 distance. Our next result describes one instance of homotopic paths which can be C^1-far apart (but still C^0-close).

LEMMA 3.9. *Let $D \in \mathscr{D}$ and $A \in \mathrm{Cyl}(\mathscr{E})$ such that* $\dim \ker D = 1$,

$$\ker D = \mathrm{span}(U),$$

and

$$(AU, U) \neq 0.$$

If $B \in \mathrm{Cyl}(\mathscr{E})$ *is such that*

$$(BU, U) = (AU, U),$$

then $\exists \varepsilon > 0$ *such that for all* $0 < |t| \leqslant \varepsilon$ *and for all* $s \in I$,

$$\gamma_s(t) = D + (1 - s)(tA) + s(tB) \in \mathscr{D}^*.$$

In particular, $\gamma_s(\cdot) \in \mathscr{P}$ *realizes an affine homotopy between* $\gamma_0(t) = D + tA$ *and* $\gamma_1(t) = D + tB$ $(|t| \leqslant \varepsilon)$.

Proof. The paths $\gamma_s(t)$ are analytic in t (being affine). In such situations, more powerful perturbation results are available. In particular, by Theorem VII 3.9 of [K], there exist $\varepsilon_1 > 0$ and analytic functions

$$\lambda_{n,s}\colon [-\varepsilon_1, \varepsilon_1] \to \mathbf{R} \qquad n \in \mathbf{Z}, s \in [0, 1]$$

such that

$$\sigma(\gamma_s(t)) = \{\lambda_{n,s}(t)/n \in \mathbf{Z}\} \qquad \text{(multiplicities included)}.$$

We labelled the eigenvalues so that $\lambda_{0,s}(0) = 0 \in \ker D$. Note that $\lambda_{n,s}(0)$ is independent of s for all $n \in \mathbf{Z}$. We will denote it by λ_n. On the other hand, we can find $a, b > 0$ such that for all $v \in C^\infty(\mathscr{E})$,

$$\|\dot{\gamma}_s(t)v\| \leqslant a\|v\| + b\|\gamma_s(t)v\| \qquad \forall |t| \leqslant \varepsilon \quad \forall s \in [0, 1].$$

Theorem VII 3.6 of [K] implies

$$|\lambda_{n,s}(t) - \lambda_n| \leqslant C(1 + \lambda_n)t,$$

where $C = C(a, b) > 0$ is independent of $n \in \mathbf{Z}$ and $s \in [0, 1]$. In particular, for

$$0 < |t| \leqslant \varepsilon_2 = \inf\left\{\frac{|\lambda_n(0)|}{2C(1 + |\lambda_n(0)|)}\Big/ n \in \mathbf{Z}\backslash\{0\}\right\} \cup \{\varepsilon_1\},$$

$$|\lambda_{n,s}(t)| \geqslant 1/2\,|\lambda_n(0)| > 0. \tag{3.7}$$

On the other hand, by the Kato Selection Theorem,

$$\dot{\lambda}_{0,s}(0) = (AU, U) \neq 0 \qquad \forall s \in [0, 1].$$

Arguing by contradiction, we can find $0 < \varepsilon < \varepsilon_2$ such that

$$\lambda_{0,s}(\pm\varepsilon) \neq 0. \tag{3.8}$$

In particular, (3.7) and (3.8) show that the operators $\gamma_s(\pm\varepsilon)$ are invertible for any s, and Lemma 3.9 is proved. □

Definition 3.10. A good path $\gamma \in \mathscr{P}$ is called *elementary* if for all $t \in Z(\gamma)$

$$\dot{\gamma}(t) = \alpha \operatorname{Id}_{\mathscr{E}}$$

for some $\alpha \in C_0^\infty(M)$, a function supported in $M_2 \backslash N$ and not changing sign (see Figure 4).

Remark 3.11. If $\alpha \in C_0^\infty(M)$ is as in Definition 3.10 and $D \in \mathscr{D}$, then the unique continuation principle for Dirac operators [BW4, Chapter 8] implies that

$$(\alpha U, U) \neq 0 \qquad \forall U \in \ker D\backslash\{0\}.$$

Lemmata 3.3, 3.4, 3.8, 3.9, Corollary 3.7, and Remarks 3.11 have the following corollary.

COROLLARY 3.12. *Any path $\gamma \in \mathscr{P}$ is homotopic to an elementary path.*

In particular, we have the following abstract result.

PROPOSITION 3.13. *Let $\phi\colon \mathscr{P} \to \mathbf{Z}$ be a continuous, additive function such that for any elementary path ω*

$$\phi(\omega) = SF(\omega)$$

and $\Phi(\gamma) = 0$ for every $\gamma\colon I \to \mathscr{D}^*$. Then for all $\gamma \in \mathscr{P}$: $\phi(\gamma) = SF(\gamma)$.

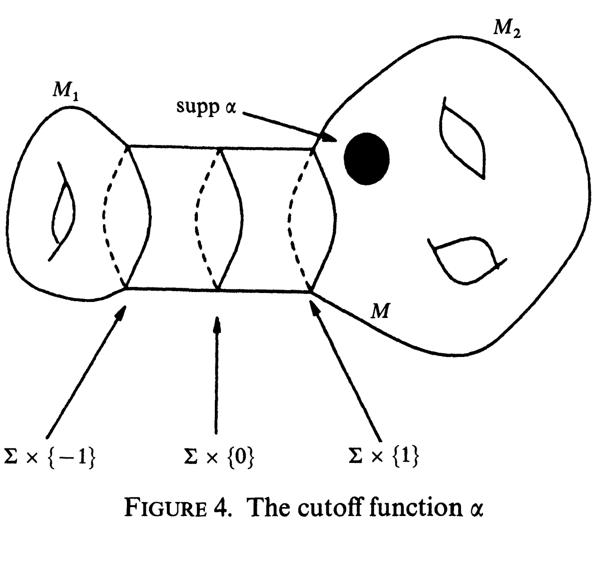

FIGURE 4. The cutoff function α

Let $\gamma \in \mathscr{P}$, $\gamma(t) = D(t)$. Denote by $D_j(t)$ the restriction of $D(t)$ to M_j $j = 1, 2$. Let $\Lambda_j(t)$ be the CD space of $D_j(t)$ $j = 1, 2$. Since $D(0)$ and $D(1)$ are invertible, we deduce $\Lambda_1(t) \cap \Lambda_2(t) = 0$ for $t = 0, 1$. The results of Section 2 show that the Fredholm pairs of Lagrangians $(\Lambda_1(t), \Lambda_2(t))$ vary smoothly with t. In particular, the Maslov index $\mu(\Lambda_1(t), \Lambda_2(t))$ is well defined. We can now state the main result of this paper.

THEOREM 3.14. *For any path $\gamma \in \mathscr{P}$ as above, we have*

$$SF(\gamma) = \mu(\Lambda_1(t), \Lambda_2(t)). \tag{3.9}$$

Proof. We have defined a map $\phi\colon \mathscr{P} \to \mathbf{Z}$,

$$\phi\colon \gamma = D(t) \mapsto \mu(\Lambda_1(D(t)), \Lambda_2(D(t))).$$

By Propositions 2.4 and 2.6, we see that ϕ is continuous and $\phi = 0$ on the paths in $\mathscr{D}^*$. By Proposition 3.13, it suffices to check (3.9) on elementary paths. Thus, fix a cylindrical Dirac operator such that

$$\ker D = \operatorname{span}(F_0), \qquad |F_0| = 1,$$

and consider the family $D(t) = D + t\alpha I$ with $|t| \leqslant \varepsilon$, where α is a smooth, not-changing-sign function, compactly supported inside M_2, away from the neck N. The operator $D_1(t)$ is not changing since α is supported outside M_1. Thus,

$$\Lambda_1(t) \stackrel{\text{def}}{=} \Lambda_0$$

is constant, and $\Lambda(t) \stackrel{\text{def}}{=} \Lambda_2(t)$ is varying.

