MATH 10550, EXAM 3 SOLUTIONS

1.Compute

$$\int_1^2 t\sqrt{t-1}\,dt$$

Solution

Let $u = t-1$. So $du = dt$ and $t = u+1$. We have $u = 0$ when $t = 1$ and $u = 1$ when $t = 2$. Therefore

$$\int_1^2 t\sqrt{t-1}\,dt = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 u^{3/2} + u^{1/2}\,du$$
$$=(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2})\Big|_0^1 = \frac{16}{15}.$$

2.Find the value of the following integral.

$$\int_{-1}^1 \sqrt{1+x^2}\sin x\,dx$$

Solution

Consider $\sin x$ is an odd function and $\sqrt{1+x^2}$ is an even function. So $\sqrt{1+x^2}\sin x$ is an odd function. Thus

$$\int_{-1}^1 \sqrt{1+x^2}\sin x\,dx = 0.$$

3.Evaluate the following limit.

$$\lim_{n\to\infty}\sum_{i=1}^n \sin\left(\frac{\pi i}{n}\right)\cdot\frac{1}{n}$$

(Hint: Identify this as a limit of a Riemann sum and compute the corresponding integral.)

Solution

Let $f(x) = \sin(\pi x)$, $a = 0$, and b=1. With n subintervals, we have

$$\Delta x = \frac{b-1}{n} = \frac{1}{n} \text{ and } x_i = \frac{i}{n}.$$

So

$$\int_0^1 \sin(\pi x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n \sin(\frac{\pi i}{n})\cdot\frac{1}{n}.$$

Thus

$$\lim_{n\to\infty}\sum_{i=1}^n \sin(\frac{\pi i}{n})\cdot\frac{1}{n} = \int_0^1 \sin(\pi x)\,dx$$
$$= -\frac{1}{\pi}\cos(\pi x)\Big|_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}.$$

4. A table of values for a function f is given below.

t	0	2	4	6
$f(t)$	1	2	4	5

Use 3 rectangles and left endpoints to estimate the value of the integral

$$\int_0^6 f(t)\,dt.$$

Solution

Let $a = 0$, $b = 6$, and $n = 3$. So

$$\Delta x = \frac{b-a}{n} = \frac{6-0}{3} = 2.$$

Using the values for the function f given above and x_i's are left endpoints , we have

$$\int_0^6 f(t)\,dt = \sum_1^3 f(x_i)\Delta x = f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 = 2(1+2+4) = 14.$$

5.Compute

$$\int_{-4}^0 \sqrt{16-x^2}\,dx.$$

Solution

The given integral represents the area of the top left quarter of the circle centered at the origin with radius 4. So

$$\int_{-4}^0 \sqrt{16-x^2}\,dx = \frac{1}{4}\pi(4^2) = 4\pi.$$

6. Compute
$$\int \frac{1+t^2}{t^2}\,dt.$$
Solution

$$\int \frac{1+t^2}{t^2}\,dt = \int t^{-2}+1\,dt = \frac{t^{-1}}{-1}+t+C = \frac{-1}{t}+t+C = \frac{t^2-1}{t}+C.$$

7. Find an expression for the area bounded by the curves $y = 4 - x^2$ and $y = 2 - x$.
Solution
Setting $4 - x^2 = 2 - x$, we will get $0 = x^2 - x - 2 = (x-2)(x+1)$. By solving the two equations, we find that the points of intersection are (-1,3) and (2,0). So the area bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ is
$$\int_{-1}^{2} (4-x^2)-(2-x)\,dx = \int_{-1}^{2} 2+x-x^2\,dx.$$

8. Solving the equation $x^3 + x^2 - 1 = 0$ using Newton's method with initial guess $x_1 = 1$, what is x_2?
Solution
The equation for Newton's method is
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
Compute $f'(x) = 3x^2 + 2x$. So
$$x_2 = x_1 - \frac{x_1^3+x_1^2-1}{3x_1^2+2x_1} = 1 - \frac{1+1-1}{3+2} = 1 - \frac{1}{5} = \frac{4}{5}.$$

9. Let
$$f(x) = \int_{x^2}^{2} \frac{1}{1+t^2}\,dt.$$
Find $f'(x)$.

Solution

$$f(x) = \int_{x^2}^{2} \frac{1}{1+t^2}\, dt = -\int_{2}^{x^2} \frac{1}{1+t^2}\, dt.$$

Let $u = x^2$. Using the fundamental theorem of Calculus and the chain rule, we have

$$\begin{aligned} f'(x) = \frac{d}{dx}\int_{x^2}^{2} \frac{1}{1+t^2}\, dt &= -\frac{d}{dx}\int_{2}^{x^2} \frac{1}{1+t^2}\, dt \\ &= -\frac{d}{du}\left(\int_{2}^{u} \frac{1}{1+t^2}\, dt\right)\frac{du}{dx} \\ &= -\frac{1}{1+u^2}\frac{du}{dx} \\ &= -\frac{1}{1+x^4}\cdot 2x = -\frac{2x}{1+x^4}. \end{aligned}$$

10. Find an expression for the volume obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $y = x - 2$ and $y = 0$ about the x-axis.

