<u>Case 2.b.2</u>: wu_i is not the base of C. Say $b = cy$ is the base of C where $c \in C$. Make every shrunk node of C an odd node of $\mathcal{S}$, define $p(C) = w$, and add wu_i to $\mathcal{S}$.

Suppose $cy \in M$, $z_b = 0$, and y is not in $\mathcal{S}$. If y is shrunk, then go to Step 3 Case 1.a taking v and w to be c and y, respectively. If y is real (see Figure 2.7), then make y an even node of $\mathcal{S}$, define $p(y) = C$, and add cy to $\mathcal{S}$. (As above, if $z_b = 0$, then y must be real and either even or not in $\mathcal{S}$.) If $cy \in M$ and $z_b > 0$, do nothing.

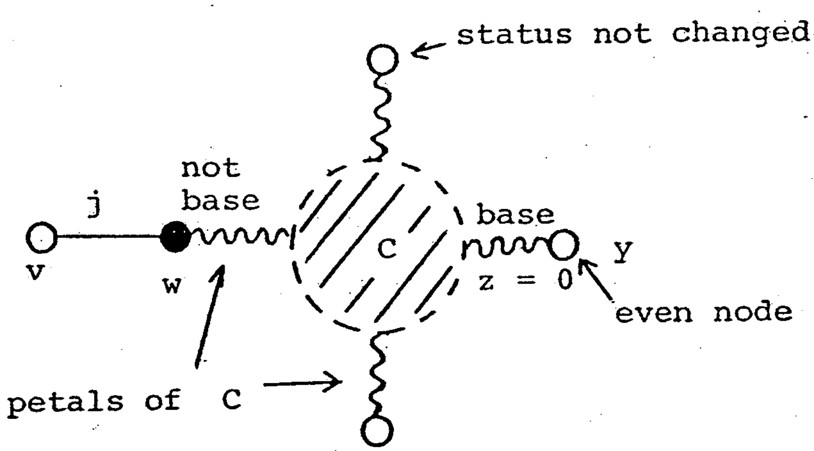


Figure 2.7

Suppose $cy \notin M$. If y is odd, then add cy to $\mathcal{S}$ and go to Step 2. (By design of the algorithm, y must be real.) Otherwise (see Figure 2.8), go to Step 2 taking v and w to be c and y, respectively.

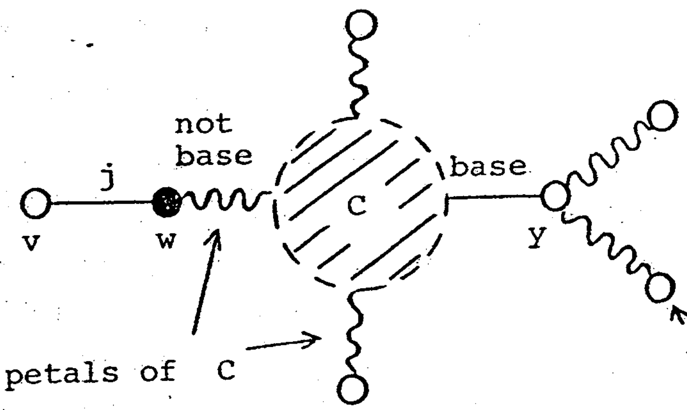


Figure 2.8

<u>Step 4</u> [Matching Augmentation]: Let $\overline{P}$ be the edge set obtained from P by removing all the blossom cluster petals (if any). Let e be a generic edge of $\overline{P}$. If $e \notin M$, then add the edge e to the set M; on the other hand, if $e \in M$, then remove the edge e from M. Let C be a generic blossom cluster encountered on the path P (if any). One node of C is incident with an edge f of $\overline{P}$ which was just added to M. Let u be this node. (Note that u is a node of G.) Change M inside C so that it is a saturating matching of the blossom deficient at node u. (This can be done since C is a critical blossom, a claim that will be proved in Proposition 2.1.) If u was the tip at a petal g of C, then change the base of C to g. If u was contained in a shrunk node of C, then change the base of C to f. Change x accordingly. Throw away $\mathcal{S}$ and go to Step 1.

Step 5 [Shrinking]: If the paths $(v,p(v),p(p(v)),\ldots,r_1)$ and $(w,p(w),p(p(w)),\ldots,r_2)$ lead to different roots r_1 and r_2, then let $P = (r_1,\ldots,p(p(v)),p(v),v,w,p(w),p(p(w)),\ldots,r_2)$ and go to Step 4. Otherwise let b be the first common node on the paths $(v,p(v),p(p(v)),\ldots,r)$ and $(w,p(w),p(p(w),\ldots,r)$. If $b = r$ and $x(\delta(r)) = 0$, then go to Step 4 where we augment M on the path $P = (r,\ldots,p(p(w)),p(w),w,v,p(v),p(p(v)),\ldots,r)$. Otherwise we define the following set S of nodes. A node $s \in V$ is in S if s is either a real node or contained in a blossom cluster encountered in the paths $(v,p(v),p(p(v)),\ldots,b)$ and $(w,p(w),p(p(w)),\ldots,b)$. (In this step, if, for example, $p(p(v))$ is a blossom cluster, then we replace $p(p(v))$ in this path with the sequence of nodes in $p(p(v))$ on the unique path through $p(p(v))$ from $p(v)$ to $p(p(p(v)))$.)

If some node $s \in S$ is deficient and $s \neq r$, then s is a real even node of one of the above paths, say $s \in \{v,p(v),p(p(v)),\ldots,b\}$. Then go to Step 4 where we augment M on the path $P = (s,\ldots,p(p(v)),p(v),v,w,p(w),p(p(w)),\ldots,r)$.

Finally, if every node $s \in S$, $s \neq r$, is saturated, shrink S into a node u of $\tilde{G}$. If S contains a root, let $T = M \cap \delta(S)$ and redefine the root to be the shrunk node u. Otherwise set $p(u) = p(b)$. If $j = up(u) \in M$, let $T = (M \cap \delta(S))\backslash\{j\}$ and if $j \notin M$, let $T = (M \cap \delta(S)) \cup \{j\}$. In all three cases, (2.6) check whether some edge t of T joins u to an odd node or even shrunk node w of $\mathcal{S}$ such that $p(w) \notin S$ and $p(u) \neq w$. If such an edge t exists, and $z_t = 0$ then go to Step 5 where v and w are taken to be the two end nodes of the edge t. Otherwise, for each petal t with tip z such that $z_t = 0$ and z is not in $\mathcal{S}$, make z an even node of $\mathcal{S}$, define $p(z) = u$

and add the edge uz to $\mathcal{S}$; for each node w of $\mathcal{S}$ such that $p(w) \in S$, reset $p(w) = u$.

Finally if two edges k and ℓ of T meet into the same node z, modify the blossom by adding z to the set S and by removing the edges k and ℓ from T. If either k or ℓ happens to be the edge j, then also modify $p(u)$ by setting it equal to $p(z)$. If u creates a new blossom cluster by enlarging an old cluster C, then let C refer to this new cluster and replace u with C in all sequences of predecessors which contain u. Go to Step 2.

Step 6 [Dual Change]: We now proceed to decrease the value of the dual solution by an amount $\sigma > 0$ as follows:

Dual change for y and π:

$y_i \leftarrow y_i - \sigma$ if surface (i) is an even node of $\mathcal{S}$.

$y_i \leftarrow y_i + \sigma$ if surface (i) is an odd node of $\mathcal{S}$.

$\pi_B \leftarrow \pi_B - 2\sigma$ if $B \in \mathcal{B}$ corresponds to an odd shrunk blossom of $\mathcal{S}$.

$\pi_B \leftarrow \pi_B + 2\sigma$ if $B \in \mathcal{B}$ corresponds to an even shrunk blossom of $\mathcal{S}$.

Dual change for z:

Given $e \in M$ and e an edge of $\tilde{G}$ we change z_e as follows:

Case 1: Neither endnode of e is in $\mathcal{S}$: do nothing.

Case 2: Exactly one endnode of e is in $\mathcal{S}$.

(a) If the endnode is real even: $z_e \leftarrow z_e + \sigma$.

(b) If the endnode is real odd: $z_e \leftarrow z_e - \sigma$.

(c) If the endnode is shrunk in a cluster C:

Endnode of $\mathcal{S}$	e is base of C	e is petal of C
shrunk even	cannot happen	$z_e \leftarrow z_e - \sigma$
shrunk odd	$z_e \leftarrow z_e - \sigma$	$z_e \leftarrow z_e + \sigma$

Case 3: Both endnodes of e are in $\mathcal{S}$.

(a) Both endnodes are real.

(i) Both even: $z_e \leftarrow z_e + 2\sigma$

(ii) Both odd: $z_e \leftarrow z_e - 2\sigma$

(iii) One even, one odd: do nothing.

(b) One endnode is real and one is shrunk in a cluster C:

Endnodes of $\mathcal{S}$	e is base of C	e is petal of C
real even & shrunk even	cannot happen	do nothing
real odd & shrunk even	do nothing	$z_e \leftarrow z_e - 2\sigma$
real even & shrunk odd	do nothing	$z_e \leftarrow z_e + 2\sigma$
real odd & shrunk odd	$z_e \leftarrow z_e - 2\sigma$	do nothing

(c) Both endnotes are shrunk:

(i) If both are even or odd and in the same cluster: do nothing.

(ii) If both are even and in different clusters: $z_e \leftarrow z_e - 2\sigma$.

(iii) If both are odd and in different clusters:

$z_e \leftarrow z_e + 2\sigma$

(iv) If one is odd and one is even and

(a') If e is the base of both clusters: do nothing.

(b') If e is the base of one cluster and a petal of the other: $z_e \leftarrow z_e - 2\sigma$.

The magnitude of σ is bounded by the feasibility conditions of the dual LP:

(2.7) $\sigma \leq y_i$ for all nodes i such that surface (i) is an even node of $\mathcal{S}$.

(2.8) $2\sigma \leq \pi_B$ for all $B \in \mathcal{B}$ which correspond to an odd shrunk blossom of $\mathcal{S}$.

(2.9) $\sigma \leq z_e$ for all $e \in E$ such that exactly one endnode x of e is in $\mathcal{S}$ and x is:

(i) real odd,

(ii) shrunk even where e is a petal of the cluster containing x,

(iii) shrunk odd where e is the base of the cluster containing x.

(2.10) $2\sigma \leq z_e$ for all $e \in E$ such that both endnodes of e are in $\mathcal{S}$ and they are:

(i) both real odd,

(ii) one real odd and one shrunk even where e is a petal of the cluster containing the shrunk even node,

(iii) one real odd and one shrunk odd where e is the base of the cluster containing the shrunk odd node,

(iv) both even shrunk nodes which are not in the same cluster,

(v) one odd shrunk and one even shrunk where e is the base of one cluster and a petal of the other.