Let $U(t)$ be a smooth path of unitary operators on $L^2(\mathscr{E}_0)$ such that

$$U(0) = I, \qquad \Lambda(t) = U(t)\Lambda(0).$$

Set $f_0 = RF_0, f_t = U(t)f_0$ being the restriction of F_0 to Σ. (We adopt the convention of using capital letters for sections of $\mathscr{E}$ defined over M, M_1, or M_2, and small letters for sections of $\mathscr{E}$ defined only over Σ.) Then f_t lies in $\Lambda(t)$, so there exists a unique $F_0 \in \ker D_2(t)$ such that

$$\begin{cases} D_2(t)F_t = 0 & \text{in } M_2 \\ RF_t = U_t f_0 & \text{on } \Sigma. \end{cases} \tag{3.10}$$

$U_t f_0$ varies smoothly with t, and the boundary estimates of Proposition 2.2 imply that F_t depends smoothly upon t as well. Derivating (3.10) at $t = 0$ (the dot will denote the t-derivative at $t = 0$) and noting that $\dot{D}_2 = \alpha I$, we get

$$\begin{cases} D_2(0)\dot{F}_0 + \alpha F_0 = 0 & \text{in } M_2 \\ R\dot{F}_0 = \dot{U} f_0 & \text{on } \Sigma \end{cases}$$

Multiplying by F_0, we get

$$-(\alpha F_0, F_0) = (D_2(0)\dot{F}_0, F_0).$$

Now if we integrate by parts in the above equality and use (3.10), we obtain

$$\int_{M_2} \langle D_2(0)\dot{F}_0, F_0\rangle = -\int_{\Sigma} \langle J\dot{F}_0, F_0\rangle + \int_{M_2} \langle \dot{F}_0, D_2(0)F_0\rangle = -(J\dot{U}f_0, f_0).$$

Thus,

$$(\alpha F_0, F_0) = (J\dot{U}(0)f_0, f_0) = \omega(\dot{U}f_0, f_0). \tag{3.11}$$

By unique continuation, $(\alpha F_0, F_0) \neq 0$. The sign of the left-hand side of (3.11) is equal to $SF(D(t); |t| \leqslant \varepsilon)$ by Theorem 3.1. The sign of the right-hand side is equal to the Maslov index $\mu(\Lambda, \Lambda(t))$ by Corollary 1.12. This completes the proof. □

4. Adiabatic limits of CD spaces. Consider a manifold with boundary M, as in Section 2, and D a neck-compatible Dirac operator on M. Define $M(r) = M \cup \Sigma \times [0, r]$. $M(r)$ is usually called an *adiabatic deformation* of M; (see Figure 5). D has a natural extension $D(r)$ as a neck-compatible Dirac on $M(r)$. Denote by $\Lambda^r \subset L^2(\mathscr{E}|_{\partial M(r)})$ the CD space of $D(r)$.

In this section, we will study the behavior of Λ^r as $r \to \infty$. On the tube $\Sigma \times [0, \infty)$, the operator D has the cylindrical form $D = c(ds)(\partial/\partial s + D_0)$, so that at least formally we may write $\Lambda^r = e^{-D_0 r}\Lambda^0$, i.e., we are dealing with a dynamics problem on a Lagrangian Grassmanian. From this representation, we see that the

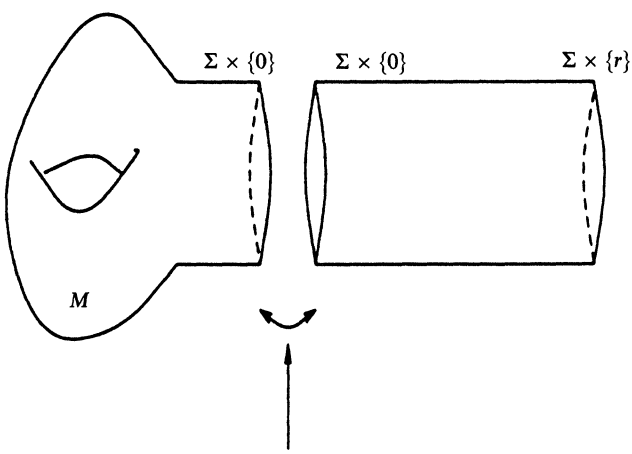

FIGURE 5. Adiabatic deformation of the neck

part of Λ^0 "interacting" with the negative spectrum of D_0 will have a dominant effect as $r \to \infty$, while we expect that the "interactions" with the positive spectrum will "soften" as r increases. We may continue our formal discussion by observing that since D_0 anticommutes with J, it lies in the "Lie algebra" of the infinite-dimensional symplectic group, so that the "flow" $e^{-D_0 r}$ is a 1-parameter group of symplectic transformations of H, and the family Λ^r is a trajectory in an infinite-dimensional Lagrangian Grassmanian. Unfortunately, these observations are purely formal, since D_0 cannot generate a semigroup. (The spectrum is unbounded from both below and above.) However, in finite dimensions, this discussion makes sense, and the first result of this section, Proposition 4.3, describes the asymptotics of this flow. The study of the infinite-dimensional situation will ultimately reduce to this result via a careful symplectic reduction.

Since we will be dealing with asymptotics of families of subspaces, it is appropriate to begin our presentation by discussing ways to measure the distance between two closed subspaces in a Hilbert space. The right notion is provided by the *gap distance* between two subspaces introduced in [K].

Let X, Y be two closed subspaces of a Hilbert space H. Define

$$\delta(X, Y) = \sup\{\operatorname{dist}(x, Y);\, x \in X\ |x| = 1\}.$$

In general, δ is not symmetric in X and Y. We symmetrize it by defining the *gap* between X and Y as

$$\hat{\delta}(X, Y) = \max\{\delta(X, Y), \delta(Y, X)\}.$$

Note that $\delta(X, Y)$ can also be characterized as the smallest number δ such that

$$\operatorname{dist}(x, Y) \leqslant \delta|x| \qquad \forall x \in X.$$

We say $X_n \to X$ if $\hat{\delta}(X_n, X) \to 0$. In particular, if P_n are the orthogonal projections onto X_n, then

$$X_n \to X \Leftrightarrow P_n \to P \qquad \text{in norm}.$$

Thus, if H is symplectic, the gap topology on the space $\mathscr{L}$ of Lagrangians is equivalent with the natural topology (defined by the identification (1.2)). Although the function $\delta(\cdot, \cdot)$ is in general not symmetric, it becomes so when restricted to $\mathscr{L}$. Indeed, by Theorem IV 2.9 of [K], we have

$$\delta(L_1, L_2) = \delta(L_2^{\perp}, L_1^{\perp}).$$

Since L_1, L_2 are Lagrangians,

$$\delta(L_2^{\perp}, L_1^{\perp}) = \delta(JL_2, JL_1) = \delta(L_2, L_1).$$

At the last step, we have used the fact that J is an isometry. Thus,

$$L_n \to L_* \quad \text{in} \quad \mathscr{L} \Leftrightarrow \delta(L_n, L_*) \to 0.$$

In studying convergence of sequences of subspaces, it is very convenient to have a method to "renormalize" them (much like the homogeneous coordinates in the projective spaces). We can achieve this if we can represent these subspaces as graphs of linear operators. This representation is possible once some obvious transversality conditions are assumed (compare with Arnold's charts on Lagrangian Grassmanians). When these renormalizations are possible, there are ways to relate the gap topology with the norm topology of linear operators. In particular, we will frequently use the following results. Their proofs can be found in [K].

LEMMA 4.1. *Let H_1 and H_2 be two separable Hilbert spaces and consider a sequence (T_n) of bounded linear operators $T_n\colon H_1 \to H_2$ with graphs $G(T_n) \subset H_1 \oplus H_2$. Then the following are equivalent:*
(i) $T_n \to T$ as $n \to \infty$ *in norm;*
(ii) $G(T_n) \to G(T)$ *as* $n \to \infty$ in gap.

