A TASTE OF TWO-DIMENSIONAL COMPLEX ALGEBRAIC GEOMETRY

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ABSTRACT. These are notes for a talk at a topology seminar at ND.

1. GENERAL FACTS

In the sequel, for simplicity we denote the complex projective space $\mathbb{CP}^N$ by $\mathbb{P}^N$, and we denote the projective coordinates by $[\vec{z}] = [z_0, \dots, z_N]$. A *smooth projective variety* is a connected compact complex submanifold of some projective space $\mathbb{P}^N$. All the smooth projective varieties are Kähler manifolds, but the converse is not true. An *algebraic surface* is a smooth projective variety of complex dimension 2.

A celebrated theorem of Chow [2, Chap. 1, Sec. 3] states that if $X \hookrightarrow \mathbb{P}^N$ is a smooth projective variety, then there exist homogeneous *polynomials* $P_1, \dots, P_\nu \in \mathbb{C}[z_0, \dots, z_N]$ such that

$$X = \left\{ [\vec{z}] \in \mathbb{P}^N;\;\; P_1(\vec{z}) = \cdots = P_\nu(\vec{z}) = 0 \right\}.$$

In other words, all projective varieties are described by a finite collection of homogeneous polynomial equations.

If X is a smooth projective variety of (complex) dimension n then the tanget bundle TX has a complex structure and thus we can speak of Chern classes

$$c_k(X) := c_k(TX), \;\; k = 1, \dots, n.$$

We also have an isomorphism of holomorphic vector bundles

$$\Lambda^m T^*X \otimes \mathbb{C} \cong \bigoplus_{p+q=m} \Lambda^{p,q} T^*X. \tag{1.1}$$

If $(u_1, \dots, u_n)$ are local holomorphic coordinates on X then the smooth sections of $\Lambda^{p,q}T^*X$ are locally described by sums of forms of the type

$$f(u) du_{i_1} \wedge \cdots \wedge du_{i_p} \wedge d\bar{u}_{j_1} \wedge \cdots \wedge d\bar{u}_{j_q}.$$

For every smooth differential form α of degree m with complex coefficients we denote by $\alpha^{p,q}$ its $\Lambda^{p,q}$-component in the *Hodge decomposition* (1.1).

The holomorphic line bundle $\Lambda^{n,0}T^*X$ is called the *canonical line bundle* of X and it is denoted by K_X. Let us point out that

$$c_1(X) = -c_1(K_X).$$

From the above equality we deduce that

$$\text{The algebraic variety } X \text{ is spinnable if and only if } c_1(K_X) = 0 \bmod 2. \tag{1.2}$$

We denote by $\boldsymbol{H}^m(X)$ the space of complex differential forms of degree m on X that are harmonic with respect to the Kähler metric, and for any closed form α we denote by $[\alpha]$ its harmonic part in the Hodge decomposition. Then Hodge theory implies that

$$\dim_{\mathbb{C}} \boldsymbol{H}^m(X) = b_m(X),$$

Last modified on March 3, 2009.

and

$$[\alpha]^{p,q}=[\alpha^{p,q}],\ \ \forall \alpha\in\Omega^m(X)\otimes\mathbb{C},\ \ d\alpha=0,\ \ \forall p+q=m.$$

We denote by $\boldsymbol{H}^{p,q}(X)$ the space of harmonic forms of type (p,q), and we set $h^{p,q}(X):=\dim_{\mathbb{C}}\boldsymbol{H}^{p,q}(X)$.

$$\boldsymbol{H}^m(X)=\bigoplus_{p+q=m}\boldsymbol{H}^{p,q}(X)\ \text{ and }\ b_m(X)=\sum_{p+q=m}h^{p,q}(X). \tag{1.3}$$

Moreover, Hodge theory also implies that

$$h^{p,q}(X)=h^{q,p}(X)=h^{n-p,n-q}(X),\quad \forall p,q. \tag{1.4}$$

If we denote by $\boldsymbol{\Omega}^p_X$ the sheaf of holomorphic sections of the holomorphic bundle $\Lambda^{p,0}T^*X$ then Dolbeault theorem implies that the space $\boldsymbol{H}^{p,q}(X)$ is naturally isomorphic to the q-th Čech cohomology of the sheaf $\boldsymbol{\Omega}^p_X$,

$$\boldsymbol{H}^{p,q}(X)\cong H^q(X,\boldsymbol{\Omega}^p_X).$$

Observe that $\boldsymbol{\Omega}^n_X$ coincides with the sheaf of holomorphic sections of the canonical line bundle so that the Hodge number $h^{n,0}(X)$ is equal to the dimension of the space of global holomorphic sections of K_X. This integer is known as the *geometric genus* of X and it is denoted by $p_g(X)$. We define the Euler characteristics

$$\chi_p(X)==\sum_q(-1)^qh^{p,q}(X).$$

When X is an algebraic surface, then the Hodge numbers can be organized in a *Hodge diamond*

$$\begin{array}{ccccc} & & h^{0,0}(X) & & \\ & h^{1,0}(X) & & h^{0,1}(X) & \\ h^{2,0}(X) & & h^{1,1}(X) & & h^{0,2}(X)\\ & h^{1,2}(X) & & h^{2,1}(X) & \\ & & h^{2,2}(X) & & \end{array}$$

