MATH 10550, EXAM 1 SOLUTIONS

1. If $f(2) = 5$, $f(3) = 2$, $f(4) = 5$, $g(2) = 6$, $g(3) = 2$ and $g(4) = 0$, find $(fg)(2) + (f \circ g)(3)$.
Solution. We evaluate each term to get:

$$(fg)(2) = f(2)g(2) = 5 * 6 = 30$$
$$(f \circ g)(3) = f(g(3)) = f(2) = 5$$

Thus

$$(fg)(2) + (f \circ g)(3) = 35.$$

2. Evaluate the following limit

$$\lim_{x \to -3} \frac{9 - x^2}{3 + x}.$$

Solution. $\lim_{x\to -3} \frac{9-x^2}{3+x} = \lim_{x\to -3} \frac{(3-x)(3+x)}{(3+x)} = \lim_{x\to -3}(3 - x) = 6$

3. For which value of the constant c is the function $f(x)$ continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} c^2x - c & x \leq 1 \\ cx - x & x > 1. \end{cases}$$

Solution. The partial functions of $f(x)$ are continuous for $x < 1$ and $x > 1$ because they are polynomials. To get $f(x)$ continuous on $(-\infty, \infty)$ we need

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x),$$

or at $x = 1$, $c^2x - c = cx - x$. This happens when $c^2 - c = c - 1$. Rearranging gives $0 = c^2 - 2c + 1 = (c - 1)^2$ and $c = 1$.

4. For $f(x) = \sqrt[3]{x^5} + \frac{6}{\sqrt[5]{x^3}}$, find $f'(x)$.
Solution. $f(x) = x^{5/3} + 6x^{-3/5}$. Thus

$$f'(x) = \frac{5}{3}x^{2/3} + 6\left(-\frac{3}{5}x^{-8/5}\right) = \frac{5\sqrt[3]{x^2}}{3} - \frac{18}{5\sqrt[5]{x^8}}.$$

5. Find the equation of the tangent line to $y = \sqrt{x^2 - 1}$ at the point $(2, \sqrt{3})$.
Solution. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2-1}}$ by the chain rule.

Then at $(2, \sqrt{3})$ the derivative of y is $\frac{2}{\sqrt{3}}$.

This gives us the slope of the tangent line, which will be of the form $y = mx + b$. Plugging in $2/\sqrt{3}$ for m and $(2, \sqrt{3})$ for (x, y) and then solving for b we find $b = \frac{-1}{\sqrt{3}}$. So our answer is $y = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}$.

6. If $\sin(\pi xy) = \pi(x+y)$ find $\frac{dy}{dx}$ at $(1, -1)$ by implicit differentiation.
Solution. We differentiate both sides with respect to x. We use the chain and product rules.

$$\begin{aligned} \frac{d}{dx}(\sin(\pi xy)) &= \frac{d}{dx}(\pi(x+y)) \\ \Longrightarrow \cos(\pi xy)(\pi x\frac{dy}{dx} + \pi y) &= \pi + \pi\frac{dy}{dx} \\ \Longrightarrow \not{\pi}\cos(\pi xy)(x\frac{dy}{dx} + y) &= \not{\pi}(1 + \frac{dy}{dx}) \end{aligned}$$

Next we solve for $\frac{dy}{dx}$ by rearranging:

$$\begin{aligned} \frac{dy}{dx}\left(x\cos(\pi xy) - 1\right) &= 1 - y\cos(\pi xy) \\ \Longrightarrow \frac{dy}{dx} &= \frac{1 - y\cos(\pi xy)}{x\cos(\pi xy) - 1} \end{aligned}$$

Plugging in $x = 1$ and $y = -1$ we obtain

$$\frac{dy}{dx}(1, -1) = 0.$$

7. Find the derivative of

$$f(x) = x^2\cos(\sqrt{x^3 - 1} + 2).$$

Solution.

$$\begin{aligned} f'(x) &= 2x\cos(\sqrt{x^3-1}+2) + x^2\frac{d}{dx}\cos(\sqrt{x^3-1}+2) \text{ (Product Rule)} \\ &= 2x\cos(\sqrt{x^3-1}+2) - x^2\sin(\sqrt{x^3-1}+2)\frac{d}{dx}(\sqrt{x^3-1}+2) \text{ (Chain Rule)} \\ &= 2x\cos(\sqrt{x^3-1}+2) - \frac{x^2}{2\sqrt{x^3-1}}\sin(\sqrt{x^3-1}+2)\frac{d}{dx}(x^3-1) \text{ (Chain Rule)} \\ &= 2x\cos(\sqrt{x^3-1}+2) - \frac{3x^4}{2\sqrt{x^3-1}}\sin(\sqrt{x^3-1}+2). \end{aligned}$$

