of $G(O(F) \cup I(F))$. It is known (see Edmonds and Fulkerson [65])
that the independence system $(O(F) \cup I(F), M')$ so defined is a
matroid, the so-called matching matroid.

Note the following relationship between M and M'. $S \in M$
if and only if $S = X \cup Y$ where $X \in M'$ and $Y \subseteq R(F)$, as a
consequence of Theorem 4.1. Since (V,M) is the direct sum of a
matching matroid, namely $(O(F) \cup I(F), M')$ and a complete matroid,
namely $(R(F), 2^{R(F)})$, it is itself a matroid.

Let $\omega : V \to R^+$ be a vector of nonnegative weights defined
on the node set of G. The <u>weight</u> of a hypomatching J is
defined as $\omega(J) = \Sigma \{\omega_i : $ node i is covered by $J\}$. Consider the
problem of finding a maximum weight hypomatching in G. Since
(V,M) is a matroid, a maximum weight hypomatching can be found
by the following <u>greedy algorithm</u>.

Order the nodes by nonincreasing weights $\omega_1 \geq \ldots \geq \omega_n$.
Start with $S^0 = \emptyset$ and $J^0 = \emptyset$. Then n iterations are performed,
say iterations $i = 1, \ldots, n$. At the beginning of iteration i, the
set S^{i-1} is a maximum weight independent subset of $\{1, \ldots, i-1\}$
and J^{i-1} is a hypomatching covering S^{i-1}. Iteration i con-
sists of either proving that $S^{i-1} \cup \{i\}$ is not independent, or
setting $S^i = S^{i-1} \cup \{i\}$ and modifying J^{i-1} (if necessary) into a
hypomatching J^i which covers the set S^i. The algorithm
terminates when $i = n$. The hypomatching J^n is a maximum weight
hypomatching in G. Its weight is

$$\omega(J^n) = \sum_{i \in S^n} \omega_i .$$

Next, we show how iteration i of this greedy algorithm can be performed by a variation of the Edmonds matching algorithm. More generally, let S be an independent set in (V,M) and J a hypomatching covering S. The next algorithm will check whether S ∪ {i} is independent, where i ∈ V − S is given, and if so, modify J so that it covers S ∪ {i}.

First, if i is covered by J we can stop immediately and conclude that S ∪ {i} is independent. Otherwise, we will construct a tree A with root i in an associated graph $\overline{G}$. Initially, $\overline{G} = G$ and the root i is the unique node of A and it is said to be an even node. Then A is grown according to the following procedure until either S ∪ {i} is found to be independent or A cannot be grown any longer in which case we will show that S ∪ {i} is not independent.

Step 1. If every edge of $\overline{G}$ which is incident with an even node of A is also incident with an odd node of A, stop: The set S ∪ {i} is not independent (this claim will be proved later). Otherwise, let j be an edge which joins an even node of A, say u, to a node v which is not an odd node of A. If v is an even node of A, go to Step 2. If v is not in A but is covered by an edge k = (vw) of J such that w ∈ S, then go to Step 3. Finally, in the other cases where v is not in A, go to Step 4.

Step 2. Let $\overline{C} = \{u,...,v\}$ be the set of nodes in the unique path of the tree A joining nodes u and v,

and let C be the set of nodes of G associated
with the node set $\overline{C}$. C is hypomatchable.

If $C \subseteq S$ and G[C] is critical, modify A (and $\overline{G}$)
by shrinking $\overline{C}$ to a single node. This shrunk node
becomes an even node of A. Go to Step 1.

Otherwise, modify J by alternating the edges in
and out of J on the path of A from i to the
closest point in $\overline{C}$. If necessary, modify the near
perfect matchings inside the shrunk even nodes on this
path so that every node of G is in at most one member
of J. (This is always possible since the shrunk nodes
of A are hypomatchable.) In addition, if there exists
$w \in C - S$, J is modified in G[C] so as to contain a
near perfect matching of G[C] leaving w uncovered;
on the other hand, if $C \subseteq S$, then G[C] is not critical
and J is modified in G[C] so as to internally cover
the nodes of G[C]. This produces a hypomatching J'
which covers $S \cup \{i\}$. Stop.

Step 3. Grow the tree A by adding the edges j and k
and the nodes v and w to A. Node v is called an
odd node of A and w an even node. Go to Step 1.

Step 4. The node v is not in A and is either (i) not
covered by J, or (ii) covered by an edge (vw) of J such
that $w \notin S$ or (iii) covered by a hypomatchable set of J.
Let J' be obtained from J by interchanging in and out

of J the edges on the path of A joining i to v.
If necessary, modify the near-perfect matchings inside
the shrunk even nodes on this path. In addition, in
case (ii), remove the edge (vw); in case (iii) replace
the hypomatchable set T of J which covers v by a
near-perfect matching of T leaving only v uncovered.
Now, in all 3 cases, J' is a hypomatching which covers
$S \cup \{i\}$. So $S \cup \{i\}$ is independent. Stop.

Proof of the Validity of the Algorithm: It is clear that this
algorithm terminates since every time it goes back to Step 1
a new edge of G is considered. When the algorithm terminates
in Steps 2 or 4, the hypomatching J' proves that $S \cup \{i\}$ is
independent. So in order to prove the validity of the algorithm
it suffices to show that, when the algorithm terminates in Step 1,
the set $S \cup \{i\}$ is not independent. By construction of A, the
even nodes of A which are shrunk only contain nodes of S
(Step 2) and the other even nodes of A belong to S (Step 3).
Also by construction the tree A contains one more even node than
odd. Finally, when the algorithm terminates every edge incident
with an even node of A has an odd node of A as its other
endpoint. As a consequence of Lemma 4.2, no hypomatching of G
can cover all the nodes inside critical components of $G[V-I]$
where I is the set of odd nodes of A, since there are $|I|+1$
such critical components. Thus, no hypomatching covers all the
nodes of $S \cup \{i\}$, proving that this set is not independent.

We conclude with a generalization of Theorem 1.7 of Berge [57].
An <u>alternating path</u> relative to a hypomatching J is a path whose
edges are alternately in and out of J. An <u>augmenting path</u> is an
alternating path, one of whose endnodes is not covered by J
and whose other endnode u is either

   (a) not covered by J, or

   (b) in a hypomatchable set of J, or

   (c) in a noncritical hypomatchable graph G[C] such
       that the nodes of C − {u} are matched among
       themselves by J. In addition, the length of
       the alternating path must be even.

Note that in cases (a) and (b) the length of the alternating
path will always be odd.

<u>Theorem 4.9</u>: A hypomatching is maximum if and only if there
exists no augmenting path.

<u>Proof</u>: If (a), (b), or (c) occurs, the hypomatching J is not
maximum. Conversely assume that J is not maximum. Let S be
the set of nodes covered by J and let i be a node such that
S ∪ {i} can be covered. By the algorithm we will find an augmen-
tation. It occurs either in Step 2, providing an augmenting
path as stated in (c), or in Step 4, providing augmenting paths
(a) or (b). (Note that Step 4 case (ii) does not occur with
our choice of S.)

Section 4. An Algorithm for Maximum Cardinality Hypomatchings

We next give an algorithm for solving the maximum cardinality hypomatching problem. It is, again, an Edmond's style algorithm and is a variation of the greedy algorithm just given.

In the course of the algorithm we grow an <u>alternating forest</u>. The nodes of the alternating forest A may be of two types. A <u>real node</u> of A is simply a node of G. A <u>shrunk node</u> of A is a node-induced subgraph of G which is critical. The edges of A are edges of G and we consider an edge j to be incident with a shrunk node of A if exactly one end of j is in the shrunk node. Each tree in the forest is <u>rooted</u> at some node (which may be a real node or a shrunk node). A node in a tree of A is said to be <u>odd</u> (<u>even</u>) if the number of edges in the path of A to the root is odd (even). Odd nodes of A will always be real nodes and be incident with two edges of A. An alternating forest A is always defined relative to a packing $P \subseteq F$ (which will not be perfect). The roots of the trees of A are precisely the nodes which are not covered by P. In every path of A starting at a root the edges must be alternately out of P and in P.

Step 0 (Initialization)

Let $P \subseteq F$ be any packing (e.g., $P = \emptyset$).

<u>Step 1</u> (Optimality test)

If P covers every node, terminate. Otherwise let A consist of the nodes of G which are not covered by P. Thus, initially, A reduces to a set of even nodes, its roots.