Solution

The line $y = x - 2$ intersects $x-$axis(or the line $y = 0$) at the point $(2, 0)$. By solving $x-2 = \sqrt{x}$, we get $x = 4$. So the point of intersection of the graph $y = x - 2$ and $y = \sqrt{x}$ is $(4, 2)$. So the volume obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $y = x - 2$ and $y = 0$ about the x-axis is

$$\begin{aligned} &\pi \int_0^2 \sqrt{x}^2\, dx + \pi \int_2^4 (\sqrt{x}^2) - ((x-2)^2)\, dx \\ =&\pi \int_0^2 x\, dx + \pi \int_2^4 (x - (x-2)^2)\, dx. \end{aligned}$$

11. The velocity of a particle (in meters per second) is given by

$$v(t) = 12 + 6t - 6t^2.$$

(a) What is the **displacement** of the particle over the interval $[0, 3]$?
(b) What is the **distance** traveled by the particle over the interval $[0, 3]$?

Solution
The displacement of the particle over the interval $[0,3]$ is

$$\int_0^3 v(t)\,dt = \int_0^3 12+6t-6t^2\,dt$$
$$=(12t+3t^2-2t^3)\Big|_0^3 = 36+27-54 = 9.$$

The distance traveled by the particle over the interval $[0,3]$ is $\int_0^3 |v(t)|\,dt$.

$$v(t) = 12+6t-6t^2 = -6(-2-t+t^2) = -6(t-2)(t+1).$$

So $v(t) \geq 0$ for $0 \leq t \leq 2$ and $v(t) \leq 0$ for $2 \leq t \leq 3$. Thus

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^2 v(t)\,dt + \int_2^3 -v(t)\,dt \\ &= \int_0^2 12+6t-6t^2\,dt + \int_2^3 -12-6t+6t^2\,dt \\ &= (12t+3t^2-2t^3)\Big|_0^2 + (-12t-3t^2+2t^3)\Big|_2^3 \\ &= 20+11 = 31.\end{aligned}$$

12. A graphic artist is designing a rectangular poster, which is to have margins of 2 inches at the top and along each side, and a 3-inch margin at the bottom. In order to save expenses, she wants the total area of the poster to be as small as possible, but the printed area (the part inside the margins) has to be 180 square inches. What dimensions of the poster will minimize the total area?

Solution
Assume that the printed area has sides of lengths x and y. So $xy = 180$, that is, $y = 180/x$. Together with the margins, the poster will have total area of

$$A(x) =(x+2+2)(\frac{180}{x}+2+3) = (x+4)(\frac{180}{x}+5)$$
$$=200+\frac{720}{x}+5x.$$

Setting $0 = A'(x) = -720x^{-2}+5$, we have $x = 12$. We can see that $A'(x) < 0$ if $0 < x < 12$ and $A'(x) > 0$ if $x > 12$. So $x = 12$ is the minimum value. When $x = 12$, $y = 180/12 = 15$. The dimensions of the poster are 16 inches and 20 inches.

13. Compute the volume obtained by rotating the region bounded by the curves $x = y^2$ and $y = x$ about $x = 1$.

Solution

The points of intersection of the graph $x = y^2$ and $y = x$ are $(0, 0)$ and $(1, 1)$. The line $x = y^2$ lies above the line $y = x$ when x is between 0 and 1. We want to use the disc method, and we want our discs to be horizontal. We'll have an outer radius and an inner radius. Because we are rotating about $x = 1$, the outer radius will be $1 - y^2$, and the inner radius will be $1 - y$. So the volume obtained by rotating the region bounded by the curves $x = y^2$ and $y = x$ about $x = 1$ is

$$\pi \int_0^1 ((1-y^2)^2 - (1-y)^2)dy = \pi \int_0^1 (y^4 - 3y^2 + 2y)dy$$

$$= \pi(y^5/5 - y^3 + y^2\big|_0^1) = \pi(1/5 - 1 + 1) = \pi/5.$$