(2.11) $\sigma \leq y_u + y_v + \sum_{B \in \mathcal{B}} \pi_B + z_e - c_e$ for all $e = (u,v) \in E$ which join an even node of $\mathcal{S}$ to a node not in $\mathcal{S}$.

Choose the largest σ which satisfies (2.7)-(2.11). If $\sigma > 0$, then make the dual change. If σ was bounded by (2.11), then a new edge has become available in $E^*(y,\pi,z)$. Go to Step 2. If σ was bounded by (2.7), then a node i which is either a real even node or contained in a shrunk even node of $\tilde{G}$ has $y_i = 0$. If M is deficient at i, "throw away" $\mathcal{S}$ and go to Step 1. Otherwise, go to Step 7 where we do a "pseudo" augmentation. If σ was bounded by (2.8), then there is an odd shrunk blossom B with $\pi_B = 0$. Go to Step 8 where we expand this shrunk node. Otherwise σ was bounded by (2.9) or (2.10).

If σ was bounded in (2.9)-(i), then take v and w to be $p(x)$ and x, respectively, and go to Case 2 of Step 3.

If σ was bounded in (2.9)-(ii), let $C = (S,T)$ be the blossom cluster containing x, let u be the collection of nodes in S and go to (2.6) in Step 5 for the petal e.

If σ was bounded in (2.9)-(iii), let C be the cluster containing x. If (p(C),C) is not in M, then take v and w to be p(C) and C, respectively, and go to Case 1.b of Step 3. If (p(C),C) is in M, then take v and w to be p(p(C)) and p(C), respectively, and go to Case 2.b.2 of Step 3.

If σ was bounded in (2.10), take v and w to be the endnodes of e and go to Step 5.

<u>Step 7</u> [Pseudo Augmentation]: There is a node i such that $y_i = 0$ and v = surface (i) is an even node of $\mathcal{S}$. If v is contained in the root of the tree of $\mathcal{S}$ which contains i, then change M inside the cluster C which contains v so that it saturates C and is deficient at i. If v is not contained in the root, then let $P = (v,p(v),p(p(v)),\ldots,r)$. If v is shrunk in a cluster C, then change M inside C so that the deficiency occurs at i. Go to Step 4.

<u>Step 8</u> [Expanding a Shrunk Node]: i is an odd shrunk node. Let (i',j) be the base of i such that surface(i') = i. Let (i",k) be the nonmatching edge of $\mathcal{S}$ incident with i such that (i",k) $\neq$ (i',j) and such that surface (i") = i. Let P be the path through i from i' to i" which an augmentation through i follows. Expand i to get $\tilde{G}'$, set $\tilde{G} = \tilde{G}'$, and make every other node along P odd and even such that i" is odd. Orient

the pointers for the nodes on P toward i". Go to Step 1.

End of Algorithm

Section 3: The Validity of the Algorithm and Some Consequences

To prove the validity of the algorithm we must show that (i) the augmentations through blossoms are valid in Step 4 (this will be obtained as a by-product of Theorem 2.1 in which we show that the blossoms defined in Step 5 are critical); and (ii) the algorithm terminates with optimal primal and dual solutions. (Note that we actually need to show in (i) that augmentations through blossom clusters are valid. But since a blossom cluster is a tree of blossoms, an augmentation through a cluster is an augmentation through a sequence of blossoms along the unique path through the cluster defined by the entering and existing nodes of the augmenting path.)

Theorem 2.1. The blossoms defined in Step 5 are critical and they are saturated by the current simple 2-matching M.

Proof: By induction. Assume that the blossoms of $\mathcal{S}$ satisfy the statements of Theorem 2.1 before going to Step 5. We have to show that the blossom B = (S,T) defined in Step 5 is critical and saturated by M.

If S does not contain the root r, then every node of S is saturated by M in G. In addition, by defintion of T, M saturates the blossom B = (S,T). The only deficient node in B is one of the endnodes of the edge j = up(u) defined in Step 5. In the case where S contains r, M also saturates B, the only deficient node in B being the node r (or a node of

G inside r if r is a shrunk node).

Now we want to find a simple 2-matching M' which saturates B but which leaves some other node deficient in B, say node z. Let $\mathcal{S}$ be the alternating structure before the set S is shrunk in Step 5.

Suppose z is the tip of a petal of B whose other end-node, say y, is either real odd or an even blossom cluster of $\mathcal{S}$. Then, if the base of B is in M or b is a root, apply the augmenting step (Step 4) to P = (z,y,p(y),...,b); if the base of B is not in M, apply the augmenting step to P = (z,y,p(y), ...,b,p(b)).

Suppose z is a real even node or contained in an even blossom cluster C of $\mathcal{S}$. Then, if the base of B is in M or b is a root, apply the augmenting step to P = (z,p(z),...,b) or P = (C,p(c),...,b); if the base of B is not in M, apply the augmenting step to P = (z,p(z),...,b,p(b)) or P = (C,p(C), ...,b,p(b)).

Note that for these two cases the augmenting step provides a saturating matching M' in every shrunk node encountered in P, by the induction hypothesis. In addition, z is deficient in B while all other nodes of B are saturated in B by M'. So we have the required saturating matching M'.

Now suppose z is the tip of a petal of B whose other end node, say y, is real even or an odd blossom cluster of $\mathcal{S}$. Assume, without loss of generality, that y is on the path (v,p(v),...,b). Then, if the base of B is in M or b is a root, apply the augmenting step to $P = (z,y,\ldots,p(b),v,w,p(w),\ldots,b)$; if the base of B is not in M, apply the augmenting step to $P = (z,y,\ldots,p(v),v,w,p(w),\ldots,b,p(b))$.

Finally, suppose z is a real odd node or contained in an odd blossom cluster C of $\mathcal{S}$. Assume, without loss of generality, that z or C is on the path $(v,p(v),\ldots,b)$. Then, if the base of B is in M or b is a root, apply the augmenting step to $P = (z,\ldots,p(v),v,w,p(w),\ldots,b)$ or $P = (C,\ldots,p(v),v,w,p(w),\ldots,b)$; if the base of B is not in M, apply the augmenting step to $P = (z,\ldots,p(v),v,w,p(w),\ldots,b,p(b))$ or $P = (C,\ldots,p(v),v,w,p(w),\ldots,b,p(b))$.

Again, by induction hypothesis, the augmenting step provides the required saturating matching M' in every blossom cluster encountered in P except, possibly, at v and w. To finish the proof we must consider all the possible ways in which v and w occur prior to an application of Step 5.

<u>Case 1</u>: The edge $vw \notin M$. Then v and w are both even in $\mathcal{S}$. Since the edge vw is not a blossom petal, the edges of $\overline{P}$ are alternately in M and out of M and thus the augmenting step is valid.

<u>Case 2</u>: The edge $vw \in M$ and one of the nodes v or w is real odd, the other being an even blossom cluster. Then vw is a petal and therefore is not in $\overline{P}$. So again the edges of $\overline{P}$ are alternately in M and out of M and the augmenting step is valid.

<u>Case 3</u>: The edge $vw \in M$ and both v and w are even blossom clusters. Then vw is a common petal of the blossom clusters v and w. In addition the edges of $\overline{P}$ alternate in and out of M except for two consecutive edges in M, namely the edges of $\overline{P}$ incident with the blossom clusters v and w. However the augmenting step is still valid since both blossom clusters can be saturated by requiring that the edge $vw \notin M'$, thus leaving the tip of the petal vw deficient in each of the two blossom clusters.

These are the cases in which v and w are chosen prior to Step 5 in the primal part of the algorithm. The remaining cases occur when v and w are chosen in the dual change step; in particular, when σ is bounded in (2.9)-(ii) or in (2.10). These cases are handled similarly.

So, in all cases, B is critical and the augmentations are valid.

Let us now discuss the termination of the algorithm.

Every time a new forest is set up in Step 1, we have a deficient node i contained in the root of each tree such that $x(\delta(i)) < 2$ and $y_i > 0$. The forest is grown until either we augment and increase $x(\delta(i))$ by 1 or 2 for some such i, or we execute a dual change making $y_i = 0$ for some such i. The algorithm never increases y_v for a node unless $x(\delta(v)) = 2$ and never decreases $x(\delta(v))$ for a node unless $y_v = 0$. Therefore after at most $2|V|$ forest growings we must have (2.4) satisfied. As in the 1-matching algorithm, with careful implementation, this algorithm can be made to work in $O(|V|^3)$.

The fact that the algorithm works gives us a polyhedral characterization of the convex hull of incidence vectors of simple 2-matchings for a graph G -- that is, the primal LP we used for the algorithm. Let us now focus upon the primal part of the algorithm.

The primal part of the algorithm seeks to find a maximum cardinality simple 2-matching on the equality subgraph. To see this, suppose we have a graph G for which we want a maximum cardinality simple 2-matching. Hence $w \equiv 1$ and a dual feasible solution is $y_i = 1/2$ for all $i \in V$, $\pi_B = 0$ for all $B \in \mathcal{B}$, and $z_e = 0$ for all $e \in E$. In fact, this dual solution sets $E(y,\pi,z) = E$. Suppose we now apply the algorithm. It grows a forest until at some point a dual change is required. After one dual change the algorithm terminates with $y_i = 0$ for all deficient nodes i. Note that we regrow no blossoms and hence have no shrunk odd nodes. Thus all odd nodes are real and saturated. The validity of this cardinality algorithm provides the following theorem.

Theorem 2.2. The graph G has a simple 2-matching of cardinality k if and only if, $\forall\ S,T \subseteq V$ such that $S \cap T = \emptyset$,

$$2(|S| + |V| - k) \geq \sum_{i \in T} (2 - |\delta_{G(V \setminus S)}(i)|) + C(S,T)$$

where $C(S,T)$ is the number of critical blossoms in the graph obtained from $G(V\setminus S)$ by splitting the nodes of T. (Splitting a node t of degree d in a graph means making d copies of that node, each copy having degree 1 and being joined to a different neighbor of t.)

Proof: The inequality is necessary since, in the right-hand side, the first sum is the minimum number of deficiencies left in T by any simple 2-matching of $G(V\setminus S)$, the term $C(S,T)$ is the additional number of deficiencies due to critical components. $2(|V| - k)$ deficiencies are allowed, the rest must be accounted for by edges coming from the set S and any simple 2-matching contains at most $2|S|$ such edges. So the inequality is necessary.