where the entries symmetric with respect to one of the two axes are equal. From this symmetry we deduce that $b_1(X)=2h^{1,0}(X)$ so that the first Betti number of an algebraic surface must be even. Note that in this case

$$\chi_0(X)=1-h^{0,1}(X)+p_g(X)=1-\frac{1}{2}b_1(X)+p_g(X).$$

This holomorphic Euler characteristic is given by the *Noether formula*

$$\chi_0(X)=\frac{1}{12}\big(\langle c_1(X)^2,[X]\rangle+\langle c_2(X),[X]\rangle\big)=\frac{1}{12}\langle c_1(X)^2,[X]\rangle+\frac{1}{12}\chi(X), \tag{1.5}$$

where $\chi(M)$ is the topological Euler characteristic.

The Hodge decomposition (1.3) is compatible with the intersection form. This is the content of the *Hodge index theorem*. For simplicity, we state it only for algebraic surfaces.

Theorem 1.1 (Hodge index theorem)**.** *If X is an algebraic surface, then it is a smooth oriented real 4-manifold. The second Betti number decomposes as $b_2(X)=b_2^+(X)+b_2^-(X)$, where $b_2^{\pm}(X)$ is the number of positive/negative eigenvalues of the intersection form of X. Then*

$$b_2^+(X)=h^{2,0}(X)+h^{0,2}(X)+1=2p_g(X)+1, \tag{1.6}$$

and the signature $\tau(X)$ of X is

$$\tau(X) = b_2^+(X) - b_2^-(X) = 2p_g(X) + 2 - h^{1,1}(X).$$

□

One consequence of The Hodge index theorem implies that $p_g(X)$ and $h^{1,1}(X)$ are *topological invariants* of X.

From the identity $p_1(X) = c_1(X)^2 - 2c_2(X)$ and the Hirzebruch signature formula we deduce

$$\tau(X) = \frac{1}{3}\langle c_1(X)^2 - 2c_2(X), [X]\rangle,$$

so that

$$\langle c_1(X)^2, [X]\rangle = 2\chi(X) + 3\tau(X).$$

Using a bit of Morse theory one can produce some information about the homotopy type of a smooth algebraic variety. The following is a special case of Lefschetz' hyperplane theorem.

Theorem 1.2. *If X is a compact, complex 2-dimensional submanifold of $\mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{P}^2$ then X is connected and simply connected.* □

2. Divisors

A hypersurface on a smooth projective variety X is a closed set Y locally described as the zero set of a (nontrivial) holomorphic function.

A point $p \in Y$ is called a smooth point if there exists an open neighborhood U of p in X such that $U \cap Y$ is a smooth submanifold of U. We denote by Y^* the set of smooth points. Then Y^* is an open and dense subset of Y. The hypersurface Y is called *irreducible* if Y^* is connected. For a general hypersurface Y we define its components to be the closures of the connected components of Y^*.

The hypersurfaces on an algebraic surface are called *curves*.

We denote by $\operatorname{Div}(X)$ the free Abelian group generated by the irreducible hypersurfaces in X. The elements in X are called *divisors* on X. Clearly, every irreducible hypersurface defines a divisor that we denote by $[Y]$. Then any divisor D on X can be written as a formal sum

$$D = \sum_{i=1}^{\nu} m_i[Y_i],$$

where Y_i are irreducible hypersurfaces on X called the components of D. The integer m_i is called the multiplicity of D along Y_i. The hypersurface $Y_1 \cup \cdots \cup Y_\nu$ is called the *support* of D and it is denoted by $\operatorname{supp}(D)$.

A divisor D on X is called *effective* if the multiplicities along its components are all nonnegative.

We will find convenient use Cartier's descriptions of divisors. Suppose we are given an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in A}$, nontrivial holomorphic functions $p_\alpha, q_\alpha \in \mathcal{O}(U_\alpha)$, and nowhere vanishing holomorphic functions $g_{\alpha\beta} \in \mathcal{O}(U_{\alpha\beta})^*$ such that, if we set

$$h_\alpha = \frac{p_\alpha}{q_\alpha}$$

then

$$g_{\alpha\beta} = \frac{h_\alpha}{h_\beta} \text{ on } U_{\alpha\beta} \setminus \{f_\beta = 0\}. \tag{2.1}$$

This defines a divisor $D = D(\mathcal{U}, \{p_\alpha, q_\alpha\})$ whose support is the hypersurface

$$Z = Z_0 \cup Z_\infty, \quad Z_0 = \bigcup_\alpha \{p_\alpha = 0\}, \quad Z_\infty = \bigcup_\alpha \{q_\alpha = 0\}.$$

If $Z_{0,i}$ is a component of Z_0. Then we define $m_{0,i}$ as follows. Choose $x \in Z_{0,i}^*$, U_α containing x and then define $m_{0,i}$ to be the order of vanishing p_α along $Z_{0,i}^* \cap U_\alpha$. The equality (2.1) guarantees that the integer is independent of the various choices. We define the integer $m_{j,\infty}$ similarly and then we set

$$D(\mathcal{U}, \{p_\alpha, q_\alpha\}) = \sum_i m_{0,i}[Z_{0,i}] - \sum_j m_{\infty,j}[Z_{\infty,j}]$$

This divisor is called the *Cartier divisor* defined by the open cover $\mathcal{U}$ and the meromorphic functions h_α on U_α.