<u>Step 2</u> (Edge selection)

Find, if one exists, an edge j joining an even node u of A to a node v which is not an odd node of A. If no such node exists, terminate. Otherwise 4 cases may occur.

<u>Case 1</u>: v is not a node of A and is incident with an edge k in P. Go to Step 3.

<u>Case 2</u>: v is not a node of A and is covered by a hypomatchable graph $\overline{H}$ of P. Go to Step 4.

<u>Case 3</u>: v is an even node of A in a different tree than u. Go to Step 5.

<u>Case 4</u>: v is an even node of A in the same tree as u. Go to Step 6.

<u>Step 3</u> (Forest growth)

Let w be the node incident with k which is different from v. Grow A by adding edges j and k and nodes v and w. Thus v becomes an odd node of A and w becomes an even node. Go to Step 2.

<u>Step 4</u> (Hypomatchable augmentation)

Remove $\overline{H}$ from P and replace it by the edges of a perfect matching of $\overline{H}[S - \{v\}]$, where S is the node set of H. Go to Step 5.

Step 5 (Simple augmentation)

Add j to P. In the path of A from u to the root, the edges which were in P are removed whereas those which were not in P are now included in P. Change P similarly in the path of A from v to its root (if Case 3 applies). After this, every shrunk node of A has exactly one edge of P incident with it. Since the shrunk nodes are hypomatchable, P can be modified appropriately inside each shrunk node to perfectly match it. We now "throw away" A and any shrunk node previously found and go to Step 1.

Step 6 (Augment or shrink)

Edge j added to A creates an odd cycle C. Let $\bar{H}$ be the subgraph of G induced by the real vertices of C and those inside shrunk nodes of C. Check whether $\bar{H}$ is critical. If it is, go to Step 6a. Otherwise go to Step 6b.

Step 6a (Shrinking)

Create a new shrunk node containing $\bar{H}$. This shrunk node becomes an even node of A. Go to Step 2.

Step 6b (Augmentation)

Find a perfect packing of $\bar{H}$. Complete the augmentation along the path of A from $\bar{H}$ to the root as in Step 5. "Throw away" A and any shrunk node and go to Step 1.

Remarks on the algorithm. (i) The algorithm cannot cycle since at most $|V|$ augmentations can occur and, between augmentations,

either the alternating forest is grown or an odd cycle is shrunk.

(ii) To prove the validity of the algorithm note that, at termination in Step 2, every edge incident with an even node of A is also incident with an odd node of A. Since the even nodes are P-critical, Lemma 4.2 implies that the perfect packing of some subset of them requires also the covering of at least as many odd nodes of A. In the current packing P the odd nodes are already all matched with even nodes, so P is maximum.

The complexity of this algorithm is polynomially equivalent to the complexity of Step 6 where we perform the following two operations:

(1) check whether $\bar{H}$ is critical and,

(2) if $\bar{H}$ is not critical, find a perfect hypomatching

of $\bar{H}$.

Lemma 4.6 is useful in carrying out these operations. For example, if F only contains the edges of G and a polynomial number of hypomatchable subgraphs, then operations (1) and (2) can be performed in polynomial time. Namely, let G[S] be a hypomatchable subgraph of G. If T is the node set of a graph in F and $G[S - T]$ has a perfect matching, then G[S] is not critical. Conversely, if $G[S - T]$ does not have a perfect matching for every graph in F, then G[S] is P-critical by Lemma 4.6.

Therefore

Theorem 4.10: If F consists of all the edges of G and a polynomial family of hypomatchable subgraphs, then the maximum cardinality hypomatching problem is polynomially solvable.

This shows that the problem of packing edges and triangles is polynomially solvable. There are other hypomatching problems, where the family of hypomatchable subgraphs in F is not polynomial, which can also be solved in polynomial time; for example, the maximum cardinality problem (Q_k') for any fixed k. (See Cornuejols and Pulleyblank [83].)

A related result which helps in characterizing the critical graphs for a hypomatching problem is the following, which is proved in Cornuejols, Hartvigsen, and Pulleyblank [82].

Theorem 4.11: Every critical graph is hypomatchable.

In fact, a variation of this theorem motivated a specialized characterization of the critical graphs for the maximum cardinality problem (Q_5') which yields a polynomial identification procedure. (See Cornuejols and Pulleyblank [83].) We discuss this in the next chapter. Note that such a characterization is not known for the edge and triangle hypomatching problem.

Section 5. Max-min and Polyhedral Theorems for Hypomatchings

The following theorem is a consequence of the validity of the algorithm.

Theorem 4.12: Given a graph $G = (V,E)$ and a hypomatching

problem on G, the cardinality of a maximum hypomatching is

equal to

$$\min_{V' \subseteq V} \frac{|V|+|V'|-O(G[V\backslash V'])}{2}$$

where $O(G[V\backslash V'])$ denotes the number of components of $G[V\backslash V']$

which are critical.

Proof: Analogous to the proof of Theorem 1.8.

Let us end this chapter with a polyhedral result.

Theorem 4.13: Consider the following hypomatching problems: The

maximum cardinality edge and triangle packing problem and the

maximum cardinality versions of problem (Q'_k). Given a graph

$G = (V,E)$ and one of the above hypomatching problems, the

following polyhedral characterization is sufficient to solve the

problem as an LP: max $1 \cdot x$

$$\text{subject to} \quad x(\delta(v)) \leq 2 \qquad \text{for all } v \in V$$

$$x(E') \leq |V'| - 1 \qquad \text{for all subgraphs}$$

$$G' = (V',E') \text{ which are}$$

$$\text{critical}$$

$$x_e \geq 0 \qquad \text{for all } e \in E.$$

Proof: We must check that the cardinality of a maximum hypomatching

is equal to the optimal value of the objective function over the

set of constraints. To do this we set up a weighted algorithm, as in the primal-dual weighted matching algorithm, by choosing primal and dual feasible solutions which satisfy a subset of the complementary slackness conditions; that is, we choose any hypomatching x for the primal solution and, for the dual solution, $y_i = \frac{1}{2}$ for every node i and $y_j = 0$ otherwise. Thus every edge is in the equality subgraph. We apply the algorithm to G and, at the end, perform a dual change which produces a dual feasible solution equal in value to the primal. (There are no nodes i such that $y_i > 0$ and $x(\delta(i)) < 2$.) Since the objective value is the cardinality of a maximum hypomatching, we are done. ∎

Chapter 5

THE MAXIMUM WEIGHT TRIANGLE AND PENTAGON-FREE 2-MATCHING PROBLEM

Section 1. Introduction and Cardinality Problem

In this chapter we consider some of the questions involved in finding a polynomial algorithm for the problem $(Q_5^!)$. First, we discuss a polynomial algorithm for the cardinality problem and we give a class of 0-1 and 0-1-2 facets for the weighted problem which we conjecture are sufficient for a complete polyhedral characterization. We then give some useful properties needed to find a primal-dual Edmond's sytle algorithm for the weighted problem.

The cardinality problem of $(Q_5^!)$ is solved using the algorithm for maximum cardinality hypomatchings given at the end of the last chapter. To make this algorithm polynomial for $(Q_5^!)$ we must be able to determine in polynomial time if a given hypomatchable graph is Q_5-critical. (In this chapter, we use "Q_5-critical" to refer to graphs which are critical with respect to triangle and pentagon-free 2-matchings. We will also refer to such matchings as "Q_5-matchings".) To do this we use a characterization of Q_5-critical graphs due to Cornuejols and Pulleyblank [83].

The characterization of Q_5-critical graphs which we use is a specialization of the following theorem due to Lovasz [72].

Theorem 5.1: A graph $G = (V,E)$ is hypomatchable iff there exists a sequence of node sets V_i, $0 \leq i \leq p$, (called an <u>ear decomposition of</u> G) such that

(a) V_0 is the node set of an odd cycle of G,

(b) $V_{i-1} \subseteq V_i$ for $1 \leq i \leq p$. Furthermore, $V_i - V_{i-1}$ has even cardinality and is the node set of a path P_i of G both of whose endnodes are adjacent to V_{i-1},

(c) $V_p = V$.

Let us say a graph G is <u>nonseparable</u> if it has no cutnode and an ear decomposition is <u>nonseparable</u> if $G[V_i]$ is nonseparable for all $i = 1, 2, \ldots, p$. Since we characterize the nonseparable Q_5-critical graphs by describing their ear decomposition, the following theorem, also due to Cornuejols and Pulleyblank [83], is useful.

Theorem 5.2: G is a nonseparable hypomatchable graph, iff G has a nonseparable ear decomposition.

We are now ready for the characterization.

