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ON THE REIDEMEISTER TORSION OF RATIONAL HOMOLOGY SPHERES

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Abstract. We prove that the $\bmod \mathbb{Z}$ reduction of the Reidemeister torsion of a rational homology 3-sphere is naturally a $\mathbb{Q}/\mathbb{Z}$-valued quadratic function uniquely determined by a $\mathbb{Q}/\mathbb{Z}$-constant and the linking form.

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1. Introduction. Recently, V. Turaev has proved in [3, Theorem 4.3.1] a certain identity involving the Reidemeister torsion of a rational homology sphere M. In this paper, we suitably interpret this identity as a second-order finite difference equation satisfied by the torsion. Roughly speaking this identity states that the finite difference Hessian of the torsion coincides with the linking form of M. This allows us to prove a general structure result for the $\bmod \mathbb{Z}$ reduction of the torsion. More precisely, in Proposition 3.3 we prove that the $\bmod \mathbb{Z}$ reduction of the torsion is completely determined by three data.

- a certain canonical spin^c-structure σ_0,
- the linking form $\mathbf{lk}$ of M,
- a constant $c \in \mathbb{Q}/\mathbb{Z}$.

By fixing the spin^c-structure σ_0, we have a natural choice of Euler structure and thus, we can identify the Reidemeister torsion with a $\mathbb{Q}$-valued function on $H := H_1(M,\mathbb{Z})$. Its $\bmod \mathbb{Z}$ reduction is a function $\tau : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ of the form

$$\tau(h) = c - \widehat{\mathbf{lk}}(h), \tag{1.1}$$

where $\widehat{\mathbf{lk}}$ denotes a *quadratic form* on H such that

$$\widehat{\mathbf{lk}}(h_1+h_2) - \widehat{\mathbf{lk}}(h_1) - \widehat{\mathbf{lk}}(h_2) = \mathbf{lk}(h_1,h_2). \tag{1.2}$$

As a consequence, the constant c is a $\mathbb{Q}/\mathbb{Z}$-valued invariant of the rational homology sphere. Experimentations with lens spaces suggest this invariant is as powerful as the torsion itself.

2. The Reidemeister torsion. We review briefly a few basic facts about the Reidemeister torsion of a rational homology 3-sphere. For more details and examples we refer to [1, 3].

Suppose that M is a rational homology sphere. We set $H := H_1(M,\mathbb{Z})$ and use the multiplicative notation to denote the group operation on H. To remove the sign ambiguities in the definition of torsion, we equip $H_*(M,\mathbb{R})$ with the canonical orientation described in [3].

Denote by $\mathrm{Spin}^c(M)$ the set of isomorphism classes of spin^c-structure on M. It is an H-torsor, that is, the group H acts freely and transitively on $\mathrm{Spin}^c(M)$,

$$H\times \mathrm{Spin}^c(M) \ni (h,\sigma) \longmapsto h\cdot\sigma \in \mathrm{Spin}^c(M). \tag{2.1}$$

We denote by $\mathscr{F}_M$ the space of functions

$$\phi : H \longrightarrow \mathbb{Q}. \tag{2.2}$$

The group H acts on $\mathscr{F}_M$ by

$$H\times \mathscr{F}_M \ni (g,\phi) \longmapsto g\cdot\phi, \tag{2.3}$$

where

$$(g\cdot\phi)(h) = \phi(hg). \tag{2.4}$$

We denote by $\int_H$ the augmentation map

$$\mathscr{F}_M \longrightarrow \mathbb{Q}, \qquad \int_H \phi := \sum_{h\in H}\phi(h). \tag{2.5}$$