Proposition 2.1. *On a smooth algebraic variety any divisor can be described as a Cartier divisor.* □

To any Cartier divisor D defined by an open cover $\mathcal{U}$ and meromorphic function h_α satisfying (2.1) we can associate a *holomorphic line bundle* L_D defined by the open cover and gluing cocycle $g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$. The functions h_α can be interpreted as defining a *meromorphic* section of L_D.

Proposition 2.2. *On a smooth algebraic variety the above coprrespondence*

$$\Big\{ \textit{Cartier divisors} \Big\} \longrightarrow \Big\{ \textit{pairs (holomorphic line bundle, meromorphic section)} \Big\}$$

is a bijection. □

Because of the above proposition, the words divisors and holomorphic line bundles are used interchangeably in algebraic geometry. In particular, the canonical line bundle is often called the canonical divisor.

A *principal (Cartier) divisor* is a divisor D such that the associated line bundle is holomorphically trivializable. We denote by $\mathrm{PDiv}(X)$ the subgroup of principal divisors. The quotient $\mathrm{Div}(X)/\mathrm{PDiv}(X)$ can be identified with the space of holomorphic isomorphism classes of holomorphic line bundles on X. This group is also known as the *Picard group* of X and it is denoted by $\mathrm{Pic}(X)$.

Any hypersurface Y on a complex n-dimensional algebraic variety is defines a homology class $[Y] \in H_{2n-2}(X, \mathbb{Z})$. This extends by linearity to a morphism of groups

$$\mathrm{Div}(X) \ni D \mapsto [D] \in H_{2n-2}(X, \mathbb{Z}).$$

For every divisor $D \in \mathrm{Div}(X)$ we let $[D]^\dagger \in H^2(X, \mathbb{Z})$ denote the Poincaré dual of the homology class $[D]$. Let us observe that the Gauss-Bonnet Chern theorem implies that

$$[D]^\dagger = c_1(L_D), \quad \forall D \in \mathrm{Div}(X).$$

Proposition 2.3. *The Poincaré dual of the homology class defined by a divisor is a cohomology class of degree* 2 *and Hodge type* $(1,1)$. □

If X is an algebraic surface then any divisor on X defines a (non-torsion) 2-dimensional homology class. The intersection pairing

$$H_2(X, \mathbb{Z})/\mathrm{Tors} \times H_2(X, \mathbb{Z})/Tors \to \mathbb{Z}$$

defines an intersection pairing on $\mathrm{Div}(X)$.

Suppose $C \hookrightarrow X$ is a smooth, connected algebraic curve in an algebraic surface X. Then as a topological space C is homeomorphic to a Riemann surface. We denote by $g(C)$ its genus. The normal bundle of $C \hookrightarrow X$ is a holomorphic line bundle and we have the following *topological* isomorphism complex vector bundles

$$TX|_C \cong TC \oplus T_C X.$$

Dualizing we deduce

$$TX^*_C = T^*C \oplus T^*_C X$$

Taking the second exterior power in the above equality, and observing that $T^*C = K_C$ we deduce the *adjunction formula*

$$K_X|_C \cong K_C \otimes T^*_C X. \tag{2.2}$$

Hence

$$c_1(K_X) = c_1(K_C) + c_1(T^*_C X) = c_1(K_X) - c_1(T_C X).$$

Integrating over C we deduce

$$\langle c_1(K_X), [C]\rangle = \langle c_1(K_C), [C]\rangle - \langle c_1(T_C X), [C]\rangle = 2g(C) - 2 - [C] \bullet [C].$$

We obtain in this fashion the *genus formula*

$$g(C) = 1 + \frac{1}{2}\Big([C] \bullet [C] + \langle c_1(K_X), [C]\rangle\Big) = 1 + \frac{1}{2}[C] \bullet \big([C] + [K_X]\big).$$

3. EXAMPLES

Before we discuss in detail several examples of algebraic surfaces we need to discuss an important construction in algebraic geometry.

Example 3.1 (The blowup construction)**.** We start with the simples case, the blowup of the affine plane $\mathbb{C}^2$ at the origin. Consider the incidence variety

$$\mathcal{I} = \big\{ (\ell, p) \in \mathbb{P}^1 \times \mathbb{C}^2; \text{ the point } p \text{ belongs to the complex line } \ell \subset \mathbb{C}^2 \big\}.$$

The variety $\mathcal{I}$ is a complex manifold of dimension 2. The fiber of the natural projection $\pi : \mathcal{I} \to \mathbb{P}^1$ over the line $\ell \in \mathbb{P}^1$ consists of the collection of points in ℓ. Thus, the projection $\pi : \mathcal{I} \to \mathbb{P}^1$ is none other than the tautological line bundle over $\mathbb{P}^1$. We denote by E the submanifold $\mathbb{P}^1 \times \{0\} \subset E$. Observe that E is exactly the zero section of the tautological line bundle.