Theorem 5.3: Let $G = (V,E)$ be a hypomatchable nonseparable graph and let $V_0, \ldots, V_p$ be any nonseparable ear decomposition of G. Then G is Q_5-critical iff the following conditions are satisfied:

(a') V_0 has cardinality 3 or 5.

(b') For each $i = 1, \ldots, p$, $V_i - V_{i-1}$ has cardinality 2, say $V_i - V_{i-1} = \{y, z\}$. Furthermore, for any pair of vertices $u \neq v$ in V_{i-1}, such that u is adjacent to y and v is adjacent to z (we distinguish our pair of such nodes u, v and call them the attachment nodes of the ear), there must exist a vertex $w \in V_{i-1}$ which is adjacent to only u and v.

It is possible using this theorem to polynomially determine if a given hypomatchable graph is Q_5-critical or not (see Cornuejols and Pulleyblank [83]). Focusing on the nonseparable Q_5-critical graphs is also important because these are the graphs with which we associate 0-1 inequalities in the weighted case.

Section 2. Some Polyhedral Results

In this section we conjecture what a complete polyhedral characterization of the convex hull, $P(G)$, of Q_5-matchings for a graph G looks like. We first define three classes of graphs.

Class 1: The nonseparable Q_5-critical graphs

Class 2: The graphs which can be built from either of the two graphs in Figure 5.1 by the addition of nonseparable ears as described in part b' of Theorem 5.3.
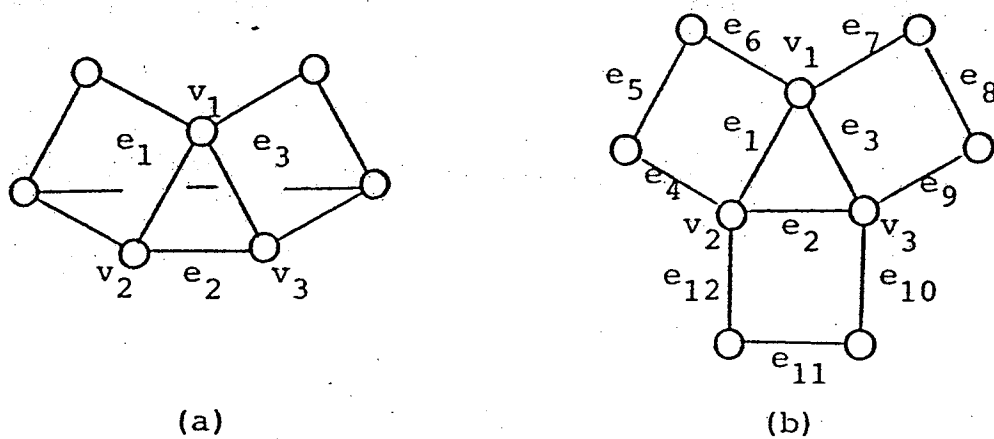
(a)   (b)

Figure 5.1

Class 3: The graphs which can be built from the graph

in Figure 5.2 by the addition of nonseparable
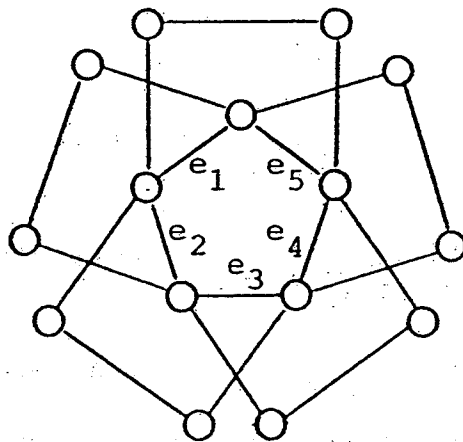
ears as described in part b' of Theorem 5.3.

Figure 5.2

We make three propositions concerning the graphs in these classes.

Proposition 5.1: Associated with every Class 1 graph $G = (V,E)$ is the inequality

$$x(E) \leq |V| - 1$$

which is a facet for $P(G)$.

Proposition 5.2: Associated with every Class 2 graph $G = (V,E)$ is the inequality

$$x(E - \{e_1, e_2, e_3\}) + 2x(\{e_1, e_2, e_3\}) \leq |V| + 1$$

which is a facet for $P(G)$ where e_1, e_2, e_3 are as in Figure 5.1.

Proposition 5.3: Associated with every Class 3 graph $G = (V,E)$ is the inequality

$$x(E - \{e_1, \ldots, e_5\}) + 2x(\{e_1, \ldots, e_5\}) \leq |V| + 3$$

which is a facet for $P(G)$ where $e_1, \ldots, e_5$ are as in Figure 5.2.

Proof of Proposition 5.1: The fact that the inequality is a facet follows from the fact that G is a nonseparable hypomatchable graph. (See Pulleyblank and Edmonds [74].)

<u>Proof of Proposition 5.2</u>:   Consider the graphs in Figure 5.1.

We first show that the inequalities corresponding to the graphs

in (a) and (b) are valid and facets.   Then we show that the

graphs obtained from these by the addition of ears as described

in part b' of Theorem 5.3 also correspond to facets.

Consider first the graph in (a).   Note that all the

Q_5-matchings are either 1-matchings or the single cycle of

length 7.   Since no 1-matching uses more than one of e_1, e_2

or e_3 and since the length 7 cycle satisfies the inequality,

the inequality is valid.

To show that the inequality corresponding to the graph in (a)

is a facet, we exhibit 10 affinely independent Q_5-matchings which

satisfy the inequality at equality.   Shrink the triangle v_1, v_2,

v_3 to a single node u.   The resulting graph G' is nonseparable

and hypomatchable and therefore there exist 7 (one for each edge)

affinely independent 1-matchings for G'.   Each of these

1-matchings may be extended to a 1-matching for the original

graph by appropriately adding one of the three edges e_1, e_2 or

e_3 to the 1-matching.   Since the 7 1-matchings of G' are

affinely independent, so are their extensions.   So we need 3

more such Q_5-matchings.   One of the 7 1-matchings chosen for G',

say x_m, must be deficient at u.   (Since G' is hypomatchable,

there exists such a 1-matching x_m.   If all 7 1-matchings

saturate u, then they all satisfy $x(\delta(u)) = 2$.   Since

$x_m(\delta(u)) = 0$, it is affinely independent of the other 7, which

is not possible.) Let x_{m_1} be the extension of x_m and assume, without loss of generality, that $x_{m_1}(e_1) = 2$. Then all 7 1-matchings satisfy $x(\delta(v_1)) + x(\delta(v_2)) = 4$. Consider the two other extensions of x_m, say x_{m_2} and x_{m_3}, where $x_{m_2}(e_2) = 2$ and $x_{m_3}(e_3) = 2$, respectively. x_{m_2} is affinely independent of the first 7 since it does not satisfy $x(\delta(v_1)) + x(\delta(v_2)) = 4$. These 8 Q_5-matchings satisfy $x(\delta(v_2)) = 2$. However, x_{m_3} does not and therefore is affinely independent of the first 8. Note that all 9 of these satisfy $x(\{e_1, e_2, e_3\}) = 2$. We may take the final Q_5-matching, say x_{m_4}, to be the length 7 cycle which contains e_2, since $x_{m_4}(\{e_1, e_2, e_3\}) = 1$.

Let us now consider the graph in (b). All the Q_5-matchings are either 1-matchings, one of the three length 7 cycles (each of which contains one of e_1, e_2, or e_3) together with an edge, or the single length 9 cycle which contains neither e_1, e_2, nor e_3. Since no 1-matching can contain more than one of the edges e_1, e_2, or e_3, all of the Q_5-matching on this graph satisfy the corresponding inequality, so it is valid.

To show that this inequality is a facet, we exhibit 12 affinely independent Q_5-matchings. Again shrink the triangle v_1, v_2, v_3 to a single node u and call the resulting graph G'. For the first 3 matchings, set $x_{m_i}(e_8) = x_{m_i}(e_{11}) = 2$, $i = 1, 2, 3$, and then take the 3 1-matchings of the triangle e_4, e_5, e_6, extending each to the triangle e_1, e_2, e_3. Notice that these 3 satisfy $x(e_7) = x(e_9) = x(e_{10}) = x(e_{12}) = 0$ and if we consider

in turn 4 1-matchings such that these edges are at value 2, we get 4 more affinely independent matchings. (For example, for e_7 we take the matching which contains e_5, e_7, e_{11} and e_2 so that $x(e_9) = x(e_{10}) = x(e_{12}) = 0$.) Note that we may get 2 more affinely independent matchings by extending the 1-matching of G' which is deficient at u into the shrunk node as we did for the graph in (a). We must produce 3 more affinely independent Q_5-matchings. For these we take the Q_5-matchings which use the length 7 cycles. For example, the matching which uses the length 7 cycle $(e_4, e_5, e_6, e_7, e_8, e_9, e_2)$ is affinely independent of the other 11 matchings because they all satisfy $x(e_1, \ldots, e_9) = 6$ whereas this matching satisfies $x(e_1, \ldots, e_9) = 7$. We argue similarly for the remaining 2 Q_5-matchings.

