RESEARCH DESCRIPTION

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My main research interest is in algebraic topology and homotopy theory. I am currently working on the homotopy theory and Quillen homology of modules and algebras over operads in symmetric spectra and unbounded chain complexes. The goal is to determine the extra structure that appears on the derived homology and the extent to which the original object can be recovered from its homology when this extra structure is taken into account. Below I elaborate on these ideas, describe my results so far, and describe proposals for future research.

1. Background

There are many interesting situations in which algebraic structure can be described by operads [27, 32, 33, 34]. In many of these, there is a notion of abelianization or stabilization [1, 2, 13, 14, 15, 16, 17, 36, 37, 40, 42] which provides a notion of (derived) homology. In this context, homology is not just a graded collection of abelian groups, but a geometric object like a chain complex or spectrum, and distinct algebraic structures tend to have distinct notions of homology. For commutative algebras this is the cotangent complex appearing in Andre-Quillen homology, and for the empty algebraic structure on spaces this is a chain complex calculating the singular homology of spaces. In other words, this is homology in a derived context, and often algebraic structure appears on the homology objects themselves. The main question I address in my research is:

Question. *How much of an algebraic object X can be recovered from its derived homology plus extra structure?*

This is inspired by Mandell's result [29] that the p-completion of a space can be recovered from its mod-p cochains, provided the cochains themselves are equipped with enough extra structure; in this case, a highly structured homotopy commutative algebra (E-infinity algebra) structure [21, 29].

For the question above, I consider algebraic structures parametrized by operads acting on symmetric sequences of unbounded chain complexes and symmetric spectra [25]; such algebraic structures are called modules over an operad [26, 38], which includes the more familiar algebras over an operad [12, 27, 31, 34, 38] as a special case. Underlying every operad is a symmetric sequence [3, 11, 12, 26, 38, 45], and it turns out that symmetric sequences provide a useful setting for studying the derived homology of algebraic structures. In addition to the classical associative algebra, commutative algebra, and Lie algebra structures, operads can describe in a useful manner various highly structured homotopy versions of these, as naturally appear for example in the algebraic analogs of n-fold loop spaces and infinite loop spaces [2, 13, 16, 21, 23, 27, 31, 32, 33, 34]. In other words, operads arise because they act on many objects.

Even in the case of a simple algebraic structure such as commutative algebras, homology provides interesting invariants; in [35] Miller proves the Sullivan conjecture on maps from classifying spaces, and in his proof derived homology of commutative algebras [14, 15, 36, 37, 39] is a critical ingredient. This suggests that homology, for the larger class of algebraic structures parametrized by an action of an operad, will provide interesting and useful invariants.

2. Results so far

Homology is taken in the sense of Quillen: derived abelianization when working with unbounded chain complexes and derived indecomposables when working with symmetric spectra. I have established in [19, 20] a framework and corresponding homotopy theory for studying the derived homology of modules and algebras over an operad, thus providing a foundation for studying invariants of many interesting algebraic structures. In particular, I have results for symmetric spectra and rational results for unbounded chain complexes. In [18] I use the homotopy theory established in [19, 20] to show that the derived homology functors are well-defined and can be calculated as hocolim or realization of a simplicial bar construction. Below I elaborate on these results and describe their relationship with previous work.

The main theorem in [20] is this.

Theorem 2.1. *Let $\mathcal{O}$ be an operad in symmetric spectra. Then the category of left $\mathcal{O}$-modules and the category of $\mathcal{O}$-algebras both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable flat positive fibrations in symmetric spectra.*

Remark 2.2. For ease of notation purposes, I have followed Schwede [43] in using the term *flat* (e.g., stable flat model structure) for what is called S (e.g., stable S-model structure) in [25, 41, 44].

The theorem remains true when the stable flat positive model structure on symmetric spectra is replaced by the stable positive model structure, which has fewer cofibrations. The theorem in [20] is this.

Theorem 2.3. *Let $\mathcal{O}$ be an operad in symmetric spectra. Then the category of left $\mathcal{O}$-modules and the category of $\mathcal{O}$-algebras both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable positive fibrations in symmetric spectra.*

One of the main theorems of Shipley [44] is that the category of commutative monoids in symmetric spectra has a natural model structure inherited from the stable flat positive model structure on symmetric spectra. Theorem 2.1 improves this result to left modules and algebras over any operad $\mathcal{O}$ in symmetric spectra.

One of the main theorems of Elmendorf and Mandell [10] is that for symmetric spectra the category of algebras over any operad $\mathcal{O}$ in simplicial sets has a natural model structure inherited from the stable positive model structure on symmetric spectra. Theorem 2.3 improves this result to left modules and algebras over any operad $\mathcal{O}$ in symmetric spectra. Their proof involves a filtration in the underlying category of certain pushouts of algebras. We have benefitted from their paper and our proofs of Theorems 2.1 and 2.3 exploit similar filtrations.

The main theorem in [19], in the context of chain complexes, is this.