According to [3], the Reidemeister torsion is an H-equivariant map

$$\tau : \mathrm{Spin}^c(M) \longrightarrow \mathscr{F}_M, \quad \mathrm{Spin}^c(M)\ni\sigma \longmapsto \tau_\sigma = \tau_{M,\sigma}\in\mathscr{F}_M \tag{2.6}$$

such that

$$\int_H \tau_\sigma = 0. \tag{2.7}$$

In particular, if M is an integral homology sphere we have $\tau_{M,\sigma} = 0$. Denote by $\mathbf{lk}_M$ the linking form of M,

$$\mathbf{lk}_M : H\times H \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{2.8}$$

V. Turaev has proved in [3] that τ_σ satisfies the identity

$$\tau_\sigma(g_1g_2) - \tau_\sigma(g_1) - \tau_\sigma(g_2) + \tau_\sigma(1) = -\mathbf{lk}_M(g_1,g_2) \mod \mathbb{Z} \tag{2.9}$$

for all $g_1,g_2\in H,\ \sigma\in\mathrm{Spin}^c(M)$. In the above identity, we replace σ by $h\cdot\sigma$ for an arbitrary $h\in H$ and using the H-equivariance of $\sigma \mapsto \tau_\sigma$, we deduce

$$\tau_\sigma(g_1g_2h) - \tau_\sigma(g_1h) - \tau_\sigma(g_2h) + \tau_\sigma(h) = -\mathbf{lk}_M(g_1,g_2) \mod \mathbb{Z} \tag{2.10}$$

for all $g_1,g_2,h\in H,\ \sigma\in\mathrm{Spin}^c(M)$.

3. A second-order differential equation. The identity (2.10) admits a more suggestive interpretation. To describe it, we need a few more notation.

Denote by $\mathcal{S}_M$ the space of functions $H \longrightarrow \mathbb{Q}/\mathbb{Z}$. Each $g \in H$ defines a first-order differential operator

$$\Delta_g : \mathcal{S}_M \longrightarrow \mathcal{S}_M, \quad (\Delta_g u)(h) := u(gh) - u(h), \quad \forall u \in \mathcal{S}_M,\ h \in H. \tag{3.1}$$

If $\Xi = \Xi_\sigma$ denotes the mod $\mathbb{Z}$ reduction of τ_σ, then we can rewrite (2.10) as

$$(\Delta_{g_1}\Delta_{g_2}\Xi)(h) = -\mathbf{lk}_M(g_1, g_2). \tag{3.2}$$

Note that the second-order differential operator $\Delta_{g_1}\Delta_{g_2}$ can be regarded as a sort of Hessian.

We prove uniqueness and existence results for this equation. We begin with the (almost) uniqueness part.

Lemma 3.1. *The second-order linear differential equation (3.2) determines Ξ up to an "affine" function, that is, the sum between a character of H and a $\mathbb{Q}/\mathbb{Z}$-constant.*

Proof. Suppose that Ξ_1, Ξ_2 are two solutions of the above equation. Set $\Psi := \Xi_1 - \Xi_2$, Ψ satisfies the equation

$$\Delta_{g_1}\Delta_{g_2}\Psi = 0. \tag{3.3}$$

Now, observe that any function $F \in \mathcal{S}_M$ satisfying the second-order equation

$$\Delta_u\Delta_v F = 0, \quad \forall u, v \in H \tag{3.4}$$

is affine, that is, it has the form

$$F = c + \lambda, \tag{3.5}$$

where $c \in \mathbb{Q}/\mathbb{Z}$ is a constant and $\lambda : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a character. Indeed, the condition

$$\Delta_u(\Delta_v F) = 0, \quad \forall u \tag{3.6}$$

implies $\Delta_v F$ is a constant depending on v, $c(v)$. Thus

$$F(vh) - F(h) = c(v), \quad \forall h. \tag{3.7}$$

The function $\lambda = F - F(1)$ satisfies the same differential equation

$$\lambda(vh) - \lambda(h) = c(v) \tag{3.8}$$

and the additional condition $\lambda(1) = 0$. If we set $h = 1$ in the above equation, we deduce

$$\lambda(v) = c(v). \tag{3.9}$$

Hence,

$$\lambda(vh) = \lambda(h) + \lambda(v), \quad \forall v, h \tag{3.10}$$

so that λ is a character of H and $F = F(1) + \lambda$. Thus, the differential equation (3.2) determines Ξ up to a constant and a character. □