We next show that the inequalities corresponding to the graphs obtained from (a) and (b) by adding on ears as in b' of Theorem 5.3 are facets. Let us assume this is true up to some point for a graph G' and let us inductively add on one more ear, say P, to get a graph $G = (V, E)$. Let v_1 and v_2 be the two nodes of P and let u_1 and u_2 be the attachments where u_i is adjacent to v_i for $i = 1, 2$.

To see that the inequality for G is valid, consider how we may match the edges in P. A Q_5-matching of G may have value 2, 3, or 4 in P. If it has value 2, then it cannot violate the inequality since we have with the addition of P added 2 to the right hand side of the inequality for G'.

If the ear is matched at value 3 or 4, then we must be covering two nodes, say u_1 and u_2, in G' with the edges in P which are in the matching. Note that there exists a node w in G' adjacent to only u_1 and u_2. Suppose w is contained in an ear $P*$. If P is matched at value 3, then $P*$ can be matched at most at value 1 (because w has degree 2) and if P is matched at value 4, then $P*$ can have no edges in the matching. Hence for these two ears we get a total value of no more than 4 and so, by inductive hypothesis, the inequality is valid. We can argue similarly if w is not in an ear.

We now exhibit $|E|$ affinely independent Q_5-matchings which satisfy the inequality at equality. By adding the edge (v_1, v_2) at value 2 to all the affinely independent Q_5-matchings for G', we get $|E| - k$ affinely independent Q_5-matchings for G where k equals the number of edges in P. We may put each edge in the ear, except (v_1, v_2), at value 2 and extend this to a near perfect matching of G' to get $k - 1$ more affinely independent Q_5-matchings. (They are independent because for any matching in which some edge e of P is at value 2, all the other matchings satisfy $x(e) = 0$.) We need only one more Q_5-matching. For this set (v_1, u_1) and (v_2, u_2) at value 2. Let w be the degree 2 node between u_1 and u_2. Suppose w is contained in an ear, say $P*$, and assume, without loss of generality, that u_1 and w are the two nodes of $P*$. Then we may extend this matching to a matching for G which satisfies the inequality at equality by near perfectly matching $G \setminus P \setminus P* \setminus u_2$.

If w is not contained in an ear, then u_1 and u_2 are nodes of one of the graphs in (a) or (b). In both cases it is easy to construct a matching which satisfies the inequality at equality. This last matching is affinely independent of the rest because it satisfies $x(P) = 4$ whereas all the others satisfy $x(P) = 2$.

<u>Proof of Proposition 5.3</u>: Exactly analogous to the proof of Proposition 5.2.

We next define an operation which combines graphs in Classes 1, 2, and 3 to give us graphs which correspond to the inequalities in our conjecture.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $E_1 \cap E_2 = \emptyset$ and $V_1 \cap V_2 = y$. Consider an edge $e = (u,v)$ such that $u \in V_1 - y$ and $v \in V_2 - y$ and suppose there exist nodes $w_1 \in V_1 - y$ and $w_2 \in V_2 - y$ such that w_1 is adjacent in G_1 to only u and y, and w_2 is adjacent in G_2 to only v and y. Then we define

(5.1) $(G_1 * e * G_2) \equiv (V', E' \cup (u,v))$.

Let us define one more class of graphs as follows.

<u>Class 4</u>: The graphs recursively generated from the graphs in Classes 1, 2, and 3 by the operation defined in (5.1). (See Figure 5.3.)
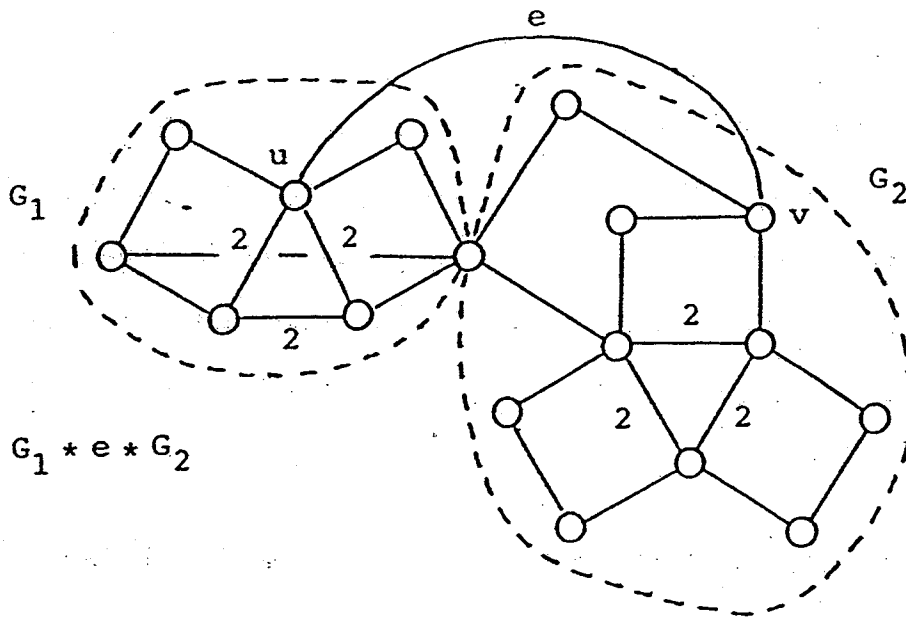
Figure 5.3

Note that Classes 1, 2 and 3 are included in Class 4.
We have the following theorem about the graphs in Class 4.

Theorem 5.4: Let $G = (V,E)$ be any graph in Class 4 such that

$$G = ((G_0 * e_1 * G_1) * e_2 * G_2) * \ldots * e_n * G_n$$

where G_i is in Classes 1, 2, or 3 and has associated with it
the inequality $a_i x \leq \alpha_i$ for $i = 0, \ldots, n$. Then,

$$ax + x(\{e_1, \ldots, e_n\}) \leq \alpha$$

is a facet for $P(G)$ where $ax = a_1 x + \ldots + a_n x$, and $\alpha = \alpha_1 + \ldots + \alpha_n$.

Proof: By induction on i. By Propositions 5.1 - 5.3 the theorem

is true for $i = 0$. Suppose it is true for $i = n-1$. Then

$G = G' * e_n * G_n$ where

$$G' = (V', E') = (G_0 * e_1 * G_1) * e_2 * G_2 * \ldots * e_{n-1} * G_{n-1},$$

$G_0, \ldots, G_n$ are in Classes 1, 2, or 3, $G_n = (V_n, E_n)$,

$E' \cap E_n = \emptyset$, $V' \cap V_n = y$, $e_n = (u,v)$, $u \in V' - y$,

$v \in V_n - y$, $w_1 \in V'$ is adjacent in G' to only u and y,

$w_2 \in V_n$ is adjacent in G_n to only v and y, and

$a'x \leq a'$ and $a_n x \leq \alpha_n$ are facets of $P(G')$ and $P(G_n)$,

respectively. We must show that

$$(5.2) \qquad a'x + a_n x + x(\{e_n\}) \leq \alpha' + \alpha_n$$

is a facet for $P(G)$. First we show that it is valid.

Let x be any Q_5-matching. Note that if $x(e_n) = 0$, then
the inequality (5.2) is valid by inductive hypothesis. So let
us assume $x(e_n) > 0$.