On the other hand, we have another natural projection

$$\sigma : \mathcal{I} \to \mathbb{C}^2 \to \mathbb{C}^2, \;\; (\ell, p) \mapsto p.$$

This map restrict to a biholomorphic map

$$\sigma : \mathcal{I} \setminus E \to \mathbb{C}^2 \setminus \{0\}.$$

The manifold $\mathcal{I}$ is called the blowup of $\mathbb{C}^2$ at the origin. Equivalently, we say that $\mathcal{I}$ is obtained via the σ-process at $0 \in\in \mathbb{C}^2$.

For any open set $U \subset \mathbb{C}^2$ we set $\mathcal{I}(U) = \sigma^{-1}(U) \subset U$. The map σ defines a surjection $\mathcal{I}(U) \to U$. Moreover, if U does not contain the origin, then $\sigma : \mathcal{I}(U) \to U$ is a biholomorphism.

Suppose $X \subset \mathbb{P}^N$ is an algebraic surface and $p_0 \in X$. We want to give two descriptions of the blowup of X at p_0.

The first construction uses the intuition from the special case discussed above. We choose local holomorphic coordinates (z_0, z_1) on a neighborhood U of p_0 so that U can be identified with a neighborhood of 0 in $\mathbb{C}^2$. Set $U^* := U \setminus \{p_0\}$.

Then the blowup of X at p_0 is obtained by removing the point p_0, and then attaching the manifold $\mathcal{I}(U)$ to $X \setminus \{p_0\}$ using the biholomorphism $\sigma : \mathcal{I}(U^*) \to U^*$. We denote by $\widehat{X}_{p_0}$ the complex surface obtained in this fashion. Its biholomorphism type could depend on U but its diffeomorphism type is independent of U. Note that we have a natural holomorphic map

$$\sigma : \widehat{X}_{p_0} \to X.$$

Its fiber over p_0 is the zero section $E \hookrightarrow \mathcal{I}$. It is a smooth curve in $\widehat{X}_{p_0}$ called the *exceptional divisor*. It is a *rational curve*, i.e., it is biholomorphic to the projective line $\mathbb{P}^1$, and in fact, it can be identified with the projectivization of the complex tangent plane $T_{p_0}X$. The points in E can thus be identified with the one-dimensional complex subspaces of $T_{p_0}X$.

The self-intersection number of E coincides with the Euler number the tautological line bundle so that

$$[E] \bullet [E] = -1.$$

To see that it is $\widehat{X}_{p_0}$ is algebraic we need to give a more algebraic description of the blowup construction. We follow the approach in [1, IV.2].

Recall that X is embedded in a projective space $\mathbb{P}^N$. Assume the point p_0 has homogeneous coordinates $[0, 0, \dots, 1]$. We obtain a map

$$X^* := X \setminus \{p_0\} \xrightarrow{\varphi} \mathbb{P}^{N-1}, \;\; X^* \ni p \mapsto [z_0(p), \dots, z_{N-1}(p)] \in \mathbb{P}^{N-1}.$$

The graph of φ is a submanifold

$$\Gamma_\varphi \subset X^* \times \mathbb{P}^{N-1} \subset X \times \mathbb{P}^{N-1}.$$

Then the blowup $\widehat{X}_{p_0}$ of X at p_0 can be identified with the closure of Γ_φ in $\mathbb{P}^{N-1}$. The blowdown map $\sigma : \widehat{X}_{p_0} \to X$ is induced by the natural projection $X \times \mathbb{P}^{n-1} \to X$.

Suppose that C is a curve on X then the *total transform* of C is the curve $\sigma^{-1}(C)$ on $\widehat{X}$. The *proper transform* of C is the closure in $\widehat{X}_{p_0}$ of $\sigma^{-1}(C \setminus \{p_0\})$. We denote it by $\sigma'(C)$. If p_0 happens to be a smooth point of C then

$$\sigma'(C) = \sigma^{-1}(C \setminus \{p_0\}) \cup [\widehat{p}_0],$$

where $[\widehat{p}_0]$ denotes the point in E corresponding to the one dimensional complex subspace $T_{p_0}C \subset T_{p_0}X$. In this case

$$\sigma^{-1}(C) = E \cup \sigma'(C).$$

The point $\widehat{p}_0$ is a smooth point on both $\sigma'(C)$ and E, and the components E and $\sigma'(C)$ intersect transversally at $\widehat{p}_0$.

More generally, the operation of proper and strict transforms extend by linearity to maps

$$\sigma^{-1}, \sigma' : \mathrm{Div}(X) \to \mathrm{Div}(\widehat{X})$$

The proper transform sends principal divisors to principal divisors and thus induces a morphism of groups

$$\sigma^{-1} : \mathrm{Pic}(X) \to \mathrm{Pic}(\widehat{X}).$$

This morphism coincides with the pullback operation on holomorphic line bundles.