The inequalities for all $S,T \subseteq V$ such that $S \cap T = \emptyset$ are sufficient, for assume that no simple 2-matching has cardinality k. Apply the algorithm to the graph and consider the alternating forest at termination. Define S as the set of odd nodes in the forest and T as the set of even real nodes of the forest. Then, since $2|S|$ edges go from S to $V\setminus S$, the RHS of the inequality minus $2|S|$ is precisely the number of deficiencies in an optimal simple 2-matching. But $2(|V| - k)$ is less than this since G does not have a simple 2-matching of cardinality k. (Note that even nodes of degree 0 in $G\setminus S$, need two edges from S. This is counted in $\Sigma_{i\in T}(2 - |\delta_{G\setminus S}(i)|)$. Even nodes of degree 1 in $G\setminus S$ are tips and as tips need 1 edge from S and the sum counts this. Even nodes of degree 2 in $G\setminus S$ are also tips, but, as tips, need nothing from S. Whatever they need is counted in $C(S,T)$. Even nodes of degree > 2 eliminate $|\delta_{C\setminus S}(i)| - 2$ blossoms of the need for an edge from S, thus the sum becomes negative.)

As a corollary of this theorem we get a special case of a well-known theorem of Tutte [52].

<u>Theorem 2.3</u>. The graph G has a 2-factor if and only if, $\forall\ S,T \subseteq V$ such that $S \cap T = \emptyset$,

$$2|S| \geq \sum_{i\in T} (2 - |\delta_{G(V\setminus S)}(i)|) + O(S,T)$$

where $O(S,T)$ is the number of odd components in the graph obtained from $G(V\setminus S)$ by splitting the nodes of T. (An <u>odd component</u> H is one such that $\Sigma_{v\in V(H)} \min(2,|\delta_H(v)|)$ is an odd integer.)

We may restate Theorem 2.2 as a max-min theorem as follows (see Edmonds [65a]).

<u>Theorem 2.4</u>. For any graph $G = (V,E)$, the maximum cardinality of a simple 2-matching is equal to

$$\min_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} \left\{ |V| + |S| - \frac{1}{2}\left[\sum_{i\in T} (2 - |\delta_{G\setminus S}(i)|) + C(S,T)\right]\right\}.$$

<u>Proof</u>: Let $\nu(G)$ be the maximum cardinality of a simple 2-matching in G. First we show that

$$(2.12)\qquad \nu(G) \leq \min_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} \left\{ |V| + |S| - \frac{1}{2}\left[\sum_{i\in T} (2 - |\delta_{G\setminus S}(i)|) + C(S,T)\right]\right\}.$$

For any $S,T \subseteq V$, $S \cap T$, the number of deficiencies (a node can be deficient by 1 or 2) is bounded below by $\Sigma_{i\in T}(2 - |\delta_{G\setminus S}(i)|) + C(S,T) - 2|S|$. Therefore, for any simple 2-matching x, we must have

$$x(V) = 2|V| - \left[\sum_{i\in T}(2 - |\delta_{G\setminus S}(i)|) + C(S,T) - 2|S|\right]$$

or

$$\nu(G) = \frac{x(V)}{2} \leq |V| + |S| - \frac{1}{2}\left[\sum_{i\in T}(2 - |\delta_{G\setminus S}(i)|) + C(S,T)\right].$$

To show that equality holds in (2.12) consider the alternating forest after applying the cardinality algorithm to G. If we let S be the set of odd nodes and T be the set of even real nodes, then $\Sigma_{i\in T}(2 - |\delta_{G\setminus S}(i)|) + C(S,T) - 2|S|$ is exactly the number of deficiencies. Hence we get equality in (2.12).

The validity of the cardinality algorithm also provides a simple proof of the following theorem.

Theorem 2.5. Every critical graph is a blossom.

Proof: Let G be a critical graph and let M be a saturating simple 2-matching. Let r be the deficient node in G. Apply the algorithm. If, at termination, there is some odd node in $\mathcal{S}$ then this node v is saturated $(x(\delta(v)) = 2)$ in every maximum simple 2-matching, a contradiction to the fact that G is critical. So, at termination, the alternating forest $\mathcal{S}$ reduces to one shrunk root and the tips of the corresponding blossom. In fact every node of G is in $\mathcal{S}$. (Otherwise these other nodes could never be deficient in a maximum simple 2-matching since one deficiency must already occur in the shrunk blossom.) So G reduces to the shrunk root and its tips, namely it is a blossom.

Chapter 3

THE MAXIMUM CARDINALITY TRIANGLE-FREE SIMPLE 2-MATCHING PROBLEM

Section 0. Introduction

In this chapter we solve the maximum cardinality triangle-free simple 2-matching problem. In Section 1 we discuss blossom trees, an important structure in the algorithm. In Section 2, we give the algorithm. The validity of the algorithm is then proved in Section 3. Section 4 contains a Tutte-type theorem which characterizes graphs with perfect triangle-free simple 2-matchings. Finally, in Section 5, we briefly discuss the weighted problem and give a conjecture concerning its polyhedral characterization.

Section 1. Blossom Trees

In this section we define the structures which will be shrunk in the course of the algorithm and then define a notion of criticality which will justify our shrinking of these structures.

Grötschel and Pulleyblank [81] defined the notion of a clique tree. They found this notion useful for studying traveling salesman problems in complete graphs. Their definition extends to general graphs although the word "clique" becomes inappropriate. Instead we will use the term blossom tree.

A *simple blossom tree* is a connected graph B such that

(i) it contains at least three nodes,

(ii) centers and petals are connected node induced subgraphs,

(iii) each petal is a block of B (i.e. a maximal connected subgraph without cutnodes),

(iv) a petal and a center can have at most one common node. Such a node is a cutnode of B,

(v) no two centers have a common node,

(vi) no two petals have a common node,

(vii) each center is adjacent to an odd number of petals,

(viii) each petal contains at least one node which belongs to no center.

For the relaxation of the Hamilton cycle problem studied in the last chapter, namely the simple 2-matching problem, the relevant structures were blossoms. Note that blossoms are exactly those simple blossom trees such that each petal contains two nodes. (Then the connectivity condition and properties (v) and (viii) imply that the simple blossom tree has only one center.)

For the triangle-free simple 2-matching problem, the relevant structures are the simple blossom trees such that

(ix) each petal contains two or three nodes.

These structures can have several centers and clearly generalize the blossoms of Section 2. [See Figure 3.1.] Note that since a petal is a block and has 2 or 3 nodes it must be induced either by an edge or by a triangle. Therefore petals will be called either edge petals or triangular petals. A graph which satisfies properties (i)-(ix) is called a simple blossom tree with edge or triangular petals. For simplicity these graphs will be refered to as blossom trees in Sections 1 - 4.

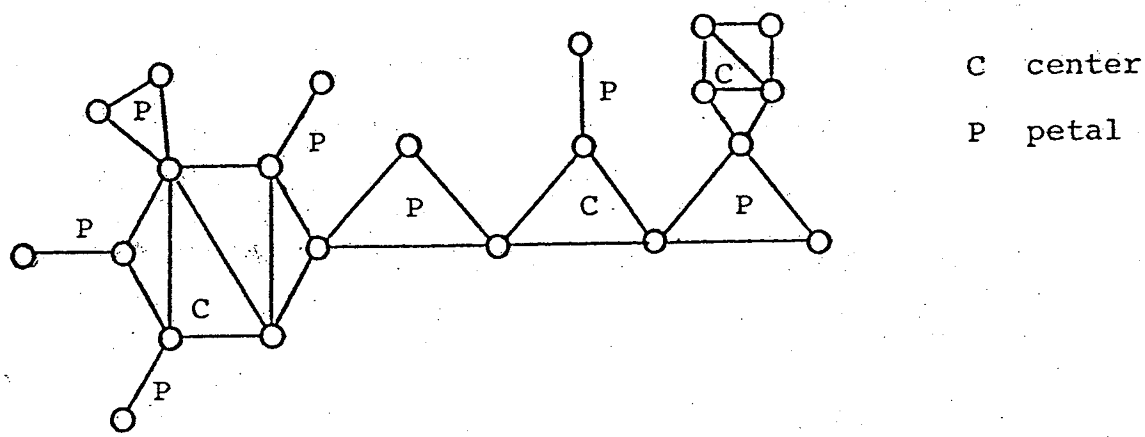


Figure 3.1. A Blossom Tree

Note that, according to the definition, a blossom tree can consist of a single petal. By properties (i) and (ix), this petal must then be a triangle.

The nodes of a blossom tree which do not belong to any center are called the tips of the blossom tree. By property (viii) each petal contains at least one tip. Therefore an edge petal is adjacent to one center whereas a triangular petal is adjacent to either zero, one or two petals. A pendent petal is adjacent to exactly one center.

Proposition 3.1. Let B be a blossom tree defined on n nodes, with edge set $E(B)$ and with p petals. Then every triangle-free simple 2-matching M with incidence vector x satisfies the inequality

(3.1) $$2x(E(B)) \leq 2n - p - 1.$$

Proof: We will prove this by induction on the number of centers. First, when B has one center, we add the inequalities (i) $x(\delta(v)) \leq 2$ for every node v of the center, (ii) $x(T) \leq 2$ for every triangular petal T, (iii) $x(e) \leq 1$ for every edge petal and the edge joining the two tips of every triangular petal. This yields the valid inequality

$$2x(E(B)) \leq 2s + 3t + u,$$

where s is the number of nodes in the center, t is the number of triangular petals and u is the number of edge petals. Equivalently, in terms of n and p, this inequality reads

$$2x(E(B)) \leq 2n - p.$$

Since the left-hand side is an even integer and p is odd, the following inequality is also valid: $2x(E(B)) \leq 2n - p - 1$.

Now we prove the result for a blossom tree B with $c \geq 2$ centers assuming the result true for less than c centers. Let T be a nonpendent triangle with edges e_1, e_2, e_3. We assume that $e_3 = (u_1 u_2)$ is the edge which joins the two centers, $e_1 = (u_1 v)$ and $e_2 = (u_2 v)$.

Case 1: $x(e_3) = 0$. Then remove the edge e_3 and split the node v. We then have two blossom trees B_1 and B_2 having each fewer than c centers, so we can use the induction

$$2x(E(B_1)) \leq 2n_1 - p_1 - 1$$

$$2x(E(B_2)) \leq 2n_2 - p_2 - 1.$$

Therefore, using the fact that $x(e_3) = 0$, we have

$$2x(e_3) + 2x(E(B_1)) + 2x(E(B_2)) \leq 2(n_1 + n_1) - (p_1 + p_2) - 2$$

$$2x(E(B)) \leq 2(n + 1) - (p + 1) - 2$$

$$2x(E(B)) \leq 2n - p - 1.$$

<u>Case 2</u>: $x(e_3) = 1$. Again use the triangle T to break the blossom tree B into 2 blossom trees B_1 and B_2, only this time T is assumed to be a pendent triangular petal of B_1 on the one hand and e_3 is assumed to be an edge petal of B_2. Thus, by adding

$$2x(E(B_1)) \leq 2n_1 - p_1 - 1$$

and

$$2x(E(B_2)) \leq 2n_2 - p_2 - 1,$$

we get the valid inequality

$$2x(e_3) + 2x(E(B)) \leq 2(n_1 + n_2) - (p_1 + p_2) - 2$$

$$2 + 2x(E(B)) \leq 2(n + 2) - (p + 1) - 2$$

$$2x(E(B)) \leq 2n - p - 1.$$

Let B be a blossom tree and M a triangle-free simple 2-matching of B with incidence vector x. M is said to <u>saturate</u> B if the inequality (3.1) holds with equality.