Now consider $H = \mathbf{R}^{2n}$ with the complex structure J induced by the identification $\mathbf{R}^{2n} \cong \mathbf{C}^n$. J defines a symplectic structure ω by $\omega(x, y) = (Jx, y)$ for all $x, y \in H$. The symplectic group is then

$$Sp(n, \mathbf{R}) = \{T \in GL(2n, \mathbf{R})/T^*JT = J\}.$$

$Sp(n, \mathbf{R})$ is a Lie group with Lie algebra

$$sp(n, \mathbf{R}) = \{A \in gl(2n, \mathbf{R})/A^*J + JA = 0\}.$$

Inside $sp(n, \mathbf{R})$ sits the subspace

$$\sigma(n) = \{A \in sp(n, \mathbf{R})/A = A^*\},$$

consisting of selfadjoint matrices anticommuting with J. Denote by $\Lambda(n)$ the Lagrangian Grassmanian of $(\mathbf{R}^{2n}, J)$. $Sp(n, \mathbf{R})$ acts (transitively) on $\Lambda(n)$. In particular, any $A \in \sigma(n)$ defines a 1-parameter group of diffeomorphisms of $\Lambda(n)$: $r \mapsto e^{-rA}$. The problem we intend to discuss is that of the asymptotic behavior of the above flow on $\Lambda(n)$. Fix $A \in \sigma(n)$ and consider

$$I_A = \{L \in \Lambda(n)/AL \subset L\},$$

the family of invariant Lagrangians of A. The Lagrangians in I_A are stationary points of the flow $r \mapsto e^{-rA}$.

Let us now describe the dynamics of e^{-rA} in a simple but instructive case.

Example 4.2. Take $n = 1$ and fix $A \in \sigma(1)\backslash\{0\}$. We can then choose $e \in \mathbf{R}^2$, $|e| = 1$, such that in the basis (e, Je) the operator A has the form $A = \operatorname{diag}(\lambda, -\lambda)$. Viewed as a (linear) flow on $\mathbf{R}^2$, e^{-rA} has the hyperbolic phase portrait depicted in Figure 6. The Lagrangians of $\mathbf{R}^2$ are the lines through the origin, so that $\Lambda(1) \cong \mathbf{RP}^1 \cong S^1$. Then e^{-rA} becomes $\operatorname{diag}(e^{-\lambda r}, e^{\lambda r})$. $H_- = \operatorname{span}(f)$ and $H_+ = \operatorname{span}(e)$ are the only stationary points of the flow. If $L \neq H_+$, then one sees from Figure 6 that

$$e^{-rA}L \to H_- \qquad \text{exponentially as } r \to \infty .$$

The phase portrait of e^{-rA} on $\Lambda(1)$ is then the one described in Figure 7. In particular, we have shown that for all $L \in \Lambda(1)$, $e^{-rA}L$ has a limit in I_A as $r \to \infty$.

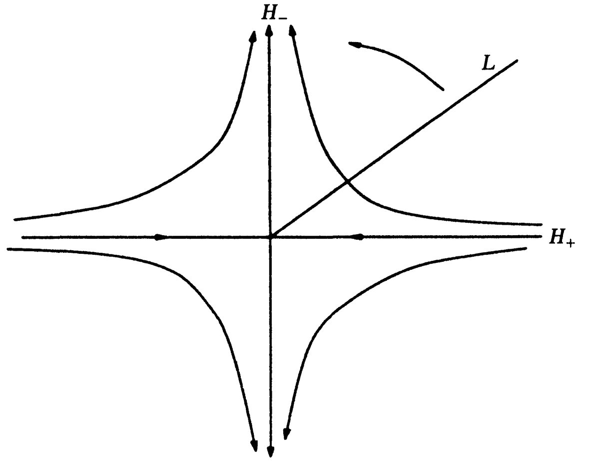

FIGURE 6. Hyperbolic flow

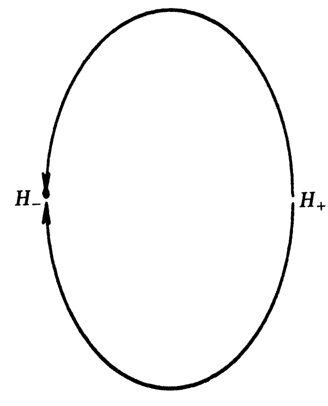

FIGURE 7. Dynamics on $\Lambda(1)$

The situation presented in the example above is a manifestation of a more general phenomenon.

PROPOSITION 4.3. *Let a be a real $n \times n$ symmetric matrix. Then for any subspace $U \subset \mathbf{R}^n$, there exists U_∞, an invariant subspace of A, such that*

$$\lim_{r \to \infty} e^{-rA} U = U_\infty .$$

Proof. Assume $\sigma(A) = \{\lambda \leqslant \cdots \leqslant \lambda_n\}$ with the corresponding orthonormal spectral basis $e_1, \ldots, e_n$. Pick $u_1, \ldots, u_m$ ($m = \dim U$) a basis of U. Then

$$u_i = \sum_{j=1}^{n} c_{ij} e_j \qquad i = 1, \ldots, m,$$

and we can form the matrix

$$C = (c_{ij})_{1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant n} .$$

We may assume C is upper triangular. Otherwise, we can reduce it to this form by performing row operations (which is equivalent to choosing a different basis for U). For each $1 \leqslant i \leqslant m$, let $j(i)$ be the smallest j such that $c_{ij} \neq 0$. Since C is upper triangular,

$$j(1) < \cdots < j(m). \tag{4.1}$$

For $r \geqslant 0$, let

$$v_i(r) = \frac{1}{c_{ij(i)}} e^{r\lambda_{j(i)}} e^{-rA} u_i .$$

Let $v_1(r), \ldots, v_m(r)$ form a basis of $e^{-rA} U$. Moreover,

$$v_i(\infty) = \lim_{r \to \infty} v_i(r)$$

exists, and for all i and

$$v_i(\infty) = e_{j(i)} + \sum_{k > j(i)} v_{ik} e_k .$$

From (4.1) we deduce that $v_i(\infty)$ are linearly independent, and therefore

$$e^{-rA} U \to U_\infty = \operatorname{span}(v_1(\infty), \ldots, v_m(\infty)).$$

Proposition 4.3 is proved. □

As a consequence we have the next corollary.

COROLLARY 4.4. *Let $L \in \Lambda(n)$ and $A \in \sigma(n)\backslash\{0\}$. Then there exists $L_\infty \in I_A$ such that*

$$\lim_{r\to\infty} e^{-rA}L = L_\infty .$$

Remark 4.5. In [N3] we showed that for a natural Riemann metric on $\Lambda(n)$, the flow $L \mapsto e^{At}L$ on $\Lambda(n)$ is the gradient flow of the $\mathbf{Z}_2$-perfect Morse function $f_A\colon L \mapsto -\operatorname{tr}(AP_L)$, where for each $L \in \Lambda(n)$, P_L denotes the orthogonal projection onto L. Moreover, the unstable manifolds realize a Schubert-type decomposition of $\Lambda(n)$, and for some choices of A, this function is also self-indexing.

We now return to our original problem. We use the notation introduced at the beginning of this section. We are interested in the adiabatic limit $\lim_{t\to\infty} \Lambda^r$. As usual, set $D_0 = D|_\Sigma$. For any real number E, we denote by $\mathscr{H}_>^E$ (resp., $\mathscr{H}_\geqslant^E$, $\mathscr{H}_<^E$, $\mathscr{H}_\leqslant^E$, $\mathscr{H}_0^E$) the subspace of $L^2(\mathscr{E}_0)$ spanned by eigenvectors corresponding to eigenvalues $> E$ ($\geqslant E$, $< E$, $\leqslant E$ and resp., in $[-|E|, |E|]$). In the sequel, we will frequently use the following technical result.

LEMMA 4.6. *For any $U \subset L^2(\mathscr{E}_0)$ finite-dimensional subspace and any real E, the pair $(\Lambda^r(D), \mathscr{H}_>^E \oplus U)$ is Fredholm.*

For a proof of this lemma, we refer to [BW4]. For nonnegative E, the space $\mathscr{H}_>^E$ is an isotropic subspace of $L^2(\mathscr{E}_0)$. By the above lemma, the pair $(\Lambda^r(D), \mathscr{H}_>^E)$ is Fredholm. Thus, according to Lemma 1.14, we can construct the symplectic reduction of Λ^r mod $\mathscr{H}_>^E$:

$$L_E^r = \frac{(\Lambda^r \cap \mathscr{H}_\geqslant^{-E})}{\mathscr{H}_>^E}. \tag{4.2}$$

(The symplectic reduction of $\Lambda = \Lambda^0$ mod $\mathscr{H}_>^E$ will be denoted by L_E.) These are Lagrangian subspaces in the symplectic vector space $\mathscr{H}_0^E$. Set $A_E = D_0|_{\mathscr{H}_0^E}$.