If C_0, C_1 are two curves on X such that p_0 is a smooth point on both, then

$$[\sigma'(C_0)] \bullet [\sigma(C_1)] = [C_0] \bullet [C_1] - 1.$$

In particular,

$$[\sigma'(C_0)]^2 = [C_0]^2 - 1. \tag{3.1}$$

For any complex curve $C \subset X$ we have the equality

$$\sigma^*([C]^\dagger) = [\sigma^{-1}(C)]^\dagger.$$

The canonical line bundle of the blowup $\widehat{X}$ is related to the canonical line bundle of X via the equality

$$K_{\widehat{X}} = \sigma^*(K_X) \otimes L_E.$$

If we think of line bundles as divisors, the last equality can be rewritten as

$$K_{\widehat{X}} = \sigma^{-1}(K_X) + E \in \mathrm{Pic}(\widehat{X}).$$

□

There is a very simple way of recognizing when an algebraic surface is the blowup of another. More precisely, we have the following remarkable result, [2, Chap 4., Sec 1].

Theorem 3.2 (Castelnuovo-Enriques)**.** *Suppose Y is an algebraic surface containing an exceptional curve, i.e., a rational curve C with self-intersection $[C]^2 = -1$. Then there exists an algebraic surface, a point $p_0 \in X$ and a biholomorphic map*

$$\phi : Y \to \widehat{X}_{p_0}$$

such that $\phi(C)$ is the exceptional divisor of the blowdown map $\sigma : \widehat{X}_{p_0} \to X$. The manifold X is called the blow down of Y along the exceptional curve C. □

In simpler terms, the Castelnuovo-Enriques theorem states that exceptional curves can be blown down.

Definition 3.3. (a) Two algebraic surfaces are said to be *birationally equivalent* if we can obtain one from the other by a sequence of blowups and blowdowns. A surface is called *rational* if it is birationally equivalent to the projective plane $\mathbb{P}^2$.

(b) A geometric invariant of an algebraic surface is called a *birational invariant* if it does not change under blowups. Thus, birationally equivalent surfaces have identical birational invariants. □

Example 3.4. The Hodge numbers $h^{1,0}$ and $h^{2,0}$ are birational invariants of an algebraic surface. More generally, if X is an algebraic surface, then we define $P_n(X)$ to be the dimension of the space of holomorphic sections of the line bundle $K_X^{\otimes n}$. The numbers $P_n(X)$ are called the *plurigenera* of X. All of them are birational invariants of X. □

Example 3.5 (The projective plane)**.** Consider the projective plane $X = \mathbb{P}^2$, the point $p_0 = [1, 0, 0]$, and the line at infinity H given by the equation $z_0 = 0$.

The homology group $H_2(\mathbb{P}^2, \mathbb{Z})$ is generated by the homology class determined by H, and $H^2 = 1$. The line bundle determined by the divisor $-H$ is the tautological line bundle $\mathbb{U} \to \mathbb{P}^2$. Using Proposition 2.3 we deduce.

$$h^{1,1}(\mathbb{P}^2) = 1 = b_2(X)$$

so that$p_g(\mathbb{P}^2) = h^{2,0}(\mathbb{P}^2) = 0$. The canonical line bundle $K_{\mathbb{P}^2}$ coincides with the line bundle associated to the divisor $-3H$,

$$K_X = -3H \Longleftrightarrow K_X \cong \mathbb{U}^{\otimes 3}.$$

In particular, this shows that $\mathbb{P}^2$ is not spinnable.

The Hodge diamond of $\mathbb{P}^2$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Any projective line ℓ through p_0 is uniquely determined by its intersection with H. For any $p \in H$ we denote by ℓ_p the line going through p_0 and p. The blowup of $\mathbb{P}^2$ at p_0 contains two distinguished

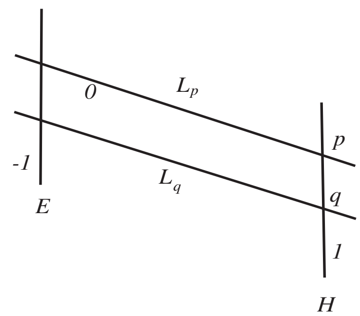

FIGURE 1. Blowing up the projective plane.

rational curves: the exceptional divisor E and the proper transform of H which we continue to denote by H. We denote by L_p the proper transform of ℓ_p. Then (see Figure 1)

$$E^2 = -1, \;\; H^2 = 1, \;\; L_p^2 = 0, \;\; \forall p \in H.$$

We obtain a holomorphic map $\widehat{\mathbb{P}}^2 \to H$ whose fiber above $p \in H$ is the rational curve L_p; see Figure 1. This is an example of ruled surface.

□

Example 3.6 (Quadrics)**.** Consider the ruled surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. If we consider the Segre embedding

$$\boldsymbol{S} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \;\; ([s_0, s_1], [t_0, t_1]) \mapsto [z_0, z_1, z_2, z_3] = [s_0t_0, s_0t_1, s_1t_0, s_1t_1]$$

we observe that the image $S(\mathbb{P}^1 \times \mathbb{P}^1)$ coincides with the quadratic hypersurface

$$\big\{ [z_0, z_1, z_2, z_3] \in \mathbb{P}^2; \;\; z_0z_3 - z_1z_2 = 0 \big\}.$$

In fact, any smooth quadratic hypersurface in $\mathbb{P}^3$ is biholomorphic to Q

The surface Q is swept by two family of projective lines

$$X(t) = \mathbb{P}^1 \times \{t\} \in \mathbb{P}^1 \times \mathbb{P}^1, \;\; Y(s) = \{s\} \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

forming the pattern in Figure 2.