Suppose $x(e_n) = 1$. Since w_1 and w_2 have degree 2,
$x(\delta(w_1)) = x(\delta(w_2)) = 2$ implies that either x contains the
pentagon u, w_1, y, w_2, v or else $x(\delta(y)) > 2$, $x(\delta(u)) > 2$,
or $x(\delta(v)) > 2$. So we must have $x(\delta(w_1)) < 2$ or $x(\delta(w_2)) < 2$.
Assume, without loss of generality, that $x(\delta(w_1)) < 2$. We
construct a new Q_5-matching $\tilde{x}$ as follows:

If $x(\delta(w_1)) = 0$ or, if $x(\delta(w_1)) = 1$ and $x(uw_1) = 1$, set

$$\tilde{x}_e = \begin{cases} x_e & \text{for } e \neq e_n, uw_1 \\ x_e-1 & \text{for } e = e_n \\ x_e+1 & \text{for } e = uw_1. \end{cases}$$

If $x(\delta(w_1)) = 1$ and $x(w_1 y) = 1$, set

$$\tilde{x}_e = \begin{cases} x_e & \text{for } e \neq e_n, uw_1, w_1 y \\ x_e-1 & \text{for } e = e_n, w_1 y \\ x_e+1 & \text{for } e = uw_1. \end{cases}$$

Suppose $x(e_n) = 2$. Then, as above, we must have $x(\delta(w_1)) < 2$ or $x(\delta(w_2)) < 2$ since w_1 and w_2 have degree 2. So let us assume, without loss of generality, that $x(\delta(w_1)) < 2$. We construct a new Q_5-matching $\tilde{x}$ as follows:

If $x(\delta(w_1)) = 0$, set

$$\tilde{x}_e = \begin{cases} x_e & \text{for } e \neq e_n, uw_1 \\ x_e-2 & \text{for } e = e_n \\ x_e+2 & \text{for } e = uw_1. \end{cases}$$

If $x(\delta(w_1)) = 1$, then $x(\delta(w_2)) \leq 1$, and we set

$$\tilde{x}_e = \begin{cases} x_e & \text{for } e \neq e_n \\ x_e-2 & \text{for } e = e_n \\ x_e+1 & \text{for } e = uw_1, w_2 v. \end{cases}$$

In all cases, x and $\tilde{x}$ have the same value in (5.2).
However $\tilde{x}(e_n) = 0$ and therefore, by inductive hypothesis,
$\tilde{x}$ satisfies (5.2). Hence x does also and (5.2) is valid.

To finish the proof we exhibit $|E|$ affinely independent
Q_5-matchings which satisfy (5.2) at equality.

By the proofs of Propositions 5.1 - 5.3, we know that at any
node of a Class 1, 2, or 3 graph there exists a near-perfect
matching which is deficient at that node. Hence this is also
true for any graph in Class 4. So, consider any near-perfect
matching $\hat{x}$ of G' which is deficient at y, zero on $E_n \cup \{e_n\}$,
and such that $a'\hat{x} = \alpha'$. Since $a_n x \leq \alpha_n$ is a facet for $P(G_n)$,
there exist $|E_n|$ affinely independent Q_5-matchings which satisfy
$a_n x = \alpha_n$. Thus $\hat{x}$ added to each of these gives $|E_n|$ affinely
independent Q_5-matchings of G which satisfy $ax = \alpha$.
Similarly, we may choose a near-perfect matching $\tilde{x}$ of G_n
which is deficient at y and extend it to $|E'|$ affinely
independent Q_5-matchings of G which satisfy $ax = \alpha$. If we
combine these two sets of Q_5-matchings, we get a set of
$|E'| + |E_n| - 1$ affinely independent Q_5-matchings of G which
satisfy $ax = \alpha$. We need two more such matchings. To get these,
take two matchings in which e is at value 2 with the added
constraint that in one (w_1, y) is at value 2 and w_2 is deficient
while in the other (w_2, y) is at value 2 and w_1 is deficient.

Clearly the first of the two is affinely independent of the $|E'| + |E_n| - 1$ others because it is the only matching using e_n. Note that all of these $|E'| + |E_n|$ matchings satisfy $a'x = \alpha'$. However the last one does not, since it is deficient at two nodes of G', and so it is affinely independent at the rest.

With this final class of facets we are ready to make the following conjecture.

Conjecture: For any graph $G = (V, E)$, $P(G)$ is characterized by the following inequalities:

$$x(\delta(v)) \leq 2 \qquad \text{for all } v \in V,$$

$$ax \leq \alpha \qquad \text{for all Class 4 subgraphs of } G,$$

$$x_e \geq 0 \qquad \text{for all } e \in E.$$

Proposed proof: Using an Edmond's style algorithm for the weighted problem (Q_5').

In the next section we give some properties of Q_5-critical graphs which will be useful in developing an Edmond's style algorithm.

Section 3. Properties of Q_5-critical Graphs

Let G be a nonseparable Q_5-critical graph with nonseparable ear decomposition $V_0, \ldots, V_p$. Let P^0 be the edges in the odd cycle with node set V_0 and let P^i be the three edges in the path determined by the nodes $V_i - V_{i-1}$ and the two associated attachment nodes. We refer to P^0 as the "original polygon" and to the P^i's, $i > 0$, as "ears".

Consider an ear P^n in G^F. Its two attachment nodes are mutually adjacent to a degree 2 node. If this node is not in P^0, then it is contained in another ear, say P^{n-1}, which has its own degree 2 node. Continuing this process until P^0 is reached yields a sequence $P^0, \ldots, P^n$ called an <u>ear sequence</u>. Every ear in G has a unique ear sequence. A maximal ear sequence is called a <u>branch</u>. Thus two ears are in the same branch if and only if the ear sequence for one is a subsequence of the ear sequence for the other.

Let $x_n \notin P^0$ be a node in G. Consider the length 2 path from x_n to the attachment, say x_{n-1}, of the ear that contains it. If $x_{n-1} \notin P^0$, then consider the length 2 path from x_{n-1} to the attachment node of the ear that contains it. Continuing this until a node $x_0 \in P^0$ is reached yields the <u>descending path</u> from x_n to P^0. (See Figure 5.4.)
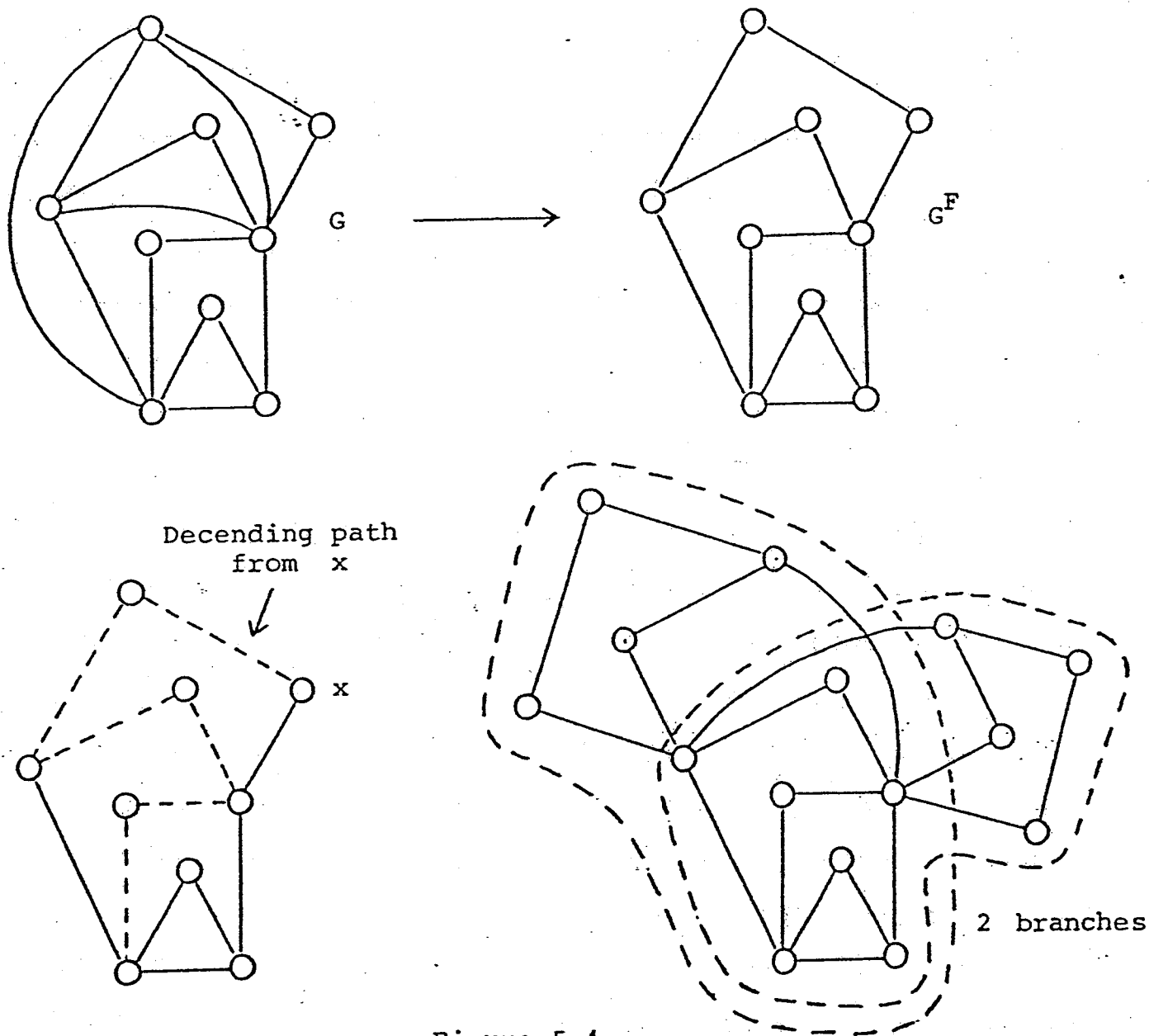
Figure 5.4

We call each Q_5-critical subgraph $G[V_0], G[V_1], \ldots, G[V_p]$ that occurred in the ear decomposition of G a <u>level</u>. Any Q_5-matching of G which has maximum cardinality for each level is called <u>levelwise correct</u>. We will say that such a matching <u>saturates</u> every level of G.