LEMMA 4.7. *The set*

$$\mathscr{N}(D) = \{E \geqslant 0 / \Lambda(D) \cap \mathscr{H}_>^E = 0\}$$

is a nonempty, closed, unbounded interval.

Proof. Consider an increasing sequence $E_n \to \infty$. Using Lemma 4.6, we obtain a decreasing sequence of finite-dimensional vector spaces

$$U_n = \Lambda \cap \mathscr{H}_>^{E_n}.$$

In particular, there exists an $m > 0$ such that

$$U_m = U_{m+1} = \cdots .$$

On the other hand, $\bigcap U_n = 0$. Thus, $U_m = 0$, and therefore $E_m \in \mathcal{N}(D)$. Since the spectrum of D_0 is discrete, we deduce that $\mathcal{N}(D)$ is closed. It is an unbounded interval because $(\mathcal{H}_>^E)_{E \geqslant 0}$ is a decreasing family of (isotropic) subspaces of $L^2(\mathcal{E}_0)$. □

Definition 4.8. The set $\mathcal{N}(D)$ is called the *nonresonance range* of D, and $\nu(D) = \min \mathcal{N}(D)$ is called the *nonresonance level* of D. When $\nu(D) = 0$ (i.e., $\mathcal{N}(D) = [0, \infty)$), the operator D is called *nonresonant.*

We can now formulate the main result of this section, which shows that the family Λ^r has a limit as $r \to \infty$.

THEOREM 4.9. *Let M and D be as above and let $E \geqslant \nu(D)$. As $r \to \infty$,*

$$\Lambda^r \to L_E^\infty \oplus \mathcal{H}_<^{-E},$$

where

$$L_E^\infty = \lim_{r \to \infty} L_E^r = \lim_{r \to \infty} e^{-rA_E} L_E.$$

Proof. Fix $E \in \mathcal{N}(D)$. The proof is carried out in several steps.

Step 1: A dynamical description of Λ^r. Let $\mathcal{E}_r$ be the extension of $\mathcal{E}$ to $M(r)$ and $\mathcal{K}(r) = \mathcal{K}_{1/2}(D(r))$. For each $0 \leqslant s \leqslant r$, let

$$T_s\colon \mathcal{K}(r) \to L^2(\mathcal{E}_0)$$

be the restriction map $U \mapsto U|_{\Sigma \times \{s\}}$ whose image lies in Λ^s. The CD space Λ^r can be equivalently described as $\Lambda^r = T_0(\mathcal{K}(r))$. By Proposition 2.4, $T_0\colon \mathcal{K}(r) \to \Lambda^r$ is bijective with continuous inverse. These traces define a *backward translation operator $G_r\colon \Lambda^r \to \Lambda^0$* defined as the composition

$$G_r\colon \Lambda^r \xrightarrow{T_r^{-1}} \mathcal{K}(r) \xrightarrow{T_0} \Lambda^0. \tag{4.3}$$

On the cylindrical portion $C_r = \Sigma \times [0, r)$ of $M(r)$, $D(r)$ has the form

$$D(r) = c(ds)\left(\frac{\partial}{\partial s} + D_0\right).$$

Thus, any $U \in \mathcal{K}(r)$ satisfies on C_r an evolution-like equation

$$DU = \frac{\partial}{\partial s} U + D_0 U = 0.$$

For any $u \in L^2(\mathcal{E}_0)$, we write $u = u_+ + u_0 + u_-$ according to the spectral decomposition

$$L^2 \mathcal{E}_0 = \mathcal{H}_>^E \oplus \mathcal{H}_0^E \oplus \mathcal{H}_<^{-E},$$

which is independent of s. Thus, we can decompose $U(s) = T_s U$ as

$$U(s) = U(s)_+ + U(s)_0 + U(s)_-.$$

Each of these three pieces satisfies the same evolution-like equation as U (formally $U(s) = e^{-sD_0}U(0)$). Since the spectrum of D_0 is discrete, we can find $\mu > 0$ such that the set $[-\mu, -E) \cup (E, \mu]$ contains no eigenvalues of D_0. Then we deduce (by standard Fourier techniques)

$$|(T_s U)_+| \leqslant \text{const.} \exp(-\mu s)|(T_0 U)_+| \tag{4.4}$$

$$|(T_s U)_-| \geqslant \text{const.} \exp(\mu s)|(T_0 U)_-|. \tag{4.5}$$

Using (4.4) and (4.5), we deduce that for all $u \in \Lambda^r$,

$$|u_+|^2 \leqslant \text{const.} \exp(-\mu r)|(G_r u)_+|^2 \tag{4.6}$$

$$|u_-|^2 \geqslant \text{const.} \exp(\mu r)|(G_r u)_-|^2. \tag{4.7}$$

Intersecting Λ^r with the coisotropic subspace $\mathscr{H}_{\geqslant}^{-E}$, we get (by Lemma 4.6) the finite-dimensional space

$$\bar{\Lambda}^r = \Lambda^r \cap \mathscr{H}_{\geqslant}^{-E},$$

which leads to the symplectic reduction L_E^r defined in (4.2). Using the Fourier decomposition for D_0, we deduce easily that for any $L \in \mathbf{R}$, D_0 restricted to $\mathscr{H}_{\geqslant}^L$ defines a C_0-semigroup, which we denote by e^{-rD_0} $r \geqslant 0$. In particular,

$$\bar{\Lambda}^r = e^{-rD_0}\bar{\Lambda}$$

$$L_E^r = e^{-rD_0}L_E = e^{-rA_E}L_E.$$

Let $L_E^\infty = \lim_{r\to\infty} e^{-rA_E}L_E$ (which exists by Corollary 4.4), which is a Lagrangian in $\mathscr{H}_0^E$.

Step 2. Asymptotic transversality. If $E \geqslant v(D)$, then for r large, Λ^r is transverse to the Lagrangian subspace

$$W = JL_E^\infty + \mathscr{H}_>^E.$$

First, suppose $u_r \in \Lambda^r \cap W$. Since $JL_E^\infty \subset \mathscr{H}_0^E$, u_r lies in $\bar{\Lambda}^r$, so its orthogonal projection $\bar{u}_r$ on $\mathscr{H}_0^E$ lies in $L_E^r \cap JL_E^\infty$. But L_E^r converges to L_E^∞, which is transverse to J_E^∞, so $\bar{u}_r = 0$ for large r. Our nonresonant choice $E \geqslant v(D)$ then implies $u_r = 0$, so for large r

$$\Lambda^r \cap W = 0. \tag{4.8}$$

Now, according to Lemma 4.6, (Λ^r, W) is a Fredholm pair of Lagrangians and so has index 0 by (1.3). Then (4.8) and the definition of the index imply that Λ^r and W span, so

$$\Lambda^r + W = L^2(\mathscr{E}_0). \tag{4.9}$$

Step 3.

$$\lim_{r \to \infty} \overline{\Lambda}^r = L_E^\infty. \tag{4.10}$$

By Step 2, $\overline{\Lambda}^r \cap W = 0$. Since $\overline{\Lambda}^r$, L_E^r, L_E^∞ have the same dimension, we can represent $\overline{\Lambda}^r$ as the graph of a bounded linear map

$$B_r \colon L_E^\infty \to W = JL_E^\infty + \mathscr{H}_>^E.$$

To describe B_r, we first represent L_E^r as the graph of a symmetric operator $S_r \colon L_E^\infty \to L_E^\infty$ (see Figure 8)

$$L_E^r = \{u + JS_r u / u \in L_E^\infty\}$$

(where $S_r \to 0$ since $L_E^r \to L_E^\infty$). Next (since Λ^r is clean mod $\mathscr{H}_>^E$), there exists a bounded linear map $h_r \colon L_E^\infty \to \mathscr{H}_>^E$ such that

$$\overline{\Lambda}^r = \{u + JS_r u + h_r(u) / u \in L_E^\infty\}, \qquad B_r(u) = (JS_r u, h_r(u)).$$

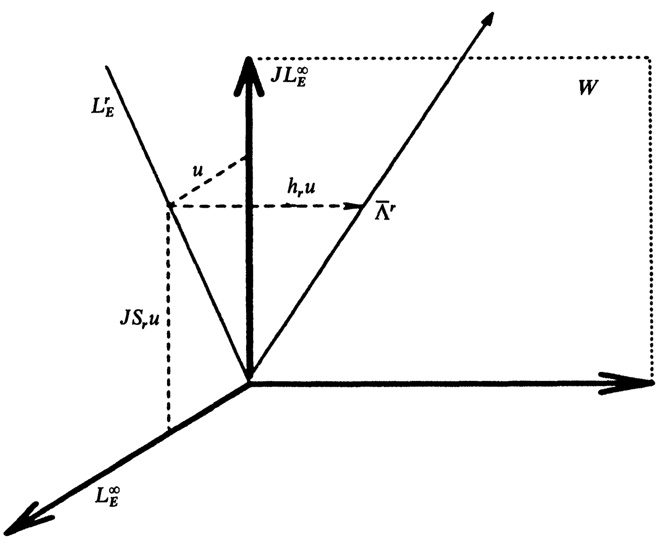

FIGURE 8. $\overline{\Lambda}^r$ is the graph of $B_r = JS_r + h_r$.