Now we will define critical blossom trees. Define the vector $b^B = (b_v^B : v \in V(B))$ as follows: $b_v^B = 1$ for exactly one tip v in each petal of B, and $b_v^B = 2$ for all the other needs. Consider a triangle-free simple 2-matching M of B which satisfies

$$(3.2) \qquad x(\delta_B(v)) \leq b_v^B \quad \text{for every} \quad v \in V(B).$$

Note that, if M saturates B, then $x(E(B)) = \lfloor b^B(V(B))/2 \rfloor$, since $b^B(V(B)) = 2|V(B)| - p$ where p is the number of petals of B. Since $b^B(V(B))$ is odd, then there exists a unique $i \in V(B)$ such that $x(\delta_B(i)) = b_i^B - 1$. For all the other nodes $v \in V(B)$, $v \neq i$, $x(\delta_B(v)) = b_v^B$. We say that M is <u>deficient</u> in B at node i. Let $V^*(B) = V(B) - \{v \in V(B) : b_v^B = 1$ and v belongs to a pendent triangular petal$\}$. If, for every node $u \in V^*(B)$, there exists a saturating triangle-free simple 2-matching M which satisfies (3.2) and which is deficient at u, then the blossom tree B is said to be <u>critical</u>.

In order to justify our name "blossom" tree, we give the following proposition.

<u>Proposition 3.2</u>. Let B be a critical blossom tree. Let $S_1, \ldots, S_k$ be the centers of B. Define $B_j = (S_j, T_j)$ as the blossom obtained from B by using the center S_j and by

replacing each petal of B adjacent to S_j by an edge petal incident with the same node of S_j. Then B_j is a critical blossom for every $j = 1,\ldots,k$.

Proof: By induction on the number of centers of B.

First consider a blossom tree B with one center and let B_1 be as defined in the proposition. Properties (iv), (vi), (vii) and (viii) of the blossom tree B show that B_1 is a blossom. Now we show that, if B is critical, then B_1 is also critical. Let M be a saturating matching of B. If M is deficient at a node i which is in the center S_1 or which is the tip of an edge petal, then M can be transformed into a saturating matching M_1 of B_1 by taking $M_1 \equiv M$ except in triangular petals of B; in each triangular petal of B, replace the two edges of M by one edge of M_1 in the corresponding edge petal of B_1. So now assume that M is deficient at the tip of a triangular petal, i.e. $x(\delta_B(i)) = x(\delta_B(j)) = 1$ where i and j are the tips of a petal i,j,k. Two cases will be considered depending on whether $ij \in M$ or not.

Case 1: $ij \in M$. Then we perform the transformation of M into M_1 exactly as above except in the triangular petal i,j,k, where the edge $ij \in M$ is not replaced by any edge of M_1 in the corresponding edge petal. So the tip of that petal is deficient in M_1.

Case 2: $ij \notin M$. Then $ik, jk \in M$. Consider a saturating matching M' of B deficient at some node $z \in S_1$ which is adjacent to k. Consider an alternating path of $M \Delta M'$

($\equiv M \cup M' - M \cap M'$) starting at z with an edge of $M \backslash M'$. This path must terminate at node i with the edge $ji \in M' \backslash M$, since i is the only node deficient in M but not in M'. Let $\tilde{M}$ be obtained by taking $\tilde{M} \equiv M$ outside the alternating path, $\tilde{M} \equiv M'$ on the alternating path, removing the edge ik from $\tilde{M}$ and adding kz to $\tilde{M}$. This new saturating matching $\tilde{M}$ is deficient at i and it contains the edge ij so we can apply the transformation from $\tilde{M}$ to M_1 defined in Case 1.

Now assume that the blossom tree B contains at least two centers. As noted previously, each B_j must be a blossom. Now assume that B is critical. We must show that B_j is critical. Let S_1 be a center different from S_j and which is adjacent to only one nonpendent triangle, say tuv, where t is the tip and $u \in S_1$. We will show that $\tilde{B}$ obtained by removing $S_1 \backslash \{u\}$ and all the pendent petals of S_1 is also a critical blossom tree, proving by induction that B_j is a critical blossom.

Consider a saturating matching M of B deficient at a node i of $\tilde{B} \backslash \{t,u\}$. Note that, whether $i = v$ or not, the restriction of M to $\tilde{B}_1$ formed by S_1 and its petals must be deficient at node v, since all the other nodes of $\tilde{B}_1$ are saturated by M in $\tilde{B}_1$. So either $tu, uv \in M$ or $tv \in M$. In both cases, consider the restriction of M to $\tilde{B}$ and in the second case replace the edge tv by tu, uv. This matching saturates $\tilde{B}$ and is deficient at node i.

Now assume that M is a saturating simple 2-matching deficient at node t. Since $x(\delta_B(t)) = 0$, $tu, tv \notin M$. Note that $uv \notin M$ either, since otherwise $\tilde{B}_1'$ obtained from $\tilde{B}_1$ by replacing the petal tuv by uv would be perfectly matched by M, a contradiction to the fact that $\tilde{B}_1'$ is a blossom tree. Now we consider the restriction of M to $\tilde{B}$ and we add the edge ut to it. This is a matching deficient at t in $\tilde{B}$. So $\tilde{B}$ is a critical blossom.

Section 2. The Alternating Structure

For finding maximum triangle-free simple 2-matchings, we make use of an alternating structure which plays a similar role to that of the alternating forest constructed in the simple 2-matching algorithm. First we define the surface graph.

(3.3) A surface graph $\tilde{G}$ is obtained from G by shrinking pairwise disjoint sets of nodes $S_1, \ldots, S_k$ of G, each set S_i being shrunk to a different node of $\tilde{G}$, say s_i. The sets S_i will be the centers of a family of critical blossom trees of G, each of which is saturated by M. After shrinking their centers, these blossom trees will be referred to as shrunk blossom trees. See Figure 3.2.

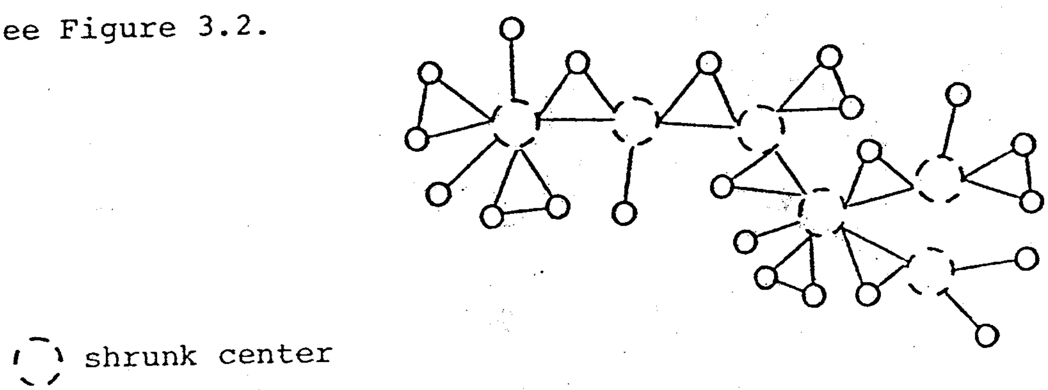


Figure 3.2. A Shrunk Blossom Tree

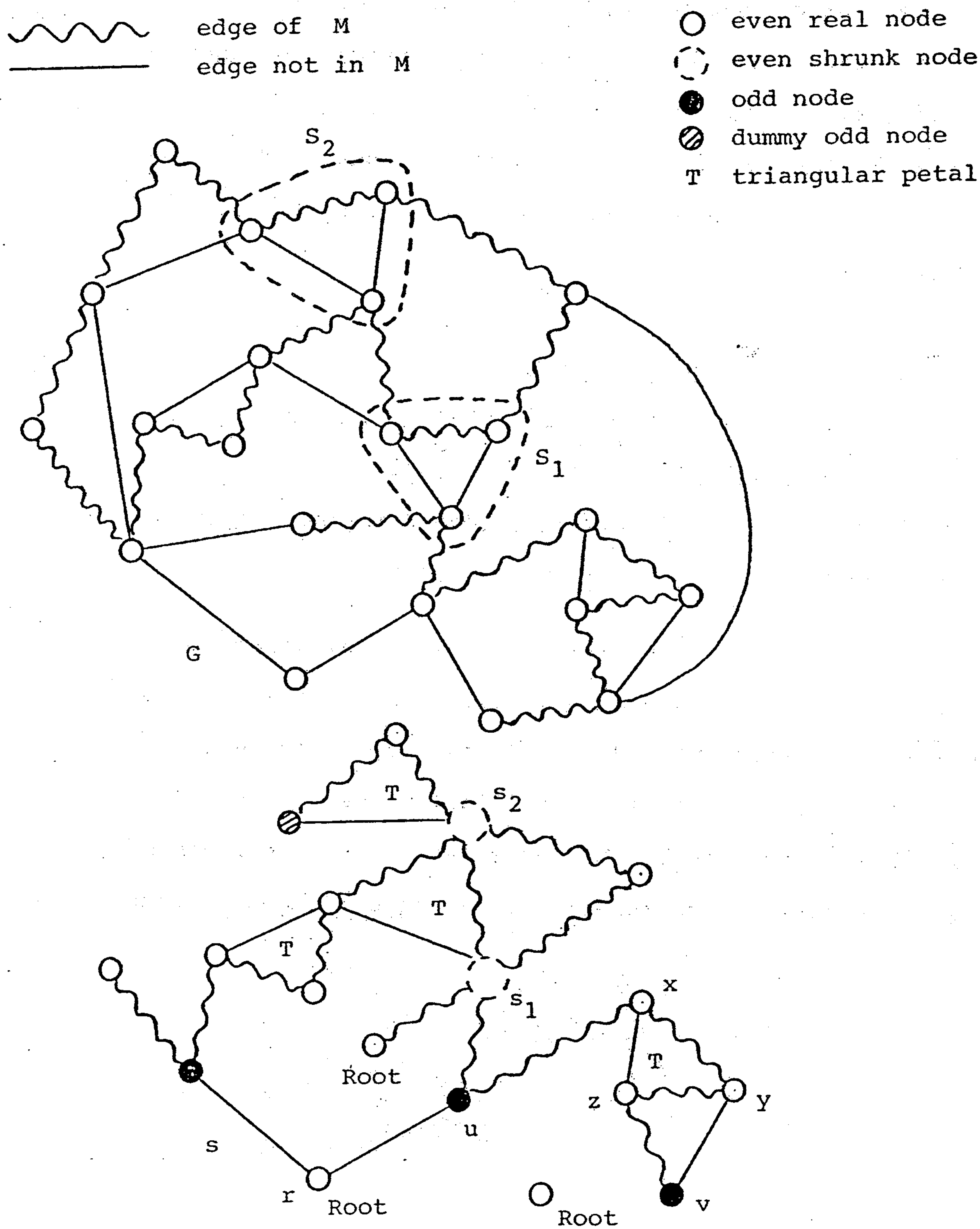


Figure 3.3. A Graph G, a Triangle-Free Simple 2-Matching M and an Alternating Structure S

An <u>alternating structure</u> is defined with respect to a triangle-free simple 2-matching M as follows (see Figure 3.3).