LEMMA 3.2. *Suppose that $b : H\times H \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a nonsingular, symmetric, bilinear form on H. Then there exists a quadratic form $q : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that*

$$\mathcal{H}q = b, \tag{3.11}$$

where

$$(\mathcal{H}q)(u,v) := q(uv) - q(u) - q(v). \tag{3.12}$$

PROOF. Let us briefly recall the terminology in this lemma. b is nonsingular if the induced map $H \longrightarrow H^\sharp := \mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ is an isomorphism. A quadratic form is a function $q : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that

$$q(1) = 0, \qquad q(u^k) = k^2 q(u), \quad \forall u \in H,\ k \in \mathbb{Z} \tag{3.13}$$

and $\mathcal{H}q$ is a bilinear form.

Suppose that b is a nonsingular, symmetric, bilinear form $H\times H \longrightarrow \mathbb{Q}/\mathbb{Z}$. Then, according to [4, Section 7], b admits a resolution. This is a nondegenerate, symmetric, bilinear form

$$B : \Lambda\times\Lambda \longrightarrow \mathbb{Z} \tag{3.14}$$

on a free abelian group Λ such that the induced monomorphism $J_B : \Lambda \longrightarrow \Lambda^* := \mathrm{Hom}(\Lambda,\mathbb{Z})$ is a resolution of H,

$$0 \hookrightarrow \Lambda \xrightarrow{J_B} \Lambda^* \overset{\pi}{\twoheadrightarrow} H \longrightarrow 0 \tag{3.15}$$

and b coincides with the induced bilinear form on $\Lambda^*/(J_B\Lambda)$ $(n := \#H)$,

$$b(\pi(u),\pi(v)) = \frac{1}{n^2} B(J_B^{-1}(nu), J_B^{-1}(nv)) \bmod \mathbb{Z}, \quad \forall u,v \in \Lambda^*. \tag{3.16}$$

Now, set

$$q(\pi(u)) = \frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) \bmod \mathbb{Z}. \tag{3.17}$$

This quantity is well defined, that is,

$$\frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) = \frac{1}{2n^2} B(J_B^{-1}(nv), J_B^{-1}(nv)) \bmod \mathbb{Z} \tag{3.18}$$

if $v = u + J_B\lambda$, $\lambda \in \Lambda$. Clearly, $\mathcal{H}q = b$. □

Denote by Q the space of solutions of the equation (3.11), that is, the space of quadratic forms q on H satisfying $\mathfrak{H}q = -\mathbf{lk}_M$. Q consists of more than one element. It is a G-torsor, where $G = \mathrm{Hom}(H,\mathbb{Z}_2)$ and the G action is given by

$$(Q\times G) \ni (q,\mu) \longmapsto q+\mu. \tag{3.19}$$

Using the linking form on M we can identify G with the 2-torsion subgroup of H. Denote by Ξ_σ the reduction mod $\mathbb{Z}$ of τ_σ.

Fix a spin^c structure σ_0 on M. We deduce that for every $q \in Q$ there exists a constant $k = k(q)$ and a character $\lambda = \lambda_q$ of H

$$\Xi_{\sigma_0}(h) = k(q) + \lambda_q(h) + q(h), \qquad \mathfrak{H}q = -\mathbf{lk}_M. \tag{3.20}$$

