Notes on Seiberg-Witten Theory

Liviu I. Nicolaescu

To my parents, with love and gratitude

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Introduction

My task which I am trying to achieve is by the power of the written word, to make you hear, to make you feel-it is, before all, to make you see. *That-and no more, and it is everything.*

Joseph Conrad

Almost two decades ago, a young mathematician by the name of Simon Donaldson took the mathematical world by surprise when he discovered some "pathological" phenomena concerning *smooth* 4-manifolds. These pathologies were caused by certain behaviour of instantons, solutions of the Yang-Mills equations arising in the physical theory of gauge fields.

Shortly after, he convinced all the skeptics that these phenomena represented only the tip of the iceberg. He showed that the moduli spaces of instantons often carry nontrivial and surprising information about the background manifold. Very rapidly, many myths were shattered.

A flurry of work soon followed, devoted to extracting more and more information out of these moduli spaces. This is a highly nontrivial job, requiring ideas from many branches of mathematics. Gauge theory was born and it is here to stay.

In the Fall of 1994, the physicists N. Seiberg and E. Witten introduced to the world a new set of equations which according to physical theories had to contain the same topological information as the Yang-Mills equations.

From an analytical point of view these new equations, now known as the Seiberg-Witten equations, are easier to deal with than the Yang-Mills equations. In a matter of months many of the results obtained by studying instantons were re-proved much faster using the new theory. (To be perfectly honest, the old theory made these new proofs possible since it created the right mind set to think about the new equations.) The new theory goes one step further, since it captures in a more visible fashion the interaction geometry-topology.

The goal of these notes is to help the potential reader share some of the excitement afforded by this new world of gauge theory and eventually become a player him/herself.

There are many difficulties to overcome. To set-up the theory one needs a substantial volume of information. More importantly, all this volume of information is processed in a non-traditional way which may make the first steps in this new world a bit hesitant. Moreover, the large and fast-growing literature on gauge theory, relying on a non-negligible amount of "folklore"[1], may look discouraging to a beginner.

To address these issues within a reasonable space we chose to present a few, indispensable, key techniques and as many relevant examples as possible. That is why these notes are far from exhaustive and many notable contributions were left out. We believe we have provided enough background and intuition to the interested reader to be able to continue the Seiberg-Witten journey on his/her own.

It is always difficult to resolve the conflict clarity vs. rigor and even much more so when presenting an eclectic subject such as gauge theory. The compromises one has to make are always biased and thus may not satisfy all tastes and backgrounds. We could not escape this bias but, whenever a proof would have sent us far astray we tried to present all the main concepts and ideas in as clear a light as possible and makeup for the missing details by providing generous references. Many technical results were left to the reader as exercises but we made sure that all the main ingredients can be found in these notes.

Here is a description of the content. The first chapter contains preliminary material. It is clearly incomplete and cannot serve as a substitute for a more thorough background study. We have included it to present in the non-traditional light of gauge theory many classical objects which may already be familiar to the reader.

The study of the Seiberg-Witten equations begins in earnest in Chapter 2. In the first section we introduce the main characters: the monopoles, i.e. the solutions of the Seiberg-Witten equations and the group of gauge transformations, an infinite dimensional Abelian group acting on the set of monopoles. The Seiberg-Witten moduli space and its structure are described in Section 2.2 while the Seiberg-Witten invariants are presented in Section

[1]That is, basic facts and examples every expert knows and thus are only briefly or not at all explained in a formal setting. They are usually transmitted through personal interactions.

2.3. We painstakingly included all the details concerning orientations because this is one of the most confusing aspects of the theory. We conclude this chapter with two topological applications: the proof by P. Kronheimer and T. Mrowka of the Thom conjecture for $\mathbb{CP}^2$ and the new proof based on monopoles of Donaldson's first theorem, which started this new field of gauge theory.

In Chapter 3 we concentrate on a special, yet very rich, class of smooth 4-manifolds, namely the algebraic surfaces. It was observed from the very beginning by E. Witten that the monopoles on algebraic surfaces can be given an explicit algebraic-geometric description, thus opening the possibility of carrying out many concrete computations. The first section of this chapter is a brief and informal survey of the geometry and topology of complex surfaces together with a large list of examples. In Section 3.2 we study in great detail the Seiberg-Witten equations on Kähler surfaces and, in particular, we prove Witten's result stating the equivalence between the Seiberg-Witten moduli spaces and certain moduli spaces of divisors. The third section is devoted entirely to applications. We first prove the nontriviality of the Seiberg-Witten invariants of a Kähler surface and establish the invariance under diffeomorphisms of the canonical class of an algebraic surface of general type. We next concentrate on simply connected elliptic surfaces. We compute all their Seiberg-Witten invariants following an idea of O. Biquard based on the factorization method of E. Witten. This computation allows us to provide the complete *smooth* classification of simply connected elliptic surfaces. In 3.3.3, we use the computation of the Seiberg-Witten invariants of $K3$-surfaces to show that the smooth h-cobordism theorem fails in 4 dimensions. We conclude this section and the chapter with a discussion of the Seiberg-Witten invariants of symplectic 4-manifolds and we prove Taubes' theorem on the nontriviality of these invariants in the symplectic world.