(3.4) The alternating structure $\mathcal{S}$ is a subgraph of $\tilde{G}$. Thus the alternating structure has two types of nodes, <u>real</u> nodes which are simply nodes of G, and <u>shrunk</u> nodes. A node of $\tilde{G}$ is deficient if either it is a deficient real node (i.e. $x(\delta(v)) < 2$) or else it is a shrunk node obtained from shrinking a set S_i which contains a deficient node of G. Every deficient node of $\tilde{G}$ is called a <u>root</u> of $\mathcal{S}$.

(3.5) The edges of the alternating structure $\mathcal{S}$ are edges of $\tilde{G}$ and therefore edges of G. If an edge j of $\mathcal{S}$ joins two nodes v and w and if, in $\tilde{G}$, an edge of M joins v to w, then $j \in M$. Certain edges or triangles of $\mathcal{S}$ will be identified as petals. If a triangle of $\mathcal{S}$ is identified as a petal, then these three edges also form a triangle in G. Each petal of $\mathcal{S}$ will belong to exactly one blossom tree of $\mathcal{S}$.

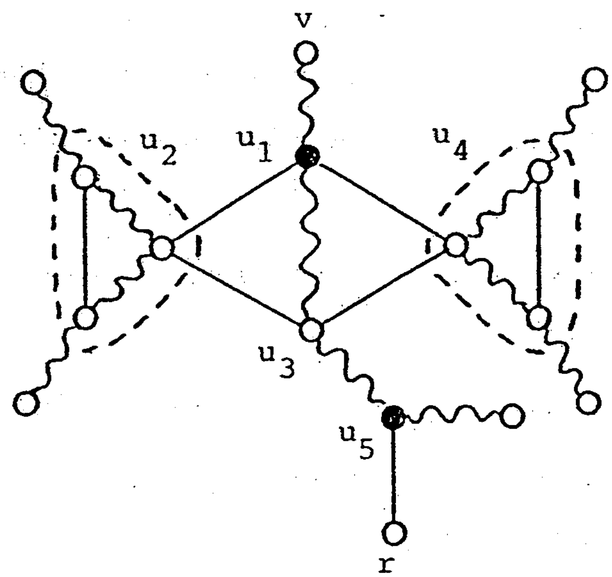
(3.6) Each node of the alternating structure $\mathcal{S}$ is called either <u>even</u>, <u>odd</u> or <u>dummy odd</u>. The roots are even nodes. The shrunk nodes are also even. So the odd and dummy odd nodes are real nodes of $\mathcal{S}$ and they are saturated in G. Odd nodes are saturated in $\mathcal{S}$ whereas dummy odd nodes are deficient in $\mathcal{S}$. Any dummy odd node v belongs to a triangular petal uvw of $\mathcal{S}$ such that the nodes u and w are even and at least one of them is a real node of $\mathcal{S}$. When growing the structure, dummy odd nodes may become even or odd.

(3.7) Every odd or dummy odd node v of $\mathcal{S}$ has a unique predecessor node $p(v)$ which is an even node of $\mathcal{S}$ such that $vp(v)$ is an edge of $\mathcal{S}$ and $vp(v) \notin M$.

Every even node v of $\mathcal{S}$ has a unique predecessor path $P_v = (v = v_1, v_2, \ldots, v_{k-1}, v_k = r)$ joining v to a root r, where v_i are nodes of $\mathcal{S}$ and $v_i v_{i+1}$ are edges of $\mathcal{S}$, for $i = 1,\ldots, k-1$. The path P_v cannot have duplicated edges, but it can have duplicated nodes, thus forming loops. However all such loops have length three. The node v_2 in P_v is called the predecessor node of v. The node v_k is a root whereas the nodes $v_1, v_2, \ldots, v_{k-2}$ are not. The node v_{k-1} may be a root. Then $v_{k-2}v_{k-1}v_k$ is a petal of $\mathcal{S}$ with $v_{k-2}v_{k-1}$ and $v_{k-2}v_k \in M$ and $v_{k-1}v_k \notin M$.

Let $\bar{P}_v$ be the set of edges obtained by removing any edge of P_v which belongs to a petal of $\mathcal{S}$. The set $\bar{P}_v$ contains an even number of edges (this is the reason why we call v and even node of $\mathcal{S}$). Along the path P_v, the edges of $\bar{P}_v$ are alternately in and out of M, the first edge of $\bar{P}_v$ being in M and the last being out of M.

The notation P_v^u denotes the portion of the path P_v between u and v. The notation P_v^{-1} will be used to represent the path $(r = v_k, v_{k-1}, \ldots, v_2, v_1 = v)$. An example:



$$P_v = (v, u_1, u_2, u_3, u_1, u_4, u_3, u_5, r)$$

As another example, let us define the predecessor nodes or paths of some of the nodes in the alternating structure of Figure 3.3.

$$p(u) = r$$
$$p(v) = y$$
$$P_{s_1} = (s_1, u, r)$$
$$P_{s_2} = (s_2, s_1, u, r) = (s_2, P_{s_1})$$
$$P_x = (x, u, r)$$
$$P_y = (y, z, x, u, r) = (y, z, P_x)$$
$$P_z = (z, v, y, z, x, u, r) = (z, v, y, z, P_x).$$

For any even node of S, the path P_v is such that, by alternating the edges of $\overline{P}_v$ in and out of M (and modifying M appropiratelv within the blossom trees traversed by P_v), another triangle-free simple 2-matching with the same cardinality as M is produced, leaving the node v deficient. Of course, P_v must satisfy the following properties with respect to any triangle e,f,g of G.

(i) If $e,f,g \notin M$, then at most two of the three edges e,f,g belong to P_v;

(ii) If $e,f \notin M$ and $g \in M$, then either $g \in P_v$ or at most one of the two edges e,f belong to P_v;

(iii) If $e \notin M$ and $e,f \in M$, then either $e \notin P_v$ or at least one of the two edges f,g belongs to P_v.

Before adding an edge to the structure, the algorithm will seek triangles that could potentially violate conditions (i), (ii) or (iii). Such triangles will be called <u>blocking</u> <u>triangles</u>. (A precise definition will be given in the course of the algorithm.)

(3.8) Let u be a shrunk node of the alternating structure. If u is different from a root, then the nodes of G belonging to u are all saturated by M; whereas if u is a root one of these nodes is deficient by one.

The tips of the petals of a shrunk blossom tree are nodes of $\mathcal{S}$. The tip of an edge petal is always an even node of $\mathcal{S}$ which may or may not be shrunk. The tip of a nonpendent triangular petal is a real even node of $\mathcal{S}$. The two tips of a pendent triangular petal are real nodes of $\mathcal{S}$ and at least one of the two is an even node of $\mathcal{S}$. The petals of the structure are edge disjoint but not necessarily node disjoint. In Figure 3.4 we give all the possible combinations of nodes that a triangular petal can have.

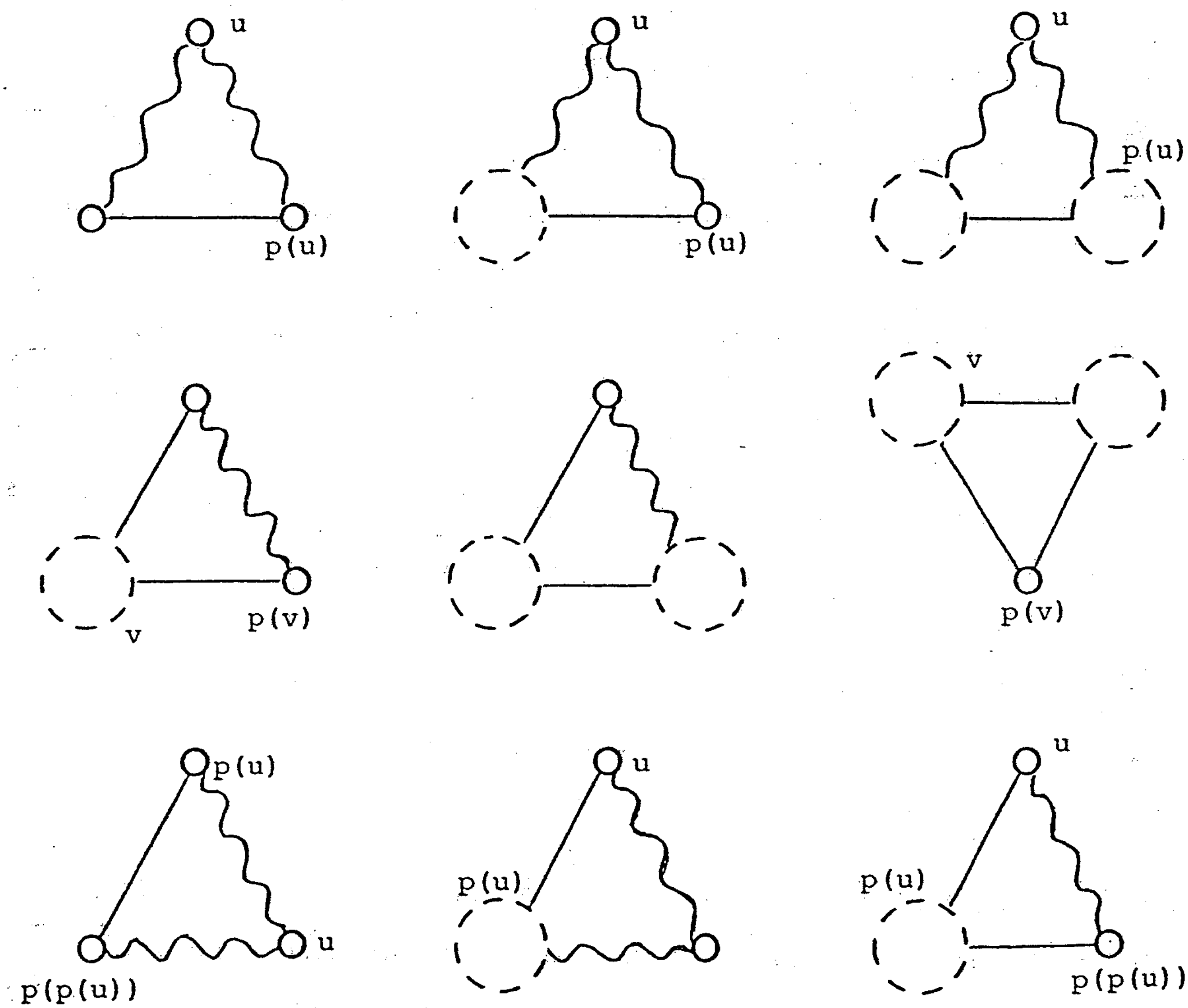


Figure 3.4. The Different Types of Triangular Petals

Finally, before stating the algorithm, we list some properties of alternating structures which will be essential in proving the validity of the algorithm.