Consider two nodes u and v of G and the descending paths from each. Suppose the path from u intersects the path from v and does so for the first time at a node x. If the paths from u to x and x to v are both even, then we call the path which they describe from u to v a _connector from u to v_. If the paths do not intersect, then they end at different nodes x_1 and x_2 of P^0. In this case we define the connector from u to v to be the path described by the two descending paths (which may be of length 0) together with the even length path from x_1 to x_2 in P^0.

Proposition 5.4: Let G be a nonseparable Q_5-critical graph and let $u,v \in V$ have a connector C between them. Then G can be matched levelwise correctly where the edges in C are given the value 1.

Proof: Suppose $u \in P^0$ and $v \in P^0$. Then perfectly match the odd path from u to v in P^0, near-perfectly match the even path from u to v in P^0, and match every pair of nodes occurring in an ear. This matching is levelwise correct.

Suppose that $u \notin P^0$ and $v \notin P^0$. Suppose the descending paths from u and v intersect at a node w; then near-perfectly match the descending path from w to, say, w* in P^0 leaving w deficient, and match every pair of nodes occurring in an ear which is not already matched. If u and v descend to u_1 and v_1 in P^0 where $u_1 \neq v_1$, then perfectly match the odd

path from u_1 to v_1 in P^0, near-perfectly match the even path from u to v in P^0, and match every pair of nodes occurring in an ear which is not already matched. Those matchings are levelwise correct.

Similarly, we can handle the case $u \notin P^0$ and $v \in P^0$.

Proposition 5.5: There is always a connector between two nodes in different branches of a nonseparable Q_5-critical graph G.

Proof: Let us call the two nodes u and v. If the descending paths from u and v extend to P^0 without intersecting, except possibly at the last node, then by the definition we get a connector.

Let us assume the descending paths intersect before P^0. This implies that the ear sequences containing u and v build upon a common degree 2 node z which is not in P^0. Let x_1 and x_2 be the two nodes of G^F which are adjacent to z. (See Figure 5.5.)
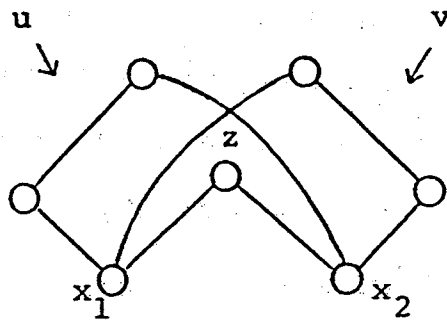

Figure 5.5

The descending path from u must contain x_1 and/or x_2 after an even number of edges as must the descending path from v. If they both contain x_1 or x_2 we have a connector. Suppose, without loss of generality, that the descending path from u contains x_1 and the descending path from v contains x_2. z must be in an ear with either x_1 or x_2. If it is in an ear with x_1, then the descending path from u to x_1 continues through z to x_2 to yield a connector. Similarly, if z is in an ear with x_2, then the descending path from v to x_2 continues through z to x_1 to yield a connector.

Suppose u and v are two nodes of two ears in the same branch of a Q_5-critical graph where v is contained in u's ear sequence. To see if a connector exists from u to v, let us say x is the first node on the descending path from u which is in one of the three edges of the ear P^i containing v. Let us say y is the degree 2 node of G^F adjacent to the attachments of P^i. We consider four cases, two for the nodes of P^i and two for the attachment nodes of P^i. (See Figure 5.6.)
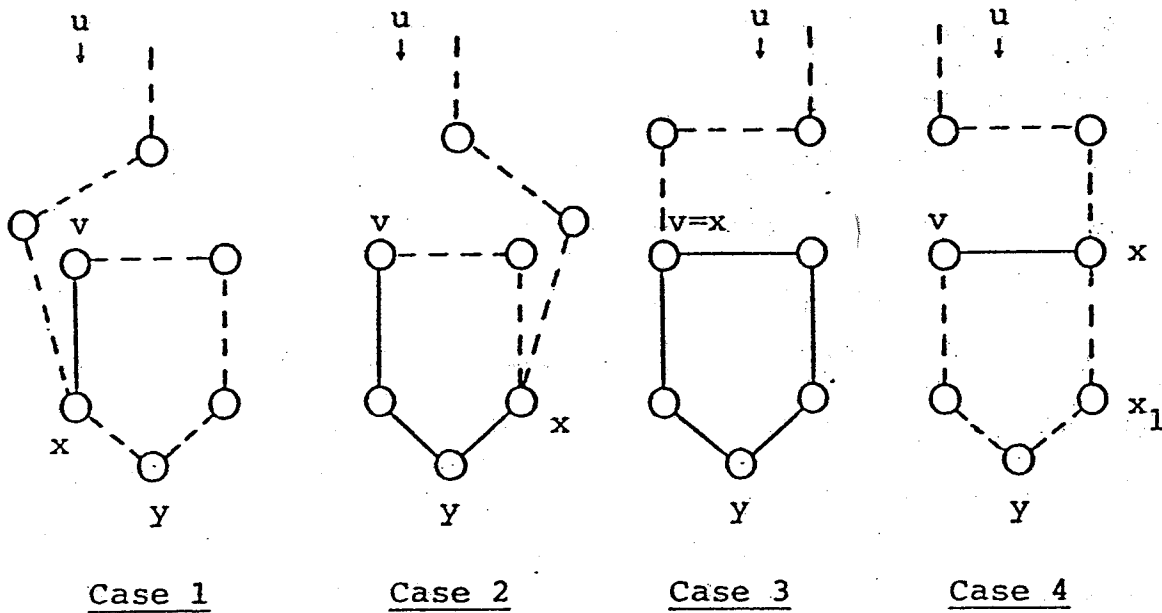
Case 1: x is an attachment node of P^i and is adjacent to v in G^F.

Case 2: x is an attachment node of P^i and is not adjacent to v in G^F.

Case 3: $v = x$.

Case 4: $v \neq x$ and x is a node of P^i.

Case 1   Case 2   Case 3   Case 4

- - - - -: Edges of the "connecting" path

Figure 5.6

In cases 1, 2, and 3 the connector is defined, but it is
not defined in case 4 because the path from u to v is of
odd length. However, if we consider the connector from u to x_1,
which is defined, and then change this path by exchanging edge
(v,x) for edge (x,x_1), we get a path from u to v with the
property that G can be matched levelwise correctly if the
edges in the path are given the value 1.

Let us call all connector paths together with those paths occurring in case 4 above <u>connecting paths in a nonseparable</u> Q_5-critical graph G.

Suppose G is a (separable) Q_5-critical graph and u and v are two nodes in different blocks, say G^u and G^v, of G. Let $G^u = G_1, G_2, \ldots, G_n = G^v$ be the sequence of blocks of G encountered on any path from u to v and let x_i be the cutnode shared by G_i and G_{i+1} for $i = 1, \ldots, n-1$. Let P be the path consisting of the connecting paths from u to x_1, x_1 to $x_2, \ldots, x_{n-1}$ to v. Then we call P a <u>connecting path</u> in a (separable) Q_5-critical graph G.

We may now state the following theorem.

<u>Theorem 5.5</u>: Between any two nodes of a Q_5-critical graph G there exists a connecting path C such that G can be matched levelwise correctly where the edges in C are given the value 1.

Note that if two nodes in a Q_5-critical graph are mutually adjacent to a degree 2 node, then the connecting path between them goes through this node.