But recall that

$$\overline{\Lambda}^r = e^{-rD_0}\overline{\Lambda} = \{e^{-rA_E}u + e^{-rA_E}JS_0u + e^{-rD_0}h_0(u)/u \in L_E^\infty\}$$
$$= \{v + Je^{rA_E}S_0e^{rA_E}v + e^{-rD_0}h_0(e^{rA_E}v)/v \in L_E^\infty\}.$$

Therefore, we have

$$S_r = e^{rA_E}S_0e^{rA_E} \qquad \text{and} \qquad h_r(v) = e^{-rD_0}h_0(e^{rA_E}v).$$

Then the estimate

$$\|h_r\| \leqslant \|e^{-rD_0}\|_{\mathscr{H}_>^E}\|h_0\| \, \|e^{rA_E}\|_{\mathscr{H}_0^E} = \leqslant e^{-r\mu}e^{rE}\|h_0\|$$

shows that $h_r \to 0$ exponentially (we chose $\mu > E$); we then deduce (4.10) using Lemma 4.1.

Step 4. Convergence. The conditions (4.8) and (4.9) can be used as in the proof of Theorem 1.2 to represent Λ^r as the graph of a bounded selfadjoint operator M_r: $L_E^\infty \oplus \mathscr{H}_<^{-E} \to L_E^\infty \oplus \mathscr{H}_<^{-E}$, i.e.,

$$\Lambda^r = \{u + JM_ru/u \in L_E^\infty \oplus \mathscr{H}_<^{-E}\}.$$

M_r has a block decomposition

$$M_r = \begin{pmatrix} S_r & (-Jh_r)^* \\ -Jh_r & C_r \end{pmatrix}, \qquad C_r: \mathscr{H}_<^{-E} \to \mathscr{H}_<^{-E}.$$

We already know that $S_r \to 0$, $h_r \to 0$, and we will now show that $\|C_r\| \to 0$. The theorem will follow from Lemma 4.1.

Remark 4.10. Let P_∞ denote the orthogonal projection onto L_E^∞, *which is a closed* D_0-invariant subspace of $L^2(\mathscr{E}_0)$. If $U \in \mathscr{K}(r)$, then $w(s) = P_\infty T_s U$ satisfies the o.d.e.:

$$\dot{w}(s) + A_E w(s) = 0 \qquad s \in [0, r).$$

In particular, if $w(r) = P_\infty T_r U = 0$, then the backward translation $w(s) = 0$ for all $s \in [0, r]$.

For any $f \in \mathscr{H}_<^{-E}$, consider

$$u = u(f) = f + JM_rf = f + J(-Jh_r)^*f + JC_rf \in \Lambda^r.$$

In particular, $P_\infty u = 0$ and any $u \in \Lambda^r$ with this property can be written in the above form. Since $J(-Jh_r)^*\colon \mathscr{H}_<^{-E} \to JL_E^\infty$, we deduce

$$u(f)_- = f \qquad \text{and} \qquad u(f)_+ = JC_r f = JC_r u_- . \tag{4.11}$$

By Remark 4.10, the backward translation of u, defined in (4.3), $v = G_r u \in \Lambda$ satisfies $P_\infty v = 0$, and as in (4.11), we deduce $v_+ = JC_0 v_-$. C_0 is continuous and we get

$$\frac{|v_+|}{|v_-|} \leqslant \text{const}. \tag{4.12}$$

On the other hand, (4.6), (4.7), and (4.11) imply

$$\frac{|v_+|}{|v_-|} = \frac{|(G_r u)_+|}{|(G_r u)_-|} \geqslant \frac{e^{r\mu}|u_+|}{e^{-r\mu}|u_-|} = e^{2r\mu}\frac{|JC_r f|}{|f|}. \tag{4.13}$$

The relations (4.12) and (4.13) imply that $\|C_r\| = O(e^{-2r\mu})$. Theorem 4.9 is proved. □

Theorem 4.9 has many interesting corollaries. We will consider only a special situation motivated by problems in topology (see [Y]). Assume D is nonresonant, i.e.,

$$\nu(D) = 0.$$

In this case, we will use the simplified notation

$$\mathscr{H}_-(D) = \mathscr{H}_<^0, \qquad \mathscr{H}_0(D) = \mathscr{H}_0^0, \qquad \mathscr{H}_+(D) = \mathscr{H}_>^0.$$

Here $\mathscr{H}_0 = \ker D$ is finite-dimensional, and the spaces $\mathscr{H}_\pm$ are spanned by the positive/negative eigenmodes of D_0. We call $\mathscr{H}_0$ the *harmonic* space of D. Both $\mathscr{H}_\pm$ are isotropic subspaces of $L^2(\mathscr{E}_0)$. The annihilator of $\mathscr{H}_\pm$ is $\mathscr{H}_0 \oplus \mathscr{H}_\pm$. The corresponding symplectic reduction

$$L(D) = (\Lambda \cap (\mathscr{H}_0 \oplus \mathscr{H}_+))/\mathscr{H}_+ \tag{4.14}$$

will be called the *reduced Cauchy data* (RCD) space of D. It can be identified with a Lagrangian in the harmonic space. To see this, consider the Atiyah-Patodi-Singer (APS) boundary value problem, i.e.,

$$(D, APS)\colon \quad Du = 0 \text{ in } M \qquad R_0 u \in \mathscr{H}_+ \oplus \mathscr{H}_0$$

with adjoint

$$(D, APS)^*\colon \quad Du = 0 \in M \qquad R_0 u \in \mathscr{H}_+ .$$

One sees that

$$\dim L(D) = \operatorname{ind}(D, APS) = 1/2 \dim \mathscr{H}_0(D).$$

This agrees with the APS formula, since D is selfadjoint (so its index density is 0) and D_0 has a symmetric spectrum (it anticommutes with J), so its eta invariant vanishes. In [APS1], $\Lambda(D) \cap (\mathscr{H}_0 \oplus \mathscr{H}_+)$ was called the *space of extended L^2 solutions* and $L(D)$ was identified with the subspace in $\mathscr{H}_0$ of asymptotic values of extended L^2 solutions. Using the reduced CD space $L(D)$, we can form the *asymptotic* CD space

$$\Lambda^\infty(D) = L(D) \oplus \mathscr{H}_-(D).$$

The definition of the asymptotic CD space is orientation-sensitive. Changing the orientation of M will have the effect of replacing $\mathscr{H}_-$ with $\mathscr{H}_+$ in the above definition. We see that D is nonresonant if and only if $(D, APS)^*$ has only the trivial solution. The pleasant thing in the nonresonance case is that the finite-dimensional dynamics is not present, since A_E is identically 0 when $E = 0$, so that $L(D) \equiv L^r(D)$, for all $r \geqslant 0$. We deduce immediately the following corollary.

COROLLARY 4.11. *Assume that D is nonresonant. Then*

$$\lim_{r\to\infty} \Lambda^r = \Lambda^\infty.$$

COROLLARY 4.12. *Let $\{D(t); 0 \leqslant t \leqslant 1\}$ be a continuous family of neck-compatible Dirac operators on M such that each $D(t)$ is nonresonant. Let $D^r(t)$ denote their extensions to $M(r)$ and $\Lambda^r(t)$ denote their CD spaces. If* $\dim \operatorname{Ker}(D_0(t))$ *is independent of t, then*

$$\lim_{r\to\infty} \Lambda^r(t) = \Lambda^\infty(t) \qquad \textit{uniformly in } t.$$

In particular, $(\Lambda^\infty(t))$ is a continuous family of Lagrangians in $L^2(\mathscr{E}_0)$.

