MATH 10550, EXAM 2 SOLUTIONS

1. The critical numbers of $f(x) = \frac{1}{6}x^2 - x^{1/3}$ are

Solution. The critical numbers are all x-values in the domain of f at which $f'(x) = 0$ or $f'(x)$ does not exist. Note

$$f'(x) = \frac{1}{3}x - \frac{1}{3}x^{-2/3}.$$

Hence $f'(x)$ does not exist at $x = 0$. To make $f'(x) = 0$, we have

$$x = x^{-2/3} \quad \Leftrightarrow \quad x^{5/3} = 1 \quad \Leftrightarrow x = 1.$$

So the critical numbers are $x = 0, x = 1$.

2. Given $f'(x) = \frac{x^2-1}{(x^2+1)^3}$, which of the following statements is true? (Note: you are given f', not f.)

Solution. $f'(x) = 0 \iff x = \pm 1$. So we need to check for local maxima and minima at -1 and 1. Picking sample x-values in the corresponding intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ to test for signs we have $f(-2) > 0$, $f(0) < 0$, and $f(2) > 0$. Therefore, by the First Derivative Test, there is a local maximum at $x = -1$ and a local minimum at $x = 1$.

3. Evaluate the following limit

$$\lim_{x\to-\infty} \frac{\sqrt{9x^2 + 7x^{1/3}}}{8x + 7}.$$

Solution.

$$\lim_{x\to-\infty} \frac{\sqrt{9x^2 + 7x^{\frac{1}{3}}}}{8x+7} = \lim_{x\to-\infty} \frac{3|x|\sqrt{1 + \frac{7x^{\frac{1}{3}}}{9x^2}}}{8x+7}$$

$$= \lim_{x\to-\infty} \frac{-3x\sqrt{1 + \frac{7x^{\frac{1}{3}}}{9x^2}}}{8x+7} = -\frac{3}{8}$$

4. Evaluate

$$\lim_{x\to-\infty} \frac{3x^8 + 23x^2 + 131}{9x^9 + 12\sin x + 74}$$

Solution. The sine function is annoying, but we know $-1 \leq \sin(x) \leq 1$, so it won't give us too much trouble. Note

$$\frac{3x^8 + 23x^2 + 131}{9x^9 + 12\sin x + 74} = \frac{1}{x} \cdot \frac{3 + \frac{23}{x^6} + \frac{131}{x^8}}{9 + 12\frac{\sin x}{x^9} + \frac{74}{x^9}}.$$

Hence the limit is zero as $x \to -\infty$.

5. The number of inflection points of $f(x) = x^4 + x^3 + x^2$ is
Solution. If x is an inflection point, $f''(x) = 0$.

$$f''(x) = 12x^2 + 6x + 2$$

Note $6^2 - 4 \cdot 12 \cdot 2 = 36 - 96 < 0$. Hence $f''(x) = 0$ has no real solutions. Hence there are no inflection points.

6. Consider the function $f(x) = x + \frac{1}{x}$, defined for $0 < x \leq 10$. Which statement below is correct?
Solution. Observe $f'(x) = 1 - x^{-2}$ Also $f'(x) = 0$ if and only if $x^{-2} = 1$ which happens only at $x = 1$ on the domain. $x = 1$ is the only critical number on the domain.

7. Suppose $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. If $f(0) = 0$ and $0 \leq f'(x) \leq \frac{1}{2}$, what is the largest $f(x)$ can be when $x = 1$?
Solution. By the mean value theorem, we have for some c in $[0, 1]$

$$f(1) - f(0) = f'(c) \cdot (1 - 0),$$

and hence

$$f(1) = f'(c).$$

To find the largest value for $f(1)$, we choose the largest possible value for $f'(c)$, which is $1/2$. Hence the largest $f(1)$ is $1/2$.

8. The linearization $L(x)$ of the function $f(x) = (x + 1)\cos x$ at $x = 0$ is
Solution. Note $f'(x) = -(x + 1)\sin x + \cos x$. Hence $f(0) = 1$ and $f'(0) = 1$. So the linearization of $f(x)$ at $x = 0$ is

$$L(x) = f(0) + f'(0)(x - 0) = 1 + x.$$

9. Using the linear approximation to estimate $\sin \frac{\pi}{100}$ one obtains the value

Solution. Let $f(x) = \sin x$. Then $f'(x) = \cos x$, and hence $f(0) = 0$ and $f'(0) = 1$. The linear approximation of $\sin(x)$ at $x = 0$ is

$$L(x) = f(0) + f'(0)(x - 0) = x.$$

The corresponding linear approximation is $L(\frac{\pi}{100}) = \frac{\pi}{100}$.

10. The largest interval where $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 10$ is **both** increasing and concave upward is

Solution. Note

$$f'(x) = x^2 - 3x + 2 = (x - 1)(x - 2),$$
$$f''(x) = 2x - 3.$$

Hence

interval	$(-\infty, 1)$	$(1, 3/2)$	$(3/2, 2)$	$(2, \infty)$
f'	$+$	$-$	$-$	$+$
f''	$-$	$-$	$+$	$+$

So the largest interval where $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 10$ is both increasing and concave upward is $(2, \infty)$.

11. Show that there is precisely *one* (real) root of the equation

$$x^7 + 2x - 1 = 0.$$

Justify your answer and identify the theorem(s) you use.

Solution. (We will use the Intermediate Value Theorem to show there exists at least one solution and Rolle's Theorem to show there exists at most one solution. Together they suffice to show there exists *exactly* one solution.) Let $f(x) = x^7 + 2x - 1$.

Note that $f(0) = -1$ and $f(1) = 1$. Since $f(0) < 0 < f(1)$ there exists at least one solution in the interval $(0, 1)$ by the Intermediate Value Theorem.

$f'(x) = 7x^6 + 2 \geq 2$, for all x-values. If there were more than one solution then according to Rolle's Theorem there is some x such that $f'(x) = 0$, but that contradicts the fact that $f'(x) \geq 2$ everywhere. So there exists at most one solution.

We conclude that there exists exactly one solution to $f(x) = 0$.

12. Find the *maximum* value of $f(x) = 2x^3 - 9x^2 + 12x + 1$ on $[-3, 3]$.

Solution. Note

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2).$$

The critical numbers of f are $x = 1$ and $x = 2$. Compute $f(-3) = -170$, $f(1) = 6$, $f(2) = 5$, and $f(3) = 10$. So the maximum value is of f on $[-3, 3]$ is 10.

13. An eccentric mathematician enjoys inflating and releasing *cube-shaped* balloons. Assume that gas is being pumped into the balloon at the rate of three cubic feet per second. When the edge is one foot long, what is the rate of change of the *surface area* of the balloon?

Solution. Let r be the side length of the cube. It is a function of time t. The cube has volume

$$V = r^3.$$

A cube has six faces each with area r^2. Then the surface area is

$$S = 6r^2.$$

Taking derivatives, we have

$$V' = 3r^2r', \quad S' = 12rr'.$$

We know $V' = 3$. When $r = 1$, $r' = 1$. Hence

$$S' = 12rr' = 12 \cdot 1 \cdot 1 = 12.$$

Hence the surface area changes at 12 square feet per second when the edge is one foot long.