Theorem 2.4. *Let $\mathcal{O}$ be an operad in unbounded chain complexes over a field of characteristic zero. Then the category of left $\mathcal{O}$-modules and the category of $\mathcal{O}$-algebras both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the homology isomorphisms and the surjections in chain complexes.*

One of the main theorems of Hinich [22] is that for unbounded chain complexes over a field of characteristic zero, the category of algebras over any operad has a natural model category structure. Theorem 2.4 improves this result to the category of left modules, and also provides a simplified conceptual proof of Hinich's original result.

I prove in [18, 20] that a morphism of operads which is an objectwise weak equivalence induces an equivalence between the corresponding homotopy categories of modules (resp. algebras). The theorem is this.

Theorem 2.5. *Suppose $\mathcal{O}$ is an operad in symmetric spectra or unbounded chain complexes over a field of characteristic zero. Let $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) be the category of left $\mathcal{O}$-modules (resp. $\mathcal{O}$-algebras) with any of the above model structures. If $f:\mathcal{O}\longrightarrow\mathcal{O}'$ is a map of operads, then the adjunctions*

$$\mathsf{Lt}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}'}, \qquad \mathsf{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'}, \tag{2.6}$$

are Quillen adjunctions with left adjoints on top and f^* the forgetful functor. If furthermore, f is an objectwise weak equivalence, then the adjunctions (2.6) *are Quillen equivalences, and hence induce equivalences on the homotopy categories.*

One of the main theorems of Elmendorf and Mandell [10], in the context of symmetric spectra, is that a morphism of operads in simplicial sets which is an objectwise weak equivalence induces a Quillen equivalence between categories of algebras over operads. Theorem 2.5 improves this result to left modules and algebras over operads in symmetric spectra.

One of the main theorems of Hinich [22] is that for unbounded chain complexes over a field of characteristic zero, a morphism of operads which is an objectwise weak equivalence induces a Quillen equivalence between categories of algebras over operads. Theorem 2.5 improves this result to the category of left modules over operads.

Using the framework and corresponding homotopy theory established in [19, 20], I have shown in [18] that the desired derived homology functors are well-defined and can be calculated using simplicial bar constructions. The theorem is this.

Theorem 2.7.

(a) *For augmented operads $\mathcal{O}$ in symmetric spectra, the left derived "indecomposables" functor is well-defined on left $\mathcal{O}$-modules and $\mathcal{O}$-algebras, and can be calculated (modulo cofibrancy conditions) as hocolim or realization of a simplicial bar construction in the underlying category.*

(b) *For augmented operads $\mathcal{O}$ in unbounded chain complexes over a field of characteristic zero, the left derived "indecomposables" functor is well-defined on left $\mathcal{O}$-modules and $\mathcal{O}$-algebras, and can be calculated as hocolim or realization of a simplicial bar construction in the underlying category.*

One of the main theorems of Fresse [11] is that for positive chain complexes over a field of characteristic zero, and for left modules and operads which are trivial at zero (e.g., such modules do not specialize to algebras over operads), then under additional conditions, the left derived "indecomposables" functor is well-defined, and can be calculated as realization of a simplicial bar construction in the underlying category. Theorem 2.7 improves this result to unbounded chain complexes over a field of characteristic zero, to algebras and arbitrary left modules over operads, and also provides a simplified homotopical proof of Fresse's original result.

3. Proposed research

A basic idea is that some kind of Tot formula is expected for recovering the algebraic object from its homology plus extra structure. It is this idea that I propose to address as part of my future research.

The "indecomposables" functor fits into a Quillen pair, and hence the derived indecomposables functor fits into an adjunction on the level of homotopy categories. In particular, this adjunction builds a cosimplicial object (from the associated triple) with a coaugmentation by X. This suggests that maybe X can be recovered as Tot of this cosimplicial object; but there is a problem – cosimplicial objects in the homotopy category may be difficult to work with.

I would like to exploit the fact that having a Quillen pair of functors (F, G) between model categories is a lot stronger than simply knowing the functors induce an adjunction on the level of homotopy categories. This extra information is reflected in the fact that the Dwyer-Kan homotopy function complexes $\mathrm{Map}(\mathbf{L}F(X), Y)$ and $\mathrm{Map}(X, \mathbf{R}G(Y))$ are weakly equivalent as spaces [6, 7, 8, 24]. In this context, working with the adjunction on the level of homotopy categories amounts to working only with the path components of these weakly equivalent homotopy function complexes.

Proposal 1. *I would like to show that by using the extra information of a Quillen pair, the above cosimplicial object has a more or less canonical rigidification.*

Proposal 2. *I would like to show that in the case of the indecomposables functor, the coaugmentation into* Tot *of the corresponding rigidification is a weak equivalence, at least under not too restrictive assumptions.*

My proposed activity for developing a rigidification of the homotopy cosimplicial object will exploit the extra information encoded by the (hammock) simplicial localization of a model category [5, 6, 7, 8, 28], which can be understood as capturing the higher order information implicit in the homotopy theory [9, 37] but which the homotopy category itself fails to see. These simplicial localizations can be very useful, and have recently been exploited in [4] for obtaining an interesting decomposition of the K-theory space of a Waldhausen category, and in [30] for proving a neat homotopy classification result. The proposed concept of a rigidification is independent of homology objects and operads, and will itself be useful in a broad range of homotopy applications.

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