(3.9) If two even nodes are adjacent in $\mathcal{S}$, and

(i) they are both strunk; then they are centers of the same blossom tree joined by a triangular petal or they are centers of different blossom trees joined by an edge petal of one of the blossom trees (i.e. one of the shrunk nodes is also the tip of an edge petal of the other center);

(ii) one is real and one is shrunk; then one is a center of a blossom tree and the other is a tip of the same blossom tree;

(iii) they are both real; then they are two tips of a triangular petal. (Note the special case where a triangular petal occurs in $\mathcal{S}$ with three real nodes. Then the three nodes are said to be tips.)

(3.10) Dummy odd nodes have degree two in $\mathcal{S}$. One of these edges is a matching edge, the other is a nonmatching edge and they form a triangular petal of $\mathcal{S}$ with a third edge of $\mathcal{S}$. Since dummy odd nodes are saturated in G, they are adjacent to one matching edge of G which does not belong to $\mathcal{S}$.

(3.11) Odd nodes have degree three or four in $\mathcal{S}$ and two of the incident edges are matching edges in $\mathcal{S}$. Odd nodes do not belong to any petal of $\mathcal{S}$.

Now we describe the algorithm.

Section 3. The Algorithm

Step 0 [Initialization of M]: Let M be any triangle-free simple 2-matching of G. For example we could take $M = \emptyset$. Go to Step 1.

Step 1 [Initialization of $\mathcal{S}$]: If M is a 2-factor, stop. Otherwise, let $\mathcal{S}$ be any non-empty alternating structure relative to M. For example we could take $\mathcal{S}$ to comprise simply the collection of deficient nodes of G; these nodes are the roots of $\mathcal{S}$ and are even. Go to Step 2.

Step 2 [Edge Selection]: If every edge of $\tilde{G}$ which is incident with an even node of $\mathcal{S}$ is either an edge of M or has an odd node of $\mathcal{S}$ as its other endnode, then the triangle-free simple 2-matching M is maximum. Stop.

Otherwise, select an edge $j \notin M$ joining an even node v of $\mathcal{S}$ to a node w which is not an odd node of $\mathcal{S}$. Let v' and w' be the endnodes of edge j in G. ($v = v'$ and/or $w = w'$ if S and/or w are real nodes of $\mathcal{S}$.) We will distinguish several cases depending on whether the node w is (i) not in $\mathcal{S}$, (ii) even or (iii) dummy odd. It will be useful to define a blocking triangle for the edge j: (i) if $w \notin \mathcal{S}$, or if w is a dummy odd node of $\mathcal{S}$, a triangle T of G is said to be blocking if $P_v \cup \{j\}$ contains all the nonmatching edges of T but not its matching edges; (ii) if w is even, a triangle T is said to be blocking if $P_v \cup \{j\}$, $P_w \cup \{j\}$, $P_v \cup \{j,(w,p(w))\}$, or $P_w \cup \{j,(v,p(v))\}$ contains the nonmatching edges of T but not its matching edges. Thus the triangle T "blocks" a matching augmentation through the edge j. (The case when w is even, is somewhat more complex because we are in effect checking if a shrinking as well as an augmentation could take place; that is, if v and w have the same roots and a shrinking should be considered as in the simple 2-matching algorithm, we must make sure the shrinking is valid as in Theorem 2.1 where augmentations may have to be made from a node in P_v, for example, through j and then along P_w to the root without creating a triangle.)

We will distinguish the cases where the edge j does not create a blocking triangle, or it creates a blocking triangle with two matching edges, one matching edge or no matching edge. In addition, when a blocking triangle is formed, we will distinguish between the cases where this triangle shares an edge with an existing triangular petal or not. (In fact, the triangle may share edges with two existing triangular petals.) Note that the blocking triangles wer are discussing have all three edges in $\tilde{G}$. It is possible that an augmentation may create a triangle which has one edge in $\tilde{G}$. We show how these can be dealt with when proving the validity of the algorithm. The exhaustive list of cases is the following.

Case 1: $w \notin S$ and j does not form a blocking triangle; go to Step 3.1.

Case 2: $w \notin S$ and j forms a blocking triangle with two matching edges which does not share an edge with an existing triangular petal; go to Step 3.2.

Case 3: $w \notin \mathcal{S}$ and j forms a blocking triangle with two matching edges which shares an edge with an existing triangular petal; go to Step 3.3.

Case 4: $w \notin \mathcal{S}$ and j forms a blocking triangle with one matching edge which does not share an edge with an existing triangular petal; go to Step 3.4.

Case 5: $w \notin \mathcal{S}$ and j forms a blocking triangle with one matching edge which shares an edge with an existing triangular petal; go to Step 3.5.

Note that the case "$w \notin \mathcal{S}$ and j forms a blocking triangle with no matching edge" cannot occur since, in such a triangle, the edge which is incident with w and which is different from j does not belong to P_v, contradicting the assumption that the triangle is blocking. So we continue with the cases where w is even.

Case 6: w is even and j does not form a blocking triangle; go to Step 3.6.

Case 7: w is even and j forms a blocking triangle with two matching edges which does not share an edge with an existing triangular petal; go to Step 3.7.

Case 8: w is even and j forms a blocking triangle which has two matching edges and which shares an edge with one or two existing triangular petals; go to Step 3.8.

Case 9: w is even and j forms a blocking triangle with one matching edge which does not share an edge with an existing triangular petal; go to Step 3.9.

Case 10: w is even and j forms a blocking triangle which has one matching edge and which shares an edge with one or two existing triangular petals; go to Step 3.10.

Case 11: w is even and j forms a blocking triangle with no matching edge which does not share an edge with an existing triangular petal; go to Step 3.11.

Case 12: w is even and j forms a blocking triangle which has no matching edge and which shares an edge with one or two existing triangular petals; go to Step 3.12.

Case 13: w is a dummy odd node; go to Step 3.13.

Step 3.1 [$w \notin \mathcal{S}$ and j does not form a blocking triangle]: The node w is incident with two edges of M, say wu_1 and wu_2. Note that u_1 and u_2 either do not belong to $\mathcal{S}$, are real even nodes of $\mathcal{S}$ or dummy odd nodes of $\mathcal{S}$ (they cannot be shrunk or odd since this would imply that $w \in \mathcal{S}$, a contradiction). Make w an odd node of $\mathcal{S}$ with $p(w) = v$. For $i = 1,2$, if $u_i \notin \mathcal{S}$ or if u_i is a dummy odd node of $\mathcal{S}$, make u_i a real even node of $\mathcal{S}$ with $P_{u_i} = (u_i, w, P_v)$. Go to Step 2.

Step 3.2 [$w \notin \mathcal{S}$ and j forms a blocking triangle with two matching edges which does not share an edge with an existing triangular petal]: The node w is incident with two edges of M, say wu_1 and wu_2. Note that u_1 and u_2 either do not belong to $\mathcal{S}$, are real even nodes of $\mathcal{S}$ or dummy odd nodes of $\mathcal{S}$ (they cannot be shrunk or odd since this would imply that $w \in \mathcal{S}$, a contradiction). Without loss of generality, assume that $u_1v' \in M$. (We show in Proposition 3.3 that both u_1v' and $u_2v' \in M$ cannot occur.)

Make w a dummy odd node of $\mathcal{S}$ with $p(w) = v$. If $u_1 \notin \mathcal{S}$ or if u_1 is a dummy odd node of $\mathcal{S}$, make u_1 a real even node of $\mathcal{S}$ with $P_{u_1} = (u_1, w, P_v)$. (Note that we do not change the status of u_2 since an augmentation through w would produce a simple 2-matching containing the three edges of the triangle u_1vw.) Call the triangle u_1vw a triangular petal of $\mathcal{S}$. (See Figure 3.5.) Go to Step 2.

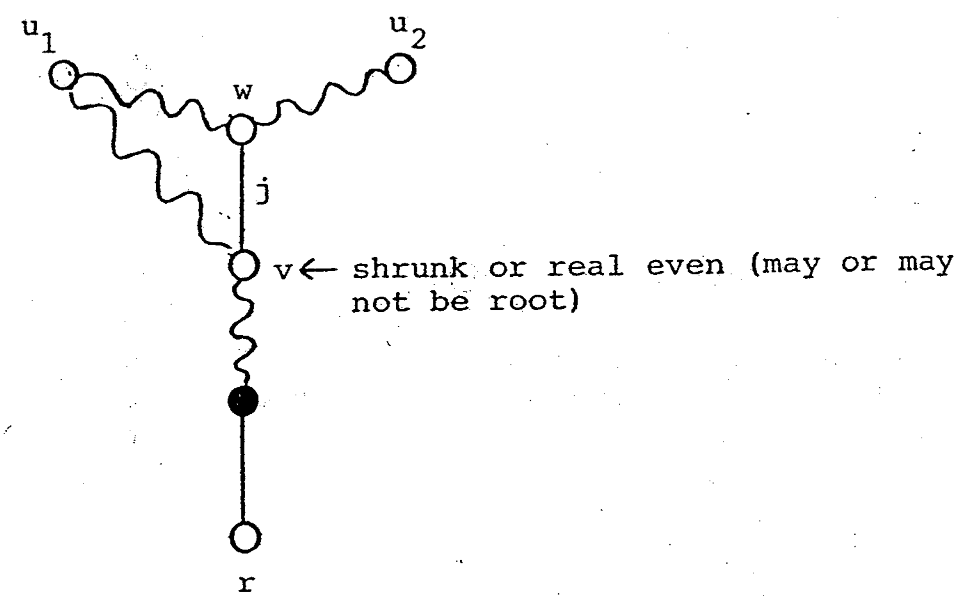


Figure 3.5

Step 3.3 [$w \notin \mathcal{S}$ and j forms a blocking triangle with two matching edges which share an edge with an existing triangular petal]: (Note that since $w \notin \mathcal{S}$ the blocking triangle can share only one edge with an existing triangular petal.)