In particular,

$$\begin{aligned}\Xi_{g\cdot\sigma_0}(h) &:= \Xi_\sigma(gh) = k(q)+\lambda_q(gh)+q(gh)\\ &= \underbrace{\big(k(q)+\lambda_q(g)+q(g)\big)}_{c(g,q)} + \underbrace{\big(\lambda_q(h)+(\mathfrak{H}q)(g,h)\big)}_{\lambda_{q,g}(h)} + q(h)\end{aligned} \tag{3.21}$$

where $\lambda_{q,g}(\bullet)=\lambda_q(\bullet)-\mathbf{lk}_M(g,\bullet)$. Since the linking form is nondegenerate we can find a *unique* $g=g(q)$ such that $\lambda_{q,g}=0$. We set $\vec{\sigma}(q)=g(q)\cdot\sigma_0$ and $c(q)=c(g(q),q)$. The above computation also shows that for every $\mu\in G$ we have

$$c(q+\mu)-c(q)=q(\mu), \qquad \vec{\sigma}(q+\mu)=\mu\cdot\vec{\sigma}(q). \tag{3.22}$$

We have thus proved the following result.

Proposition 3.3. *Suppose M is a rational homology sphere. Then there exist functions*

$$c: Q \longrightarrow \mathbb{Q}/\mathbb{Z}, \qquad \vec{\sigma}: Q \longrightarrow \mathrm{Spin}^c(M) \tag{3.23}$$

so that

$$\tau_{\vec{\sigma}(q)}(h) := q(h)+c(q) \mod \mathbb{Z}, \quad \forall h\in H. \tag{3.24}$$

Moreover,

$$c(q+\mu)-c(q)=q(\mu), \quad \vec{\sigma}(q+\mu)=\mu\cdot\vec{\sigma}(q), \quad \forall\mu\in G. \tag{3.25}$$

Remark 3.4. (a) Note that $q(\mu)\in(1/4)\mathbb{Z}$, $\forall q\in Q$, $\mu\in\mathbb{Z}$ so that $4c(q)$ is *independent* of q. It is a topological invariant of M!

(b) One can show that the image of the one-to-one map $\vec{\sigma}$ is $\mathrm{Spin}(M)$, the set spin^c structures induced by the spin structures on M. We can thus regard c as a map $c: \mathrm{Spin}(M) \longrightarrow \mathbb{Q}/\mathbb{Z}$.

4. Examples. We want to show on some simple examples that the invariant c is nontrivial. First, we need some notation.

We denote by $\mathbb{Z}_n$ the cyclic group with n elements. The functions $f:\mathbb{Z}_n \longrightarrow \mathbb{Q}$ can be conveniently described as polynomials $f\in\mathbb{Q}[x]$, where $x^n=1$. Given two such polynomials f,g, we define the equivalence relation $\sim$ by

$$f\sim g \iff \exists m\in\mathbb{Z}: f=\pm x^m g. \tag{4.1}$$

We will not keep track of Euler structures and/or homology orientations and that is why in the sequel only the $\sim$-equivalence class of the torsion will be well defined. In particular, the constant c constructed in the previous section will be defined only up to a sign.

(a) Suppose that $M=L(8,3)$. Then its torsion is (see [2])

$$T_{8,3}\sim -\frac{9}{32}x^7-\frac{3}{32}x^6-\frac{9}{32}x^5+\frac{5}{32}x^4+\frac{7}{32}x^3-\frac{3}{32}x^2+\frac{7}{32}x+\frac{5}{32}, \tag{4.2}$$

where $x^8=1$ is a generator of $\mathbb{Z}_8$. Then

$$q(x^n)=\frac{-3n^2}{16}. \tag{4.3}$$

The set of possible values $(-3m^2/16) \mod \mathbb{Z}$ is

$$A := \Big\{0, \frac{-3}{16}, \frac{4}{16}, \frac{5}{16}\Big\}. \tag{4.4}$$

The set of possible values of $\Xi(h)$ is

$$B := \Big\{-\frac{9}{32}, -\frac{3}{32}, \frac{5}{32}, \frac{7}{32}\Big\}. \tag{4.5}$$