The curves in the family $X(t)$ define the same homology class $x \in H^2(X, \mathbb{Z})$, while the curves in the family $Y(s)$ define the same homology class $y \in H^2(X, \mathbb{Z})$. The classes x and y form an integral basis of the homology group $H^2(Q, \mathbb{Z})$. The intersection form is determined by the relations (see Figure 2)

$$x \bullet x = y \bullet y = 0, \;\; x \bullet y = 1.$$

From Proposition 2.3 we deduce that

$$2 = b_2(X) \geq h^{1,1}(X) \geq 2.$$

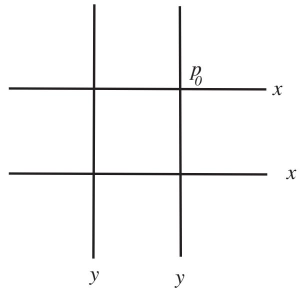

FIGURE 2. The intersection pattern of the x and y classes on a ruled surface.

Hence This shows that $h^{1,1} = 2$ and $p_g = 0$ so that the Hodge diamond of Q is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

If let $\ell, r : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the natural projections

$$\ell(s,t) = s, \quad r(s,t) = r$$

then we deduce that the canonical line bundle of Q satisfies

$$K_Q = \ell^* K_{\mathbb{P}^1} \otimes r^* K^1_{\mathbb{P}}, \quad c_1(K_Q) = \ell^* c_1(K_{\mathbb{P}^1}) + r^* c_1(K_{\mathbb{P}^1}) = -2(x^\dagger + y^\dagger).$$

From (1.2) we deduce that Q is spinnable.

Consider again the projective plane $\mathbb{P}^2$ and two points $p_0, p_1 \in \mathbb{P}^3$. Fix a projective line $L \subset \mathbb{P}^2$ that does not contain p_0 and p_1. For any $s \in L$ we denote by $\ell_x(s)$ the projective line $[p_0 s]$ and by $\ell_y(s)$ the projective line $[p_1 s]$. If q denotes the intersection of the line $L_0 = [p_0 p_1]$ with L we deduce that

$$L_0 = \ell_x(q) = \ell_y(q).$$

Note that for any $p \in \mathbb{P}^2 \setminus \{p_0, p_1\}$ there exists a unique pair $(s,t) = (s(p), t(p)) \in L \times L$ such that p is the intersection of the line $\ell_x(t)$ with the line $\ell_y(s)$. We obtain in this fashion a (rational) map

$$\pi : \mathbb{P}^2 \setminus \{p_0, p_1\} \to L \times L, \quad p \mapsto \big(s(p), t(p) \big).$$

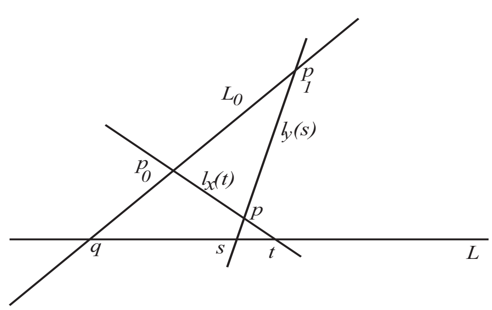

FIGURE 3. Blowing up the projective plane.

Let X be the blowup $\mathbb{P}^2$ at the points points p_0, p_1. We denote by $\ell'_x(s)$ the strict transform of $\ell_x(s)$, by $\ell'_y(s)$ the strict transform of of $\ell_y(s)$, and by L'_0 the strict transform of the line $L_0 = [p_0p_1]$. Then

$$[\ell'_x(s)]^2 = [\ell'_y(s)]^2 = [\ell_x(s)]^2 - 1 = 0, \quad [L'_0]^2 = [L_0]^2 - 2 = -1.$$

his shows that L'_0 is an exceptional curve.

After the blowups the points p_0 and p_1 are replaced by exceptional divisors E_0, E_1. Observe that the points in E_0 can be identified with the lines ℓ_x so that we have a natural biholomorphic map $\pi_0 : E_0 \to L$ such that the point $q_0 \in E$ corresponds to the line $\ell_x\big(\pi_0(q_0)\big)$ we define in a similar way a biholomorphic map $\pi_1 : E_1 \to L$ We can now define

$$\widehat{\pi} : X \to L \times L$$

as follows.