The node w is incident with two edges of M, say wu_1 and wu_2. Without loss of generality, assume that $u_1v' \in M$. Note that, since $w \notin \mathcal{S}$, the edge u_1v' is the only edge of the triangle wvu_1 which can belong to an existing triangular petal. Let u_1vx be this triangular petal. (See Figure 3.6.)

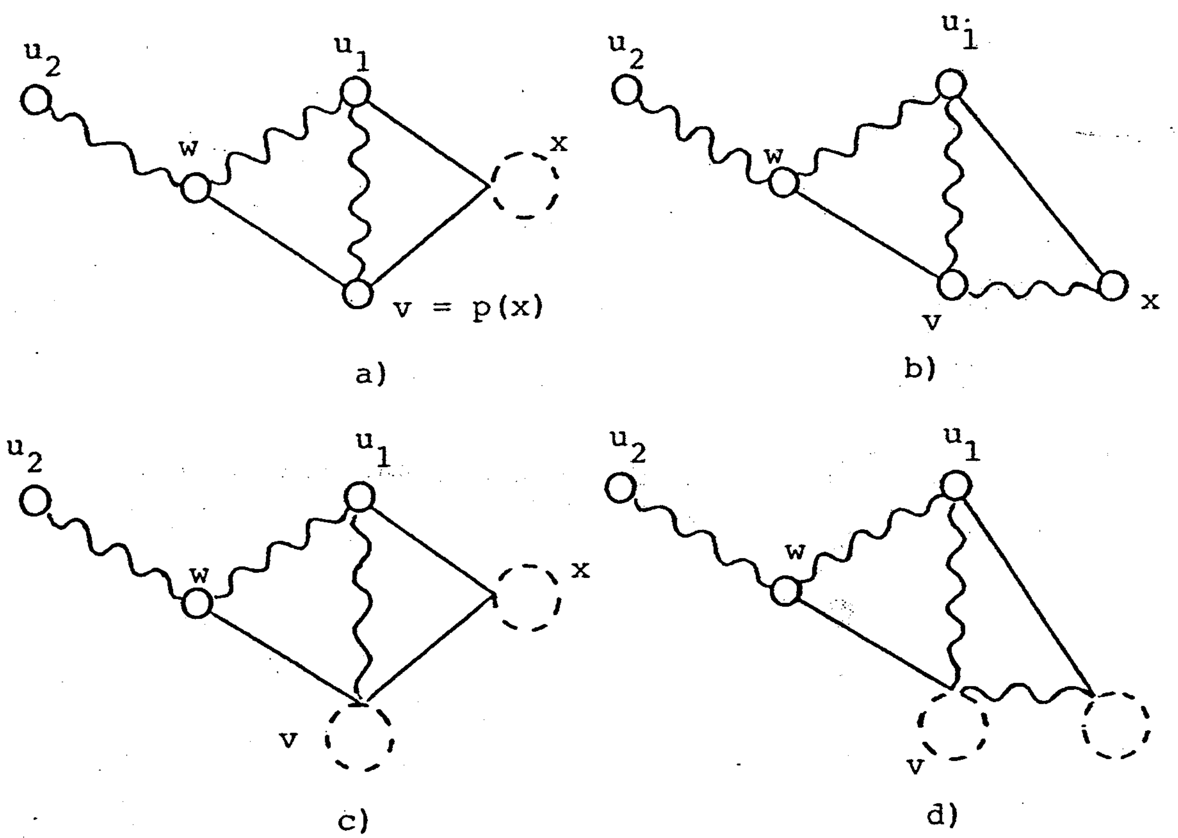


Figure 3.6

Note that in a) and b) of Fig. 3.6, v must be real even whereas in c) and d) v must be shrunk even. x is shrunk even in a), c), and d) but may be either real or shrunk even in b). In a) and c), v is the only node whose predecessor is not among the others (it may be a root). In b) and d), x is the only node whose predecessor is not among the others (again, it may be a root).

Make w an odd node of $\mathcal{S}$ with $p(w) = v$. If $u_2 \notin \mathcal{S}$ or if u_2 is a dummy odd node of $\mathcal{S}$, make u_2 a real even node of $\mathcal{S}$ with $P_{u_2} = (u_2,w,v,u_1,P_x)$ in cases a), b), and d) and $P_{u_2} = (u_2,w,v,u_1,x,P_v)$ in case c). If u_1 is a dummy odd node of $\mathcal{S}$, make u_1 a real even node of $\mathcal{S}$ with $P_{u_1} = (u_1,w,P_v)$. Go to Step 2.

Step 3.4 [$w \notin \mathcal{S}$ and j forms a blocking triangle with one matching edge which does not share an edge with an existing triangular petal]: The node w is incident with two edges of M, say wu_1 and wu_2. Let vwu_1 be the petal formed by j. Note that u_1 is a real even node of $\mathcal{S}$ whereas v is a shrunk node of $\mathcal{S}$. (This is the case since petals with one matching edge contain at least one shrunk node, and u_1 cannot be shrunk since otherwise we would have $w \in \mathcal{S}$, a contradiction. So v is shrunk and $p(v) = u_1$. Therefore u_1 is even. See Figure 3.7.)

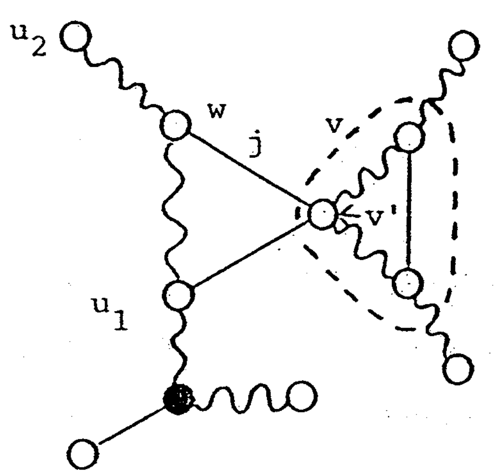


Figure 3.7

Make w a dummy odd node of $\mathcal{S}$ with $p(w) = v$. (Note that we do not change the status of u_2 since an augmentation through w would produce a simple 2-matching containing the three edges of the triangle wvu_1.) Call the triangle wvu_1 a triangular petal of $\mathcal{S}$. Go to Step 2.

Step 3.5 [$w \notin \mathcal{S}$ and j forms a blocking triangle with one matching edge which shares an edge with an existing triangular petal]: Note, since $w \notin \mathcal{S}$, the blocking triangle can share only one edge with an existing triangular petal.

Let us call the existing triangular petal u_1vx. Then we have only the following two possibilities:

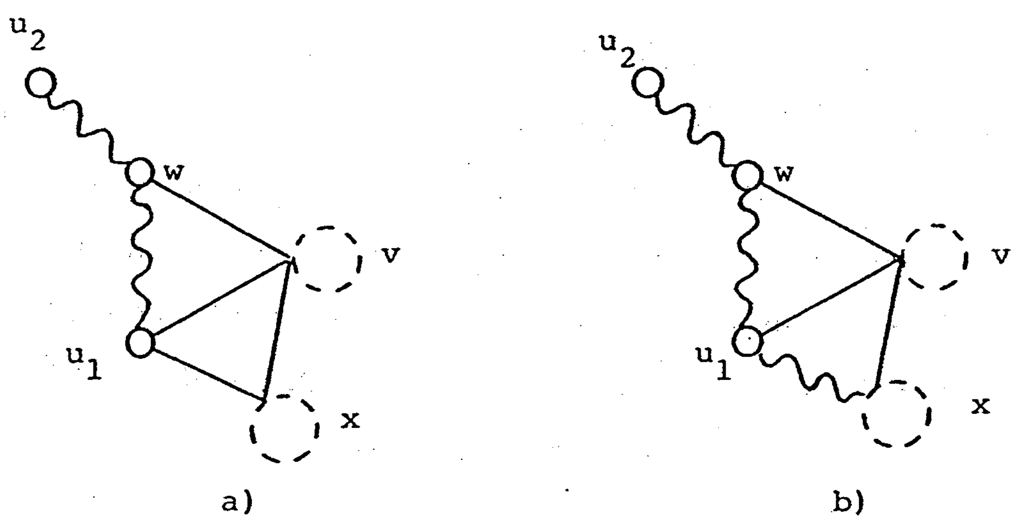


Figure 3.8

Note that u_1vx cannot have two matching edges since v must be shrunk and hence contain both matching edges incident with v'. By the same argument as in Step 3.4, the node u_1 is a real even node and $p(v) = u_1$. Make w an odd node of $\mathcal{S}$ with $p(w) = v$. If $u_2 \notin \mathcal{S}$ or if u_2 is a dummy odd node of $\mathcal{S}$, make u_2 an even node of $\mathcal{S}$. In case a), set $P_{u_2} = (u_2,w,v,x,P_{u_1})$ and in case b), set $P_{u_2} = (u_2,w,v,P_x)$. Go to Step 2.

<u>Step 3.6</u> [Coming from Step 2, w is even and j does not form a blocking triangle; let P_{vw} be reduced to the edge j].

<u>Case 3.6a</u>: If $\text{root}(v) \neq \text{root}(w)$, then let $P = (P_v^{-1},P_{vw},P_w)$ and go to Step 4.

Now suppose $\text{root}(v) = \text{root}(w)$. If P_v and P_w have a common edge, let b be the first node such that $bp(b)$ is a common edge on the paths P_v and P_w. If P_v and P_w do not have a common edge, let $b = \text{root}(v)$ $(= \text{root}(w))$.

<u>Case 3.6b</u>: If $b = \text{root}(v) = \text{root}(w)$ and if this node is a real node incident with no matching edge (i.e. it is deficient by two), then let $P = (P_v^{-1},P_{vw},P_w)$ and go to Step 4.

Let P_v^b and P_w^b be the portions of the paths P_v and P_w from v to b and from w to b, respectively.

<u>Case 3.6c</u>: If a root s is a node of a triangular petal sxy whose two other nodes x and y belong to P_v^b or P_w^b, then an augmenting path can again be found as follows. Let sx be the matching edge of the triangular petal which is incident with s and, assuming without loss of generality that $x \in P_v^b$, let P_v^x be the portion of P_v^b from v to x. Then let $P = (s,y,(P_v^x)^{-1},P_{vw},P_w)$ if sxy has two matching edges and let $P = (s,(P_v^x)^{-1},P_{vw},P_w)$ if sxy has one matching edge. Go to Step 4. (We show that P is valid in Theorem 3.1.)