8. If $f(x) = x^2\cos x + \sin x$, find $f''(x)$.

Solution. Using Chain Rule, we get

$$\begin{aligned} f'(x) &= 2x\cos x - x^2\sin x + \cos x, \\ \text{and} \quad f''(x) &= 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x - \sin x \\ &= 2\cos x - 4x\sin x - x^2\cos x - \sin x. \end{aligned}$$

9. A ball is thrown straight upward from the ground with the initial velocity $v_0 = 96$ft/s. Find the highest point reached by the ball. Hint: The height of the ball at time t is given by $y(t) = -16t^2 + 96t$.
Solution. Velocity of the ball at time t is given by

$$v(t) = y'(t) = -32t + 96.$$

The ball reaches the highest point when $v(t) = 0$, i.e. when $t = 3$ seconds, so the height of the ball at 3 seconds is

$$\begin{aligned} y(3) &= -16(3)^2 + 96(3) \text{ ft.} \\ &= -144 + 288 \text{ ft.} \\ &= 144 \text{ ft.} \end{aligned}$$

10. Find the limit

$$\lim_{x\to 0} \frac{1-\cos x}{x\tan x}.$$

Solution.

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos x}{x\tan x} &= \lim_{x\to 0} \frac{1-\cos x}{x\tan x} \cdot \frac{1+\cos x}{1+\cos x} \\ &= \lim_{x\to 0} \frac{1-\cos^2 x}{x\tan x \cdot (1+\cos x)} \\ &= \lim_{x\to 0} \frac{\sin^2 x}{x\frac{\sin x}{\cos x} \cdot (1+\cos x)} \\ &= \lim_{x\to 0} \frac{\sin x\cos x}{x \cdot (1+\cos x)} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\cos x}{1+\cos x} \\ &= 1 \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

11. Find the equation of the tangent line to the curve $y = \frac{x^3}{3} - x^2 + 1$ which is parallel to the line $y + x = 2$.

Solution. The line parallel to the line $y + x = 2$ will have the same slope, namely -1. So we need to find the point on the curve which has slope -1. $y' = x^2 - 2x$. We solve for x given $y' = -1$:

$$\begin{aligned} x^2 - 2x &= -1 \\ \implies (x-1)(x-1) &= 0 \\ \implies x &= 1 \end{aligned}$$

Plugging into the equation for the curve we see that $y = 1/3$ at this point. Setting $y = mx + b$, plugging in $(1, 1/3)$ for (x, y) and solving for b we see $b = 4/3$. So the equation of the line we are looking for is $y = -x + 4/3$.

12. Show that there are at least *two* roots of the equation

$$x^4 + 2x - 1 = 0.$$

Justify your answer and identify the theorem you use.

Solution. Let $f(x) = x^4 + 2x - 1$. Then $f(-2) = 11$, $f(0) = -1$ and $f(1) = 2$. Since $f(x)$ is a polynomial, f is continuous on the real line. We have $f(-2) > 0 > f(0)$. So, by the **Intermediate Value Theorem**, there exists a number c between -2 and 0 such that $f(c) = 0$. Similarly, there exists a number d between 0 and 1 such that $f(d) = 0$.

Note: The choices $x = -2, 0, 1$ are not the only possibilities.

13. Given

$$y = \frac{1}{x^2 + 1},$$

find y' using the **definition** of the derivative.

Solution.

$$
\begin{aligned}
\text{Let } f(x) &= \frac{1}{x^2+1}. \\
\text{Then } y' = f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{(x^2+1) - ((x+h)^2+1)}{((x+h)^2+1)\cdot(x^2+1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x}^2 + \cancel{1} - \cancel{x}^2 - 2xh - h^2 - \cancel{1}}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{-2x-h}{((x+h)^2+1)(x^2+1)} \\
&= \frac{-2x-0}{((x+0)^2+1)(x^2+1)} \\
&= -\frac{2x}{(x^2+1)^2}.
\end{aligned}
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