- On the region $X \setminus (E_0 \cup E_1) = \mathbb{P}^2 \setminus \{p_0, p_1\}$ we set $\widehat{\pi} = \pi$.
- If $q_0 \in E_0$ then we set

$$\widehat{\pi}(q_0) = (\pi_0(x), q), \quad q = L \cap L_0$$

- If $q_1 \in E_1$ then we set

$$\widehat{\pi}(q_1) = (q, \pi_1(q_1)).$$

The resulting map $\widehat{\pi} : \widehat{\mathbb{P}}^2_{p_0,p_1} \to L \times L$ is holomorphic, the fiber over (q, q) is the exceptional curve L'_0, and the induced map

$$\widehat{\mathbb{P}}^2_{p_0,p_1} \setminus L'_0 \to (L \times L) \setminus \{(q, q)\}$$

is a biholomorphism. This proves that the ruled surface $L \times L$ is obtained by blowing down the exceptional curve L'_0. Thus the quadric Q can be obtained from the projective plane $\mathbb{P}^2$ by performing two blowups followed by a blowdown. Thus any quadric is a rational surface. □

Example 3.7 (Hirzebruch surfaces)**.** For every $n \geq 0$ we consider holomorphic line bundle $L_n \to \mathbb{P}^1$ of degree n, i.e.,

$$\langle c_1(\mathbb{P}^1), [\mathbb{P}^1] \rangle = n.$$

We form the rank 2 complex vector bundle $E_n = \underline{\mathbb{C}} \oplus L_n$, and denote by F_n its projectivization. In other words, the bundle $F_n \xrightarrow{\pi} \mathbb{P}^1$ is obtained by projectivizing the fibers of E_n, so that the fiber $F_n(p)$ of F_n over $p \in \mathbb{P}^1$ is the projectivization of the vector space $\mathbb{C} \oplus L_n(p)$, where $L_n(p)$ is the fiber of L_n over p. Equivalently, $F_n(p)$ can be viewed as the one point compactification of the line $L_n(p)$.

The zero section of L_n determines a section of F_n that associates to each $p \in \mathbb{P}^1$ the the line $\mathbb{C} \oplus 0 \subset \mathbb{C} \oplus L_n(p)$. This determines a rational curve S_0 in F_n with selfintersection number

$$[S_0]^2 = n$$

because the self intersection number of the zero section of L_n equals the degree of this line bundle. We denote by x_0 the homology class determined by this curve and f the homology class determined by a fiber of F_n. They satisfy the intersection equalities

$$x_0^2 = n, \quad f^2 = 0, \quad x_0 \bullet f = 1.$$

Using these equalities and the Leray-Hirsch theorem we deduce that x_0 and f span $H_2(F_n, \mathbb{Z})$. The Hodge diamond of F_n is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The section at ∞ is the section of the bundle $F_n \to \mathbb{P}^1$ that associates to each $p \in \mathbb{P}^1$ b the line $L_n(p) \subset \mathbb{C} \oplus L_n(p)$. It defines a rational curve S_∞ which determines a homology class x_∞.

We can write $x_\infty = ax_0 + bf$, $a, b \in \mathbb{Z}$. Using the equalities

$$x_\infty \bullet x_0 = 0, \;\; x_\infty \bullet f = 1$$

we deduce

$$x_\infty = x_0 - nf, \;\; x_\infty^2 = -n.$$

Denote by $y \in H^2(F_n, \mathbb{Z})$ the homology class whose Poincaré dual is $c_1(K_{F_n})$. Again we can write $y = ax_0 + bf$, $a, b \in \mathbb{Z}$. Using the genus formula we deduce

$$0 = g(\text{zero section}) = 1 + \frac{1}{2} x_0 \bullet (x_0 + y) = 1 + \frac{1}{2}(n + na + b),$$

$$0 = g(\text{fiber}) = 1 + \frac{1}{2} f \bullet (f + y) = 1 + \frac{1}{2} a.$$

Hence $a = -2$, $b = n - 2$ so that

$$y = -2x_0 + (n-2)f, \;\; c_1(K_{F_n}) = -2x_0^\dagger + (n-2)f^\dagger. \tag{3.2}$$

This shows that F_n is spinnable if n is even and non-spinnable if n is odd. Let us point out that F_1 is biholomorphic to the blowup of $\mathbb{P}^2$ at a point, while F_0 can be identified with the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Using the genus formula coupled with (3.2) one can prove that if $m, n > 1$ and $m \neq n$, then the Hirzebruch surface F_n does not contain a smooth complex rational curve with self-intersection number m. This shows that F_n and F_m are not biholomorphic if $n \neq m$. On the other hand F_n is diffeomorphic to F_m if $n \equiv m \bmod 2$. Moreover, any Hirzebruch surface is rational. □

Example 3.8 (Hypersurfaces in $\mathbb{P}^3$)**.** The zero locus X_d of of generic degree d homogeneous polynomial $P_d \in \mathbb{C}[z_0, z_1, z_2, z_3]$ is a smooth hypersurface in $\mathbb{P}^3$. By Lefschetz theorem we deduce that X_d is simply connected. Using a higher dimensional version of the adjunction formula (2.2) coupled with Noether's formula and basic properties of the Chern classes we deduce that the Hodge diamond of X_d is (see [3])

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ \binom{d-1}{3} & & h(d) & & \binom{d-1}{3} \\ & 0 & & 0 & \\ & & 1 & & \end{array},$$

where

$$\binom{d-1}{3} = \frac{(d-1)(d-2)(d-3)}{6}, \;\; h(d) = d(d^2 - 4d + 6) - 2 - 2\binom{d-1}{3}.$$

The surface X_d is spinnable if and only if d is even. The signature of the intersection form is negative,

$$\tau(X_d) = -\frac{d(d-2)(d+2)}{3}.$$

The hypersufaces of degree $d \geq 5$ are said to have *general type*.