One can use the existence of an adiabatic limit when computing the spectral flow. We analyze what happens to the terms in Theorem 3.9 as we "stretch the neck." Assume we have a path $\gamma = D(t) \in \mathscr{P}$ such that, for every t, the operators $D_1(t)$ and $D_2(t)$ *are nonresonant.* We can form the adiabatic deformation $M(r)$ of (M, g) by replacing the neck $N \cong \Sigma \times (-1, 1)$ by a longer one, $N_r \cong \Sigma \times (-r, r)$. Let $D^r(t)$ be the obvious extension of $D(t)$ to $M(r)$. Denote by $\Lambda_j^\infty(t)$ the asymptotic CD space of $D_j(t)$. We have the following result.

COROLLARY 4.13. *Let $D(t)$ be a nonresonant path of neck-compatible Dirac operators such that* $\dim \mathscr{H}_0(t)$ *is independent of t. Assume*

$$\Lambda_1^\infty(j) \cap \Lambda_2^\infty(j) = 0 \qquad j = 0, 1. \tag{4.15}$$

Then, for r large enough, $D^r(0)$ and $D^r(1)$ are invertible and

$$SF(D^r(t)) = \mu(\Lambda_1^\infty(t), \Lambda_2^\infty(t)). \tag{4.16}$$

Proof. The fact that $D^r(0)$ and $D^r(1)$ are invertible for large r follows easily from (4.14) using "adiabatic analysis," as in Theorem 4.9. Alternatively, we can quote the results of [CLM2] from which the above conclusion follows trivially. Thus, (4.15) follows from Theorem 3.9 combined with Corollary 4.12. □

The nonresonance of the operators $D(t)$ can be translated symplectically by saying that $\Lambda_1(t)$ is clean mod $\mathscr{H}_+(D_1(t))$ and $\Lambda_2(t)$ is clean mod $\mathscr{H}_-(D_2(t))$. Using the invariance of the Maslov index under clean reductions, we deduce the next corollary.

COROLLARY 4.14. *Let $D(t)$ be as in Corollary 4.13. Then*

$$SF(D^r(t)) = \mu(L_1(t), L_2(t))$$

for r large enough, where $L_i(t) = L(D_i(t))$ is the RCD space of $D_i(t)$.

This last corollary generalizes a result of [Y]. In that case, the Dirac operators arise as the deformation complexes of the flat-connection equation on a homology 3-sphere.

Finally, we want to address a natural question. Assume that $D(t)$ is a path of neck-compatible Dirac operators on M_1, and suppose that some of them have positive nonresonace levels. For simplicity, suppose that $\nu(D(t)) = \nu_0 > 0$ for all t and the boundary operators $D_0(t) = D(t)|_\Sigma$ are independent of t. Then by Theorem 6.1, we can find Lagrangians $L^\infty(t)$ in $\mathscr{H}_0^{\nu_0}$ such that

$$\lim_{r\to\infty} \Lambda^r(t) = L^\infty(t) \oplus \mathscr{H}_<^{\nu_0} \qquad \forall t. \tag{4.17}$$

Is the convergence in (4.17) uniform in t?

We sketch a simple heuristic argument which suggests that the answer one should expect is, *in general, negative*. Let us further specialize and assume that the restriction of D_0 to $V_0 = \mathscr{H}_{\nu_0}$ (henceforth denoted by A) has only *simple eigenvalues*. In particular, A is invertible because it anticommutes with J. Denote by $L^r(t)$ the symplectic reduction of $\Lambda^r(t)$ mod $\mathscr{H}_>^{\nu_0}$. We have seen that

$$\gamma_r(t) = L^r(t) = e^{-Ar}L^0 t = e^{-Ar}\gamma_0(t) \qquad \forall t.$$

Denote by $\text{Lag}(V_0)$ the Lagrangian Grassmanian associated to the symplectic space V_0. The results of [N3] (see Remark 4.5) show that e^{-Ar} is the negative gradient flow of some function $\text{Lag}(V_0)$. Since A has only simple eigenvalues, all the critical points are nondegenerate. The function has a unique critical point P of index 1. The stable manifold of this point is a codimension 1 submanifold $\mathscr{Z}$ of

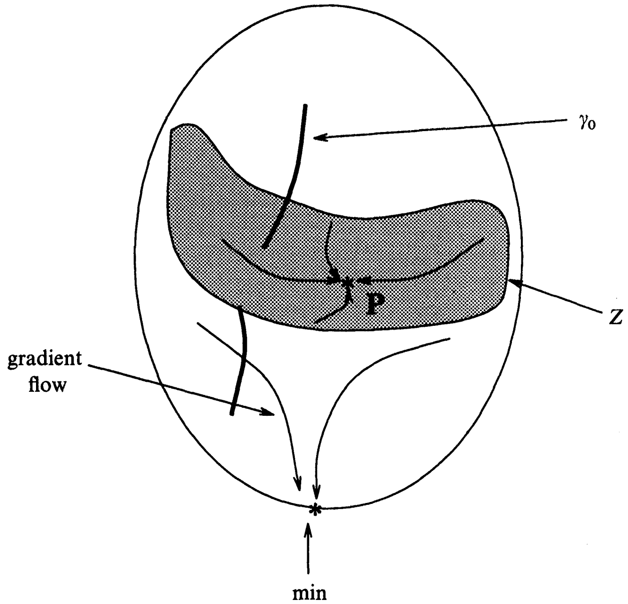

FIGURE 9. A Morse flow on the Lagrangian Grassmanian

$\mathrm{Lag}(V_0)$ whose closure $\bar{\mathscr{Z}}$ is the Poincaré dual of the Maslov index (see Figure 9). Now if we let the path γ_0 flow along the gradient lines, it will "disintegrate" as $r \to \infty$ into a finite set of critical points. Hence, the only time $\gamma_r(t)$ can converge uniformly in t is when γ_0 lies entirely in the stable manifold of some critical point. Generically, this has to be the region of attraction of the minimum, which is the complement of $\bar{\mathscr{Z}}$. Via a small perturbation, we may assume $\gamma_0(0)$, $\gamma_1(1)$ lie in this attraction region. Thus, we obtain a Maslov index

$$\mu(\gamma_0) = \# \gamma_0 \cap \bar{\mathscr{Z}}.$$

This number is stable under small perturbations. In particular, if $\mu(\gamma_0) \neq 0$, then the endpoints of γ_0 will flow towards the minimum, and some point on this curve will flow inside $\mathscr{Z}$ towards the critical point P. Hence, we do not have uniform convergence.

APPENDIX

Proof of Proposition 3.6. Proposition 3.6 is a consequence of the Sard-Smale theorem. We roughly follow an outline given by Floer (Proposition 3.1 in [F]), making several necessary modifications. (Floer overlooked the hypothesis in Lemma A.2; fixing this requires applying Sard-Smale to a modified map.) To define it, choose k large enough so that $L^2_k([0, 1] \times M) \hookrightarrow C^2([0, 1] \times M)$ ($N =$

dim M) and set

$$\mathscr{A} = \{A \in L_k^2(\mathrm{End}([0,1] \times \mathscr{E}))/A(t) \in \mathrm{Cyl}(\mathscr{E})\ \forall t \in [0,1]\}.$$

We will parametrize the 2-dimensional planes in $L_1^2(\mathscr{E})$ by

$$W = \{(\xi, \eta) \in L_1^2(\mathscr{E}) \times L_1^2(\mathscr{E})/\langle \xi, \eta \rangle_{L^2} = 0, |\xi|_{L^2} = |\eta|_{L^2} = 1\}.$$

This is a Banach manifold. Its tangent space at (ξ, η) consists of all pairs ϕ, $\psi \in L_1^2(\mathscr{E})$ that satisfy

$$\langle \xi, \psi \rangle + \langle \phi, \eta \rangle = \langle \xi, \phi \rangle = \langle \eta, \psi \rangle = 0. \tag{A.1}$$

In the proof of our genericity results, we will need the following lemmata.