Let us now consider one final result which will be of use to us.

An ear of a Q_5-critical graph is said to be <u>extreme</u> if it is not contained in the ear sequence of any other ear.

<u>Proposition 5.6</u>: For all nodes x in nonextreme ears of a
Q_5-critical graph G,

$$d_{G^F}(x) = 2 \quad \Rightarrow \quad d_G(x) = 2.$$

<u>Proof</u>: Suppose there exists an x such that $d_{G^F}(x) = 2$ and
$d_G(x) > 2$. Let P^x be the ear which contains x. Then, since
no ear can have x as an attachment (since $d_{G^F}(x) = 2$),
$d_G(x) > 2$ must be the result of some edge (x,y) which is
in G but not G^F. Let P^y be the ear or original polygon
which contains y.

If P^y occurs after P^x in the ear sequence for the branch
B containing P^x, then consider the connecting path from x to y.
We must be in case 4 of Figure 5.3 which means that this connecting
path has length ≥ 7. Hence G could not be Q_5-critical, a
contradiction.

Suppose P^y occurs before P^x in the ear sequence for the
branch B, or occurs in a different branch. Let P^z be the ear
just after P^x in the branch B. Consider the cycle C consis-
ting of the edges of P^z, the edge between the two nodes (say x
and t) of P^x, (x,y), and the connecting path from y to the
attachment of P^z which is not adjacent to t. (See Figure 5.7)
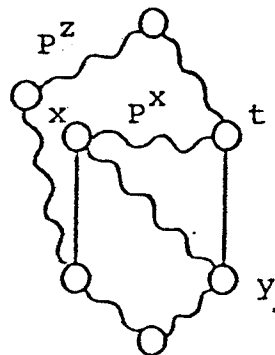
This cycle has length ≥ 7 and clearly can be completed into a perfect Q_5-matching of G. (Note that this Q_5-matching contains only one edge at value 1 of P^x.) ▢

Observation 5.1: Every nonextreme ear in a Q_5-critical graph contains exactly one degree 2 node.

Section 4. An Informal Look at an Edmond's Style Algorithm for (Q_5^1)

Let us now consider what an Edmond's style algorithm for the weighted case might look like. Just as we did for the weighted 1-matching problem in Section 5 of the Introduction, we take our conjectured system of inequalities, given earlier in this chapter, as our primal LP. From this we derive the dual LP and complementary slackness conditions. We begin to construct an algorithm exactly as in the 1-matching case: Step 0 is the initialization step where we choose feasible primal and dual solutions which satisfy all but a certain collection of complementary slackness conditions; Step 1 is the optimality check and node selection step; Step 2 is the edge selection step where new edges are considered; Step 3 is the forest growth step; Step 4 is the augmenting step; Step 5 is the shrinking step where subgraphs corresponding to Class 4 graphs are shrunk; Step 6 is the dual change step which we go to when there are no more edges to choose in Step 2; Step 7 is a pseudo augmentation step; and Step 8 is a node expansion step.

The objective of the algorithm is to modify the primal and dual optimal solutions until the complementary slackness conditions are satisfied.

At the beginning of the algorithm, shrunk nodes correspond to Q_5-critical graphs (not necessarily nonseparable). For each nonseparable Q_5-critical graph, the algorithm first constructs a triangle or pentagon and then adds ears to this which have three edges and whose attachment nodes are adjacent to degree 2 nodes. The algorithm may also add edges between nodes of these graphs. A dual variable is associated with each such nonseparable Q_5-critical graph and all of its levels. After a few passes through the dual change step some of these dual variables may be positive, as in the 1-matching algorithm. A complementary slackness condition requires that if a Q_5-critical graph has a positive dual variable associated with it, then it must be saturated by the matching.

Suppose an edge (u,v) is considered in the algorithm such that u and v occur in a Q_5-critical graph and such that the connecting path from u to v has length ≥ 6. Then there exists a perfect matching of this graph which is levelwise correct; i.e., the matching does not violate the root. Similarly, suppose the algorithm considers an edge in Step 2 which creates a (nonseparable) ear on a Q_5-critical graph. Then, if the ear has more than two nodes or if the connecting path between its attachments has length ≥ 4, then

an augmentation can be made. Let us now consider the complementary possibilities which are responsible for the difficulties.

Let us say that a Q_5-matching x is deficient on an ear P^i if $x(P^i) < 2$ and that x is deficient on an original polygon P^0 if $x(P^0) < 2$ for P^0 a triangle or $x(P^0) < 4$ for P^0 a pentagon.

The cases remaining to consider are: (i) when an edge (u,v), between two nodes of a Q_5-critical graph, is considered in Step 2 such that the connecting path from u to v is of length 2 or 4, and (ii) when an ear with two nodes is considered such that the connecting path between its attachment nodes is of length 2. When one of these cases occurs and an augmentation is possible, we will see that the augmentation is not levelwise correct, so, in general, we must lower to zero any positive dual variable associated with a level which is not saturated after the augmentation. This will allow us to augment without violating a complementary slackness condition. The matchings after the augmentation are of two types: those deficient on an ear and those deficient on an original polygon. If we were faced solely with augmentations of the first type, then any dual variables which must be lowered could be lowered using just 0 - 1 inequalities from Class 4. Unfortunately, the augmentations of the second type require the introduction of 0 - 1 - 2 inequalities from Class 4. We will examine briefly the dual change when augmentations of both types are found. Before we do this, we

need to look more closely at what happens when a nonseparable Q_5-critical graph loses its criticality.

Suppose we add an edge (u,v) to a nonseparable Q_5-critical graph G such that the connecting path from u to v is of length 2 or 4 and such that u and v occur in ears which are in the same branch of G^F. If the connecting path from u to v is of length 2, then let x denote the node in G^F on this path. If the connecting path is of length 4, then let x_1 and x_2 denote the second and fourth nodes in G^F on this path. Note that if G, with (u,v) added, has a perfect Q_5-matching, then the matching must include (u,v) since G is Q_5-critical. So, if the connecting path is of length 2 and $d_G(x) = 2$, or if the connecting path is of length 4 and $d_G(x_1) = d_G(x_2) = 2$, then G has no perfect Q_5-matching. (See Figure 5.8.)
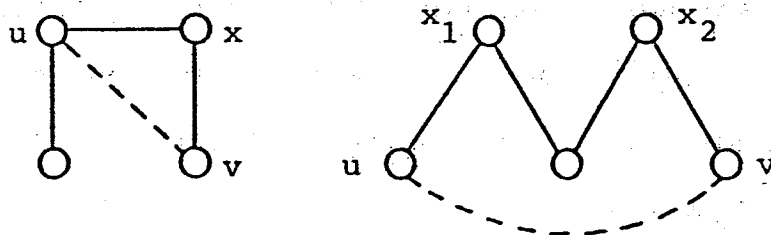

Figure 5.8

So suppose, in the first case, that $d_G(x) > 2$ and suppose, without loss of generality, that u and x occur in an ear together. If the ear containing u and x is not extreme, then by Observation 5.1, u must have been degree 2. Therefore (u,v) creates an augmentation as outlined in the proof of Proposition 5.6 (and shown in Figure 5.7) which is deficient on the ear containing u and x.

Suppose the ear containing u and x is extreme. If this graph admits a perfect Q_5-matching then it must contain (u,v) in a cycle. It must also contain one of the edges incident with x, say (x,y), where $(x,y) \notin G^F$. (Otherwise the cycle would be the triangle containing u, x and v.) Therefore, the cycle contains (u,x). It must also, by our degree 2 node arguments, contain the connecting path from y to v. So the connecting path from y to v must be of length > 2 and we get an augmentation deficient on the ear containing u and x. (See Figure 5.9.) If all such connecting paths have length 2 (that is, if for all edges (x,y), y is the attachment node of the ear containing u and x which is not adjacent to x), then we have no perfect Q_5-matching and the graph with (u,v) added is still Q_5-critical.
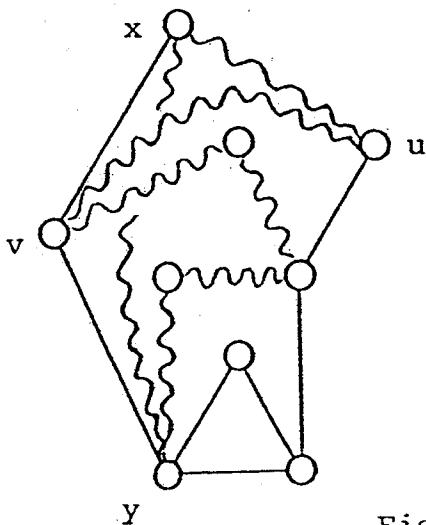

Figure 5.9

We may argue similarly for the second case where the connecting path from u to v is of length 4. In this case if either $d(x_1) > 2$ or $d(x_2) > 2$, we have an augmentation which is deficient on some ear. (5.3) Note also that these length 4 connecting paths may have endnodes in different nonseparable Q_5-critical graphs which share a single node. This case works exactly the same as above and results in the shrinking of nontrivial Class 4 graphs.