We need to find a constant $c \in \mathbb{Q}/\mathbb{Z}$ such that

$$B \pm c = A. \tag{4.6}$$

Equivalently, we need to figure out orderings $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ of A and B such that $b_i - a_i \mod \mathbb{Z}$ is a constant independent of i. A little trial and error shows that

$$\vec{A} = \Big(0, -\frac{3}{16}, \frac{4}{16}, \frac{5}{16}\Big), \qquad \vec{B} = \Big(-\frac{3}{32}, -\frac{9}{32}, \frac{5}{32}, \frac{7}{32}\Big) \tag{4.7}$$

and the constant $c = -3/32$. This is the coefficient of x^2. We deduce that (modulo $\mathbb{Z}$)

$$F := T_{8,3}(x) + \frac{3}{32} \sim -\frac{3}{16}x^7 - 0\cdot x^6 - \frac{3}{16}x^5 + \frac{1}{4}x^4 + \frac{1}{4}x^3 - 0\cdot x^2 + \frac{1}{4}x + \frac{1}{4}. \tag{4.8}$$

The translation of F by x^{-2} is

$$x^{-2}\Big(T_{8,3} + \frac{3}{32}\Big) = \frac{1}{4}x^7 + \frac{1}{4}x^6 - \frac{3}{16}x^5 - \frac{3}{16}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x. \tag{4.9}$$

(b) Suppose that $M = L(7,2)$. Then, its torsion is (see [2])

$$T_{7,2} \sim -\frac{2}{7}x^6 + \frac{1}{7}x^5 + \frac{2}{7}x^3 + \frac{1}{7}x - \frac{2}{7}, \tag{4.10}$$

where $x^7 = 1$ is a generator of $\mathbb{Z}_7$. We see that in this form $T_{7,2}$ is symmetric, that is, the coefficient of x^k is equal to the coefficient of x^{6-k}. The constant c in this case must be the coefficient of the middle monomial x^3, which is $2/7$.

(c) Suppose that $M = L(7,1)$. Then

$$T_{7,1} \sim \frac{2}{7}x^6 + \frac{1}{7}x^5 - \frac{1}{7}x^4 - \frac{4}{7}x^3 - \frac{1}{7}x^2 + \frac{1}{7}x + \frac{2}{7}. \tag{4.11}$$

This is again a symmetric polynomial and the coefficient of the middle monomial is $-4/7$. We see that this invariant distinguishes the lens spaces $L(7,1)$ and $L(7,2)$. It is known that these two spaces are homotopic but nonhomeomorphic lens spaces. Thus, the invariant c distinguishes their homeomorphism types, just as the torsion does.

(d) For $M = L(9,2)$, we have

$$T_{9,2} \sim -\frac{10}{27}x^8 + \frac{2}{27}x^7 - \frac{1}{27}x^6 + \frac{8}{27}x^5 + \frac{2}{27}x^4 + \frac{8}{27}x^3 - \frac{1}{27}x^2 + \frac{2}{27}x - \frac{10}{27}. \tag{4.12}$$

Again, this is a symmetric function, that is, the coefficient of x^k is equal to the coefficient of x^{8-k}, $x^9 = 1$. The constant is the coefficient of x^4, which is $2/27$. We deduce that $\bmod \mathbb{Z}$, we have

$$T_{9,2} - \frac{2}{27} = -\frac{2}{3}x^8 - \frac{2}{9}x^7 - \frac{1}{3}x^6 - \frac{2}{9}x^7. \tag{4.13}$$

(e) Finally, when $M = L(9,7)$ we have

$$T_{9,7} \sim -\frac{8}{27}x^8 - \frac{2}{27}x^7 + \frac{10}{27}x^6 + \frac{1}{27}x^5 - \frac{2}{27}x^4 + \frac{1}{27}x^3 + \frac{10}{27}x^2 - \frac{2}{27}x - \frac{8}{27} \tag{4.14}$$

the polynomial is again symmetric so that the constant c is the coefficient of x^4 which is $-2/27$.

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