The fourth and last chapter is by far the most technically demanding one. We present in great detail the cut-and-paste technique for computing Seiberg-Witten invariants. This is a very useful yet difficult technique, but the existing written accounts of this method can be unbalanced as regards their details. In this chapter we propose a new approach to this technique which in our view has several conceptual advantages and can be easily adapted to other problems as well. Since the volume of technicalities can often obscure the main ideas we chose to work in a special yet sufficiently general case when the moduli spaces of monopoles on the separating 3-manifold are, roughly speaking, Bott nondegenerate.

Section 4.1 contains preliminary material mostly about elliptic equations on manifolds with cylindrical ends. Most objects on closed manifolds have cylindrical counterparts which often encode very subtle features. We discovered that a consistent use of cylindrical notions is not only æsthetically desirable, but also technically very useful. The cylindrical context highlights and coherently organizes many important and not so obvious aspects of the whole gluing problem. An important result in this section is the Cappell-Lee-Miller gluing theorem. We adapt the asymptotic language of [**105**], which is extremely convenient in gluing problems. This section ends with the long subsection 4.1.6 containing many useful and revealing examples. These are frequently used in gauge theory and we could not find any satisfactory reference for them.

In Section 4.2 we study the finite energy monopoles on cylindrical manifolds. The results are very similar to the ones in Yang-Mills equations and that is why this section was greatly inspired by [**91, 128**].

Section 4.3 is devoted to the local study of the moduli spaces of finite energy monopoles. The local structure is formally very similar to that in Yang-Mills theory with a notable exception, the computation of the virtual dimensions, which is part of the folklore. We presented in detail this computation since it is often relevant. Moreover, we describe some new exact sequences relating the various intervening deformation complexes to objects covered by the Cappell-Lee-Miller gluing theorem. These exact sequences represent a departure from the mainstream point of view and play a key role in our local gluing theorem.

Section 4.4 is devoted to the study of global properties of the moduli spaces of finite energy monopoles: generic smoothness, compactness (or lack of) and orientability. The orientability is no longer an elementary issue in the noncompact case and we chose to present a proof of this fact only in some simpler situations we need for applications.

Section 4.5 contains the main results of this chapter dealing with the process of reconstructing the space of monopoles on a 4-manifold decomposed into several parts by a hypersurface. This manifold decomposition can be analytically simulated by a neck stretching process. During this process, the Seiberg-Witten equations are deformed and their solutions converge to a singular limit. The key issue to be resolved is whether this process can be reversed: given a singular limit can we produce monopoles converging to this singular limit?

In his dissertation [**94**], T. Mrowka proved a very general gluing theorem which provides a satisfactory answer to the above question in the related context of Yang-Mills equations. In 4.5.2, we prove a local gluing theorem, very similar in spirit to Mrowka's theorem but in an entirely new context. The main advantage of the new approach is that all the spectral estimates needed in the proof follow immediately from the Cappell-Lee-Miller gluing theorem. Moreover, the Mayer-Vietoris type local model is just a reformulation of the Cappell-Lee-Miller theorem. The asymptotic language of [**105**] has allowed us to provide intuitive, natural and explicit descriptions of the various morphisms entering into the definition of this Mayer-Vietoris model.

The local gluing theorem we prove produces monopoles converging to a singular limit at a certain rate. If all monopoles degenerated to the singular limit set at this rate, then we could conclude that the entire moduli space on a manifold with a sufficiently long neck can be reconstructed from the local gluing constructions. This issue of the surjectivity of the gluing construction is conspicuously missing in the literature and it is quite nontrivial in non-generic situations. We deal with it in 4.5.3 by relying on Łojasewicz's inequality in real algebraic geometry.

In 4.5.4 we prove two global gluing theorems, one in a generic situation and the other one in a special, obstructed setting.

The final section 4.5 contains some simple topological applications of the gluing technique. We prove the connected sum theorem and the blow-up formula. Moreover, we present a new and very short proof of a vanishing theorem of Fintushel and Stern.

These notes were written with a graduate student in mind but there are many new points of view to make it interesting for experts as well (especially our new approach to the gluing theorem). The minimal background needed to go through these notes is a knowledge of basic differential geometry, algebraic topology and some familiarity with fundamental facts concerning elliptic partial differential equations. The list of contents for Chapter 1 can serve as background studying guide.

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Personal note I have spent an exciting time of my life thinking and writing these notes and I have been supported along the way by many people.

The book grew out of a year long seminar at McMaster University and a year long graduate course I taught at the University of Notre Dame. I want to thank the participants at the seminar and the course for their patience, interest, and most of all, for their many useful questions and comments.

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