Note that with the proof of Proposition 5.6 we can handle the case of ears with three edges being added to another ear of a Q_5-critical graph when the connecting path between attachment nodes is of length 2 but the intermediate node on this path is not degree 2. In this case we get augmentations deficient on the ear to which the new ear is being added as in Figure 5.7. Note also that we are neglecting the case that u and v are in different branches. The arguments and augmentations are analogous to those just looked at. However, the dual change becomes more complex so we choose not to elaborate any further on this case.

Section 5. A Brief and Informal Look at the Dual Change

In reference to the two cases examined, (when u and v are in the same branch and $d(x)$, $d(x_1)$ or $d(x_2) > 2$) let us note that the augmentations which are not levelwise correct are

deficient on exactly one ear and a level is matched below its maximum if and only if this ear is extreme in this level. So the positive dual variables associated with these levels are the ones which must be reduced to zero before the augmentation can be made.

The dual change is performed as follows for such dual variables. For each dual variable with associated subgraph G' which we wish to reduce, we create a new 0-1 inequality corresponding to G' with the new edge or ear added on. This inequality corresponds to a Q_5-critical graph since the ear to which the new edge or ear is added was extreme. In the dual change we then raise the dual variable associated with the new 0-1 inequality and lower the dual variable we wish to lower. This also keeps all the edges in G', as well as the new edge or ear in the equality subgraph.

Let us now look briefly at some of the remaining types of augmentations, namely those which are deficient on the original polygon of a Q_5-critical graph.

Suppose an edge (u,v) is considered such that the connecting path from u to v is of length 4, u is in an ear, v is on the original polygon which is a triangle, and v is the degree 2 node of the triangle upon which the branch containing u was built. Then we get an augmentation, as illustrated in Figure 5.10 because the degree at v is > 2.
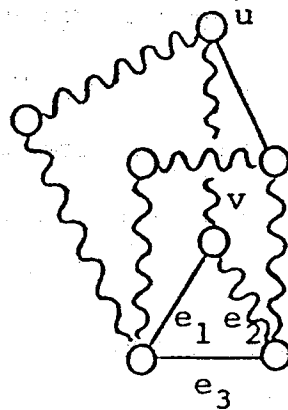
Figure 5.10

This augmentation is deficient on the triangle. Hence every positive dual variable associated with a Q_5-critical graph, which contains the triangle but not the first ear in the branch containing u, must be set to zero before the augmentation can be made. The triangle, for example, can be set to zero by associating a dual variable π with the 0-1-2 inequality obtained by assigning 2's to the edges on the triangle in Figure 5.10 and 1's to the other edges. (We have here the graph in Figure 5.1(a).) That is,

$$x(E - \{e_1, e_2, e_3\}) + 2x(\{e_1, e_2, e_3\}) \leq 8.$$

Then in the dual change, we raise π and lower the triangle, thus keeping all the edges, including the new one, in the equality subgraph and lowering the dual variable we want to lower.

Suppose the degree 2 node of a triangle upon which a branch is built has its degree increased by the addition of an ear as in Figure 5.11(a).
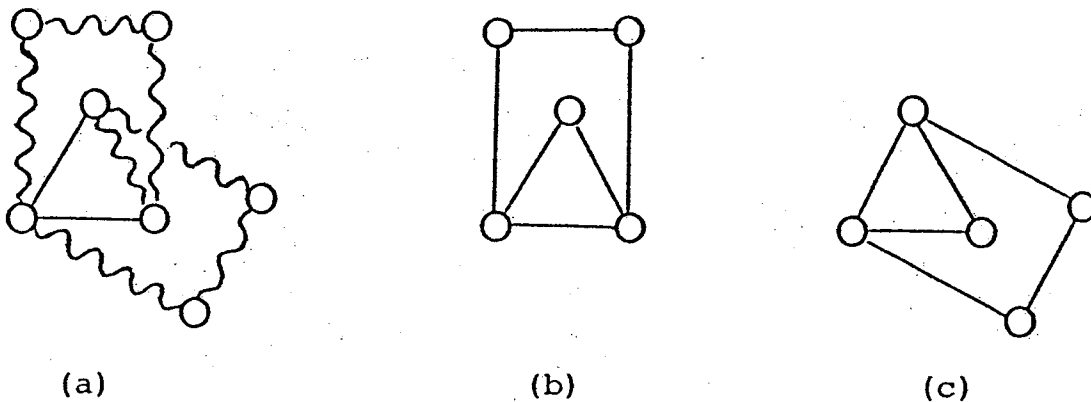

(a)   (b)   (c)

Figure 5.11

In this case we again get an augmentation deficient on the triangle, as illustrated, but the dual change to lower the triangle can be accomplished with 0-1 inequalities. Associate 0-1 inequalities with the Q_5-critical graphs in Figure 5.11(b) and (c). Raise these and then lower the triangle.

Suppose, as the dual variable associated with the triangle is being lowered, that a third ear is added to the triangle as in Figure 5.12(a). When this happens the dual change must be elaborated. To lower the triangle to zero, we associate a dual variable π with the 0-1-2 inequality obtained by assigning 2's to the edges

on the triangle and 1's to the other edges.   (Here we have the
graph of Figure 5.1(b).)   That is,

$$x(E - \{e_1,e_2,e_3\} + 2x(\{e_1,e_2,e_3\}) \leq 10.$$

Then in the dual change we raise π and lower the triangle.
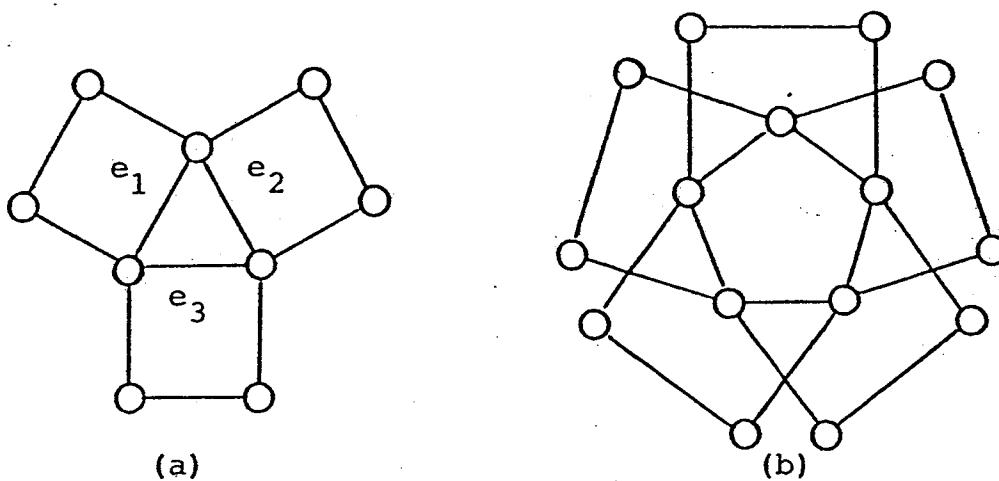

Figure 5.12

The analogous situation for pentagons is illustrated in
Figure 5.12(b).

Backing up a bit, suppose with the graph in Figure 5.11(a)
and, while the triangle is being lowered, an additional edge
(u,v), as in Figure 5.13, is considered. Although no new

Figure 5.13

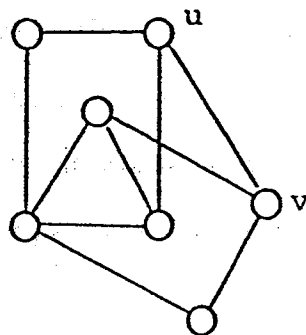augmentation is created, the dual change must be altered by the introduction of a $0 - 1 - 2$ inequality just as was done for the graph in Figure 5.10. (Note that the graph in Figure 5.10 is isomorphic to the graph in Figure 5.13.)

Let us just mention before ending this section that these types of situations can become much more complex and we have not yet been able to exhaust all the possibilities.

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