LEMMA A.1. *Let D be a cylindrical Dirac and $(\xi, \eta) \in W$ such that $D\xi = D\eta = 0$. Then there exists an open subset $U \subset M_2$ away from the neck such that ξ and η are pointwise linearly independent over U.*

Proof. By unique continuation, the set

$$S = \{x \in M/\xi(x) \neq 0 \text{ and } \eta(x) \neq 0\}$$

is open and dense as an intersection of two open and dense sets. Set $S_2 = S \cap (M_2 \backslash \text{neck})$. The set

$$\mathscr{I} = \{x \in S_2/\xi(x)\ \&\ \eta(x) \text{ are linearly independent}\}$$

is open if nonempty. The lemma is proved if we show that $\mathscr{I} \neq \varnothing$. Assume the contrary. This means there exists $\alpha \in C^\infty(S_2)$ such that

$$\xi(x) = \alpha(x)\eta(x) \qquad \forall x \in S_2,$$

$\xi, \eta \neq 0$ on S_2 so that $\alpha \neq 0$. Since $\xi \perp \eta$, we deduce from the unique continuation that α is not constant on S_2, i.e., $d\alpha \not\equiv 0$ on S_2. On the other hand, since D is a Dirac operator, we deduce (see [BGV]):

$$0 = D\xi = D\alpha\eta = D\eta + [D, \alpha]\eta = c(d\alpha)\eta.$$

This is a contradiction since the Clifford multiplication $c(d\alpha)$ is an isomorphism when $d\alpha \neq 0$. Lemma A.1 is proved. □

For $k \geqslant 0$, let S_k denote the linear space of real, symmetric $k \times k$ matrices ($S_0 \equiv 0$).

LEMMA A.2. *Let $\xi, \eta \in \mathbf{R}^k$ ($k \geqslant 2$) be two linearly independent vectors. Then for* *any vectors $u, v \in \mathbf{R}^k$ satisfying*

$$\langle \xi, v \rangle = \langle \eta, u \rangle,$$

there exists $A \in S_k$ such that $(A\xi, A\eta) = (u, v)$.

Proof. Define

$$H_{\xi,\eta}: S_k \to \mathbf{R}^{2k} \qquad A \mapsto (A\xi, A\eta).$$

We have to prove that

$$\text{Range } H_{\xi,\eta} = V_{\xi,\eta} = \{(u, v) \in \mathbf{R}^k \times \mathbf{R}^k / \langle \xi, v \rangle = \langle \eta, u \rangle\}.$$

Note that $V_{\xi,\eta} = (\text{span}(-\eta, \xi))^{\perp}$ and for any $A \in S_k$

$$\langle (A\xi, A\eta), (-\eta, \xi) \rangle = -\langle A\xi, \eta \rangle + \langle A\eta, \xi \rangle = 0$$

so that

$$\text{Range } H_{\xi,\eta} \subset V_{\xi,\eta}. \tag{A.2}$$

On the other hand,

$$\dim \text{Range } H_{\xi,\eta} = \dim S_k - \dim \text{Ker } H_{\xi,\eta}.$$

Since ξ and η are linearly independent, we can identify $\text{Ker } H_{\xi,\eta} \cong S_{k-2}$. Thus,

$$\dim \text{Range } H_{\xi,\eta} = k(k+1)/2 - (k-2)(k-1)/2 = 2k - 1 = \dim V_{\xi,\eta}. \tag{A.3}$$

Lemma A.2 follows from (A.2) and (A.3). □

Proof of Proposition 3.6. We will apply the Sard-Smale theorem to the smooth function

$$F: X = \mathscr{A} \times (0, 1) \times W \times \mathbf{R} \to Y = L^2(\mathscr{E}) \times L^2(\mathscr{E})$$

defined by

$$(A(\cdot), t, \xi, \eta, \lambda) \mapsto (D(t)\xi - \lambda\eta, D(t)\eta + \lambda\xi).$$

Let $Z = F^{-1}(0)$. The proof of Proposition 3.6 is done in two steps.

Step 1. Z is a smooth Banach manifold. To prove this, we will use the implicit function theorem. Given $z \in Z$, we will show that $DF(z): T_zX \to Y$ is onto. More precisely, we will show that $DF(z)$ has closed range and its cokernel is zero. Let $z = (A, t, \xi, \eta, \lambda) \in Z$. Note that this implies $\lambda = 0$. Indeed, we have $D(t)\xi = \lambda\eta$ and $D(t)\eta = -\lambda\xi$, so that

$$D(t)^2\eta = -\lambda D(t)\xi = -\lambda^2\eta.$$

Since $D(t)$ is selfadjoint, we deduce $|D(t)\eta|^2 = -\lambda^2|\eta|^2$, which is possible if and only if $\lambda = 0$.

Now consider the variation on the direction $(a, \tau, \phi, \psi, \mu) \in T_zX$. The partial derivatives of F are

$$D_AF(z)(a) = (a(t)\xi, a(t)\eta) \tag{A.4}$$

$$D_tF(z)(\tau) = \tau(\dot{A}(t)\xi, \dot{A}(t)\eta) \tag{A.5}$$

$$D_{(\xi,\eta)}F(z)(\phi, \psi) = (D(t)\phi, D(t)\psi) \tag{A.6}$$

$$D_\lambda F(z)(\mu) = \mu(-\eta, \xi), \tag{A.7}$$

where ϕ and ψ satisfy (A.1).

Since the operator $D(t)$ is elliptic, we deduce that the range of $DF(z)$ is closed. Let $(u, v) \in \operatorname{Coker} DF(z_0)$. From (A.4) and (A.6), we deduce

$$\langle a(t)\xi, u\rangle + \langle a(t)\eta, v\rangle = 0 \qquad \forall a \in \mathscr{A} \tag{A.8}$$

$$\langle D(t)\phi, u\rangle + \langle D(t)\psi, v\rangle = 0, \tag{A.9}$$

for all ϕ, ψ satisfying (A.1). Let $(e_n)_{n\in\mathbf{Z}}$ be the eigenvectors of $D(t)$ corresponding to the *nonzero eigenvalues*. If we let $\phi = e_n$ and $\psi = 0$ in (A.9), we deduce

$$\langle e_n, u\rangle = 0 \qquad \forall n \in \mathbf{Z},$$

so that $u \in \operatorname{Ker} D(t)$. We deduce similarly that $v \in \operatorname{Ker} D(t)$.

From Lemmata A.1, A.2, and (A.8), we deduce that on an open set $U \subset M_2$ away from the neck

$$(u(x), v(x)) = c(-\eta(x), \xi(x)), \qquad \forall x \in U$$

for some $c \in \mathbf{R}$. By unique continuation, the above equality holds for all $x \in M$. Pairing (u, v) with (A.7), we get that $c = 0$, i.e., $\operatorname{Coker} F(z) = 0$. Step 1 is completed.

Step 2. The natural projection $\pi: Z \to \mathscr{A}$ is Fredholm with index -1. It is a standard fact that π is Fredholm if and only if

$$G = (\pi, F): X \to \mathscr{A} \times Y$$

is Fredholm. Moreover, $\pi|_Z$ and G have the same index. It suffices to study DG at a point $z \in Z$ of our choice. Thus, let $z_0 = (A_0, t_0, \xi_0, \eta_0, 0) \in Z$ such that $\dot{A}_0(t_0)$ is a positive cylindrical endomorphism. Hence, $(\xi_0, \eta_0) \in W$ and $D(t_0)\xi_0 = D(t_0)\eta_0 = 0$. The derivatives of G are given by (A.4) to (A.7) and

$$D_A\pi(z_0)(a) = a, \qquad a \in \mathscr{A}. \tag{A.10}$$

Again, the ellipticity of $D(t_0)$ implies that $DG(z_0)$ has closed range.