Using M. Freedman's classification of simply connected topological 4-manifolds we deduce that the above data completely determine the homeomorphism type of X_d. In fact, they are all diffeomorphic. □

Example 3.9 ($K3$ surfaces)**.** These are simply connected surfaces X such that the canonical line bundle K_X is holomorphically trivial. For example smooth degree 4 hypersurface in $\mathbb{P}^3$ is a $K3$-surface.

In this case $p_g(X) = 1$ because the space of holomorphic sections of K_X is one dimensional. Since $b_1(X) = 0$ we deduce that $h^{0,1}(X) = 0$. Noether's formula then implies that

$$1 + p_g(X) = \chi_0(X) = \frac{1}{12}\chi(X).$$

Hence

$$24 = 2 + b_2(X) \Longrightarrow 22 = b_2(X) = 2p_g + h^{1,1}(X) \Longrightarrow h^{1,1}(X) = 20.$$

Hence, the Hodge diamond of a $K3$-surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Since $c_1(K_X) = 0$ we deduce that a $K3$ surface is spinnable. This implies that all $K3$-surfaces are homeomorphic. It is much harder to prove that in fact they are all diffeomorphic. □

Example 3.10 (Elliptic surfaces)**.** An algebraic surface X is called elliptic if there exists a holomorphic map

$$\pi : X \to \mathbb{P}^1$$

such that the generic fiber of π is a smooth algebraic curve of genus 1. The theory of elliptic surfaces is very rich so we limit ourselves to a special class of elliptic surfaces. Fix two degree 3 homogeneous polynomials $A_0, A_1 \in \mathbb{C}[z_0, z_1, z_2]$ and consider the hypersurface $E(n) \subset \mathbb{P}^1 \times \mathbb{P}^2$ described by the equation

$$([t_0, t_1], [z_0, z_1, z_2]) \in \mathbb{P}^1 \times \mathbb{P}^2, \;\; t_0^n A_0(z_0, z_1, z_2) = t_1^n A_1(z_0, z_1, z_2) = 0.$$

For generic choices of A_0 and A_1 this is a smooth hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$. We set

$$A_{t_0,t_1} = t_0^n A_0(z_0, z_1, z_2) = t_1^n A_1(z_0, z_1, z_2)$$

The natural projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ induces a holomorphic map $\pi : E(n) \to \mathbb{P}^1$ whose fiber over $[t_0, t_1]$ is the elliptic curve

$$\{A_{t_0,t_1}(z_0, z_1, z_2) = 0\} \subset \mathbb{P}^2.$$

The Hodge diamond of this surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ (n-1) & & 10n & & (n-1) \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The biholomorphism type of $E(n)$ depends on the choice of polynomials A_0 and A_1. The Euler characteristic of $E(n)$ is $12n$, and $E(n)$ is spinnable if and only if n is even. The elliptic surface $E(2)$ is a $K3$ surface, but not all $K3$-surfaces are elliptic.

Not all the fibers of the map $\pi : E(n) \to \mathbb{P}^1$ are smooth. For generic A_0 and A_1 there are $12n$ singular fibers. □

4. The Kodaira classification

The Kodaira classification has as starting point the plurigenera of $P_n(X)$ of an algebraic surface. The sequence $\big(P_n(X)\big)_{n\geq 1}$ displays one of the following types of behavior.

($-\infty$) The sequence of plurigenera is identically zero, $P_n(X) = 0$, $\forall n \geq 1$. In this case we say that the Kodaira dimension of X is $-\infty$ and we write this $\mathrm{kod}(X) = -\infty$.

(**0**) The sequence of plurigenera is bounded, but not identically zero. In this case, we say that the Kodaira dimension of X is 0, $\mathrm{kod}(X) = 0$.

(**1**) The sequence of plurigenera grows linearly, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C} n \leq P_n(X) \leq Cn, \ \ \forall n \geq 1.$$

In this case, we say that the Kodaira dimension of X is 1, $\mathrm{kod}(X) = 1$.

(**2**) The sequence of plurigenera grows quadratically, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C} n^2 \leq P_n(X) \leq Cn^2, \ \ \forall n \geq 1.$$

In this case, we say that the Kodaira dimension of X is 2, $\mathrm{kod}(X) = 2$.

Example 4.1. The rational surfaces have Kodaira dimension $-\infty$, the $K3$ surfaces have Kodaira dimension 0 the elliptic surfaces $E(n)$, $n \geq 3$, have Kodaira dimension 1 and the hypersurfaces X_d, $d \geq 5$ have Kodaira dimension 2. □

References

[1] D. Eisenbud, J. Harris: *The Geometry of Schemes*, Grad. Texts in Math., vol. 197, Springer Verlag, 2000.
[2] P. Griffiths, J. Harris: *Principles of Algebraic Geometry*, John Wiley, 1978.
[3] L.I. Nicolaescu: *Hodge numbers of projective hypersurfaces*, Notre Dame, 2005.

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