Let $(a, \tau, \phi, \eta, \mu) \in \operatorname{Ker} DG(z_0)$. This means $a = 0$, $\mu = 0$, $\tau\dot{A}_0(t_0)\xi_0 = \tau\dot{A}_0(t_0)\eta_0 = D(t_0)\xi_0 = D(t_0)\eta_0 = 0$. In particular, since ϕ and ψ satisfy (A.1), they lie in a codimension-3 subspace of $\ker D(t_0) \times \ker D(t_0)$. Because $\dot{A}_0(t_0)$ is positive, we deduce $\dot{A}_0(t_0)\xi_0$, $\dot{A}_0(t_0)\eta_0 \neq 0$, and therefore

$$\dim \operatorname{Ker} DG(z_0) = 2 \dim \operatorname{Ker} D(t_0) - 3. \tag{A.11}$$

Let $(\alpha, u, v) \in \operatorname{Coker} DG(z_0)$. We deduce from (A.4), (A.6), and (A.10) that

$$\langle a(t_0)\xi_0, u\rangle + \langle a(t_0)\eta_0, v\rangle + \langle a, \alpha\rangle_{\mathscr{A}} = 0 \qquad \forall a \in \mathscr{A} \tag{A.12}$$

$$\tau(\langle \dot{A}_0(t_0)\xi_0, u\rangle + \langle \dot{A}_0(t_0)\eta_0, v\rangle) = 0 \qquad \forall \tau \in \mathbf{R} \tag{A.13}$$

$$\langle D(t_0)\phi, u\rangle + \langle D(t_0)\psi, v\rangle = 0 \tag{A.14}$$

$$\mu(\langle \xi, v\rangle - \langle \eta, u\rangle) = 0 \qquad \forall u \in \mathbf{R} \tag{A.15}$$

for all ϕ and ψ satisfying (A.1). In particular, we deduce from (A.14) and (A.15) that $u, v \in \operatorname{Ker} D(t_0)$ and $(u, v) \perp (-\eta_0, \xi_0)$. For any $u, v \in L^2(\mathscr{E})$, define $\alpha = \alpha(u, v) \in \mathscr{A}$ by

$$\langle a, \alpha\rangle_{\mathscr{A}} = -\langle a(t_0)\xi_0, u\rangle_{L^2(\mathscr{E})} + \langle a(t_0)\eta_0, v\rangle_{L^2(\mathscr{E})} \qquad \forall a \in \mathscr{A}. \tag{A.16}$$

The existence and uniqueness of a such an α is a consequence of the Riesz-Frechet representation theorem. Set

$$E = \{(\alpha(u, v), u, v)/u, v \in \ker D(t) \;\&\; (u, v) \perp (-\eta_0, \xi_0)\}.$$

We can now describe the cokernel of $DG(z_0)$ as

$$\operatorname{Coker} DG(z_0) = \{(\alpha(u, v), u\ v)/(u, v) \in E, \langle \dot{A}_0(t_0)\xi_0, u\rangle + \langle \dot{A}_0(t_0)\eta_0, v\rangle = 0\}.$$

Since $(\dot{A}_0(t_0)\xi_0, \dot{A}_0(t_0)\eta_0) \perp (-\eta_0, \xi_0)$ and $\dot{A}_0(t_0)$ is positive, we get

$$\dim \operatorname{Coker} DG(z_0) = \dim E - 1 = 2 \dim \operatorname{Ker} D(t_0) - 2. \tag{A.17}$$

Step 2 follows from (A.11) and (A.17). Proposition 3.6 is now a consequence of Sard-Smale theorem. □

REFERENCES

[Ar] V. I. ARNOLD, "Une classe charactéristique intervenant dans les conditions de quantification," appendix to *Théorie des perturbations et methodes asymptotiques*, Dunod, Paris, 1972.

[APS1] M. F. ATIYAH, V. K. PATODI, AND I. M. SINGER, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[APS2] ———, *Spectral asymmetry and Riemannian geometry II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 405–432.

[APS3] ———, *Spectral asymmetry and Riemannian geometry III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.

[AS] M. F. ATIYAH AND I. M. SINGER, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. **37** (1969), 5–26.

[B] B. BLACKADAR, *K-Theory for Operator Algebras*, Math. Sci. Res. Inst. Publ. **5**, Springer-Verlag, New York, 1986.

[BGV] N. BERLINE, E. GETZLER, AND M. VERGNE, *Heart Kernels and Dirac Operators*, Springer-Verlag, Berlin, 1992.

[BW1] B. BOOSS AND K. WOJCIECHOWSKI, *Desuspension of splitting elliptic symbols I*, Ann. Global Anal. Geom. **3** (1985), 337–387.

[BW2] ———, *Desuspension of splitting elliptic symbols II*, Ann. Global Anal. Geom. **4** (1986), 349–400.

[BW3] ———, *Pseudo-differential projections and the topology of certain spaces of elliptic boundary value problems*, Comm. Math. Phys. **121** (1989), 1–9.

[BW4] ———, *Elliptic Boundary Problems for Dirac Operators*, Birkhauser, Boston, 1993.

[Bu1] U. BUNKE, *A glueing formula for the η-invariant*, preprint, Humboldt-Universitat, Berlin, 1993.

[Bu2] ———, *Splitting the spectral flow*, preprint, Humboldt-Universitat, Berlin, 1993.

[CLM1] S. CAPELL, R. LEE, AND E. Y. MILLER, *On the Maslov index*, preprint.

[CLM2] ———, *Selfadjoint elliptic operators and manifold decomposition Part I: Low eigenvalues and stretching*, preprint.

[C] H. O. CORDES, *Elliptic Pseudodifferential Operators*, Lecture Notes in Math. **756**, Springer-Verlag, New York, 1979.

[D1] J. J. DUISTERMAAT, *Fourier Integral Operators*, Courant Inst., New York, 1973.

[D2] ———, *On the Morse index in variational calculus*, Adv. Math. **21** (1976), 173–195.

[DP] P. DAZORD AND G. PATTISSIER, "La premiere classe de Chern comme obstruction a la quantification asymptotique," in *Symplectic Geometry, Grupoids and Integrable Systems*, ed. by P. Dazord and A. Weinstein, Math. Sci. Res. Inst. Publ. **20**, Springer-Verlag, New York, 1991.

[DRS] S. DOSTOGLOU, J. W. ROBIN, AND D. SALAMON, *The spectral flow and the Maslov index*, preprint.

[F] A. FLOER, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math. **51** (1988), 393–407.

[GS] V. GUILLEMIN AND S. STERNBERG, *Geometric Asymptotics*, Math. Surveys Monographs **14**, Amer. Math. Soc., Providence, 1977.

[Kar] M. KAROUBI, *K-Theory: An Introduction*, Springer-Verlag, Berlin, 1977.

[K] T. KATO, *Perturbation Theory for Linear Operators*, 2d ed., Springer-Verlag, Berlin, 1984.

[Ku] N. KUIPER, *The homotopy type of the unitary group of a Hilbert space*, Topology **3** (1964–65), 19–30.

[LiMa] J. L. LIONS AND E. MAGENES, "Problèmes aux limites nonhomogènes et applications," in vol. 1, *Travaux et Recherches Mathematiques* **17**, Dunod, Paris, 1968.

[N1] L. I. NICOLAESCU, *The spectral flow, the Maslov index and decompositions of manifolds*, C.R. Acad. Sci. Paris Sér. I Math. **317** (1993), 515–519.

[N2] ———, *The index of families of boundary value problems for Dirac operators*, preprint, 1994 (announced in C.R. Acad. Sci. Paris Sér. I Math. **320** (1995), 347–352).

[N3] ———, *Morse theory on Grassmanians*, to appear in Anal. Sti. Univ. Iasi.

[P1] R. S. PALAIS, *On the homotopy type of certain groups of operators*, Topology **3** (1964–65), 271–280.

[P2] ———, *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Stud. **57**, Princeton University Press, Princeton, 1965.

[RS] J. ROBBIN AND D. SALAMON, *The Maslov index for paths*, Topology **32** (1993), 827–844.

[S] R. SEELEY, *Singular integrals and boundary value problems*, Amer. J. Math. **88** (1966), 781–809.

[V] C. VITERBO, *Intersections de sous varietes lagrangiennes, fonctionelles d'action et indice des systemes hamiltoniens*, Bull. Soc. Math. France **115** (1987), 361–390.

[W] K. WOJCIECHOWSKI, *A note on the space of pseudodifferential projections with the same principal symbols*, J. Operator Theory **15** (1986), 207–216.

[Y] T. YOSHIDA, *Floer homology and splittings ofmanifolds*, Ann. of Math. **134** (1991), 277–324.

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