Case 3.6d: Now assume that the assumptions of Cases 3.6a,b and c do not hold. Let S be defined as follows. A node $s \in G$ is in S if either

(i) s is a real node or inside a shrunk node on the path P_v^b or P_w^b, or

(ii) s is a real node or inside a shrunk node of a triangular petal whose two other nodes belong to P_v^b or P_w^b.

Go to Step 5.

Step 3.7 [w is even and j forms a blocking triangle with two matching edges which does not share an edge with an existing triangular petal].

Let uvw be the blocking triangle containing the edge j.

Case 3.7a: If $u \notin \mathcal{S}$, make u an even node of $\mathcal{S}$. If v or w is a root, say w is a root, then let $P_u = (u,v,w)$. If neither v nor w is a root, then arbitrarily set $P_u = (u,v,P_w)$. In both cases, call the triangle uvw a triangular petal. Go to Step 2.

Case 3.7b: If u is an even node of $\mathcal{S}$ and at least one of the nodes u,v,w is real, call the triangle uvw a triangular petal. Go to Step 2.

Case 3.7c: If u is an even node of $\mathcal{S}$ and u,v and w are three shrunk nodes of $\mathcal{S}$, then let S be the set of nodes of G contained in u,v and w. Go to Step 3.6 with $P_{vw} = (v,u,w)$.

Case 3.7d: (See Figure 3.9.) The node u is an odd node of $\mathcal{S}$ and at least one of the two nodes v or w does not have u as its predecessor.

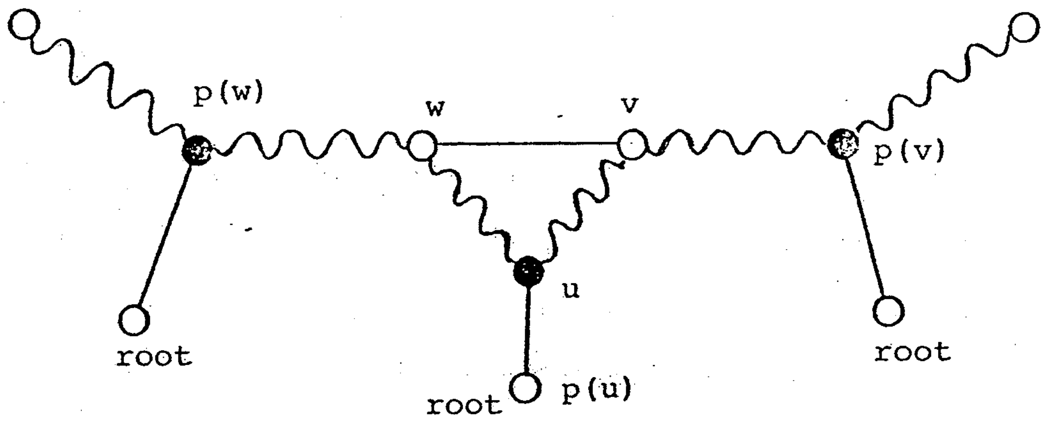


Figure 3.9

If $p(v) \neq u$, set $P_v = (v,u,P_{p(u)})$. If $p(w) \neq u$, set $P_w = (w,u,P_{p(u)})$. Go to Step 3.6 with vw.

Step 3.8 [w is even and j forms a blocking triangle which has two matching edges and which shares an edge with one or two existing triangular petals].

Let uvw be the blocking triangle containing the edge j; let uvx be the existing triangular petal, if there is just one, otherwise let uvx and uwz be the two existing triangular petals.

Case 3.8a: The node x is a dummy odd node. We have the following possibilities:

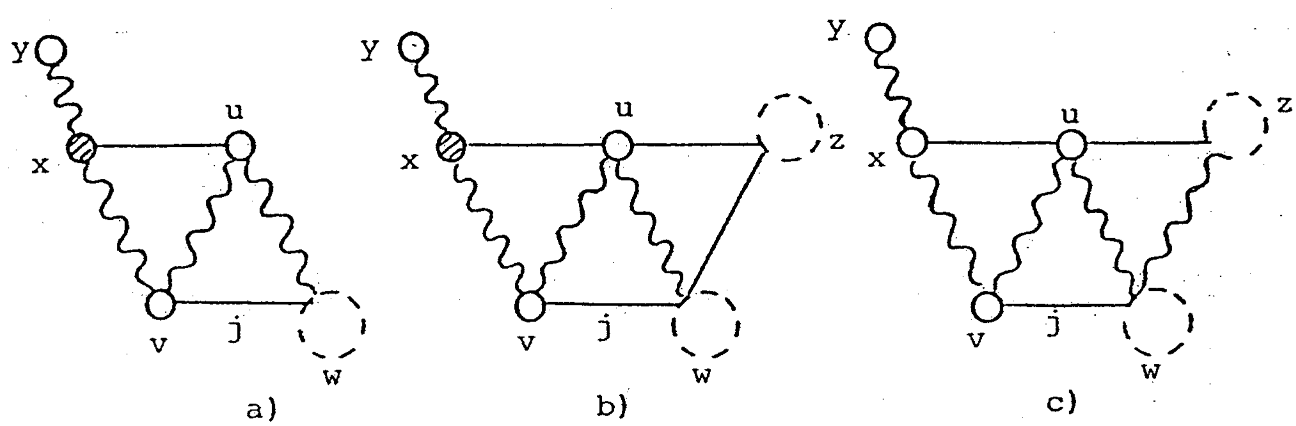


Figure 3.10

Note that $p(x) = u$. Hence u is an even node which is either real or shrunk. In either case $p(u) = w$ since v is a real even node. In order for u to be even, therefore, w must be shrunk. This implies that the predecessor of w is not u, v, or x. If there exists a node z as in c), then $p(w) = z$. So in b) $p(w) \neq z$.

In case a), make x an odd node of $\mathcal{S}$. If y is not in $\mathcal{S}$ or is a dummy odd node of $\mathcal{S}$, make y an even node with $P_y = (y,x,u,v,P_w)$. Make uvw a triangular petal of $\mathcal{S}$ and no longer call uvx a triangular petal. If $p(v) = u$, set $P_v = (v,P_x)$. Go to Step 2.

In case b), set $P_x = (x,v,w,u,z,P_w)$, $P_y = (y,x,u,v,P_w)$, $S = (u,v,w,x,z)$, $b = w$ and go to Step 5.

In case c), set $P_x = (x,v,w,u,P_z)$, $P_y = (y,x,u,v,w,P_z)$, $S = (u,v,w,x,z)$, $b = z$, and go to Step 5.

Case 3.8b: The node u is a dummy odd node. In this case there can be only one existing petal which shares an edge with uvw. We have the following possibilities:

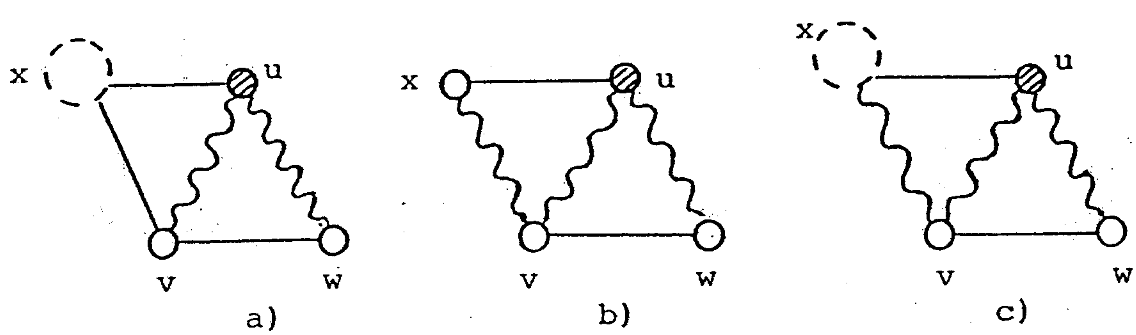


Figure 3.11

Note that v and w are real even nodes since u is dummy odd. x must be a shrunk even node when uvx has one matching edge, but may be either real or shrunk even when uvx has two matching edges.

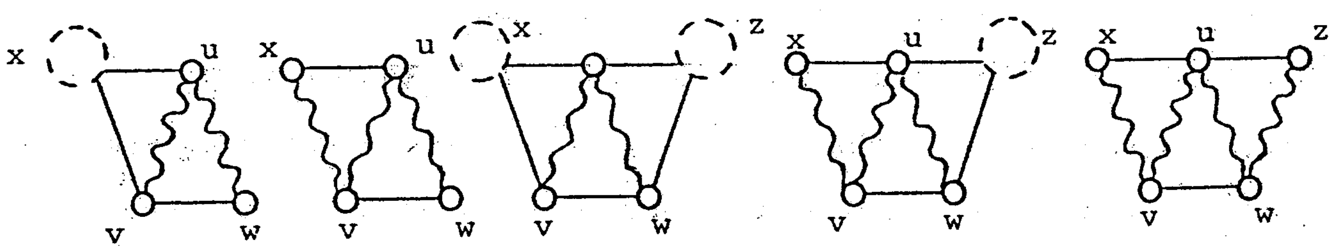
In all three cases, the two nodes which have degree 1 in the matching in the figure also have predecessors which are not among the other nodes (they may be roots). With the following, we go to Step 3.6 and either shrink or augment.

a): Set $P_{vw} = (v,x,u,v,w)$

b) and c): Set $P_{xw} = (x,u,v,w)$.

Also take $P_u = (u,v,P_w)$ in case, after Step 3.6, we go to Step 5.

<u>Case 3.8c</u>: uvw shares edges with one or two existing triangular petals uvx and, possibly, uwz where x or x and z are even. We have the following five possiblities:



a) b) c) d) e)

Figure 3.12

Note that a node which is saturated by the matching in the figure must have as its predecessor one of the two other endnodes of the matching edges incident with it. If the node is shrunk, then one matching edge is or was the base while the other is or was a petal. If the node is real, then its predecessor must be along a matching edge, and that predecessor must be shrunk since it is even. Therefore in each case there are one or two nodes whose predecessors are not among the other nodes and they must be nodes which have degree 1 in the matching in the figure. (The other nodes are shrunk even with a nonmatching base edge whose other endnode is among the other nodes.)

Assume for each case there are two nodes whose predecessors are not among the others (they may be roots). With the following, we go to Step 3.6 and either shrink or augment.

a) Set $P_{vw} = (v,x,u,v,w)$;

b) Set $P_{xw} = (x,u,v,w)$;

c) Same as for a);

d) Same as for b);

e) Set $P_{xz} = (x,u,v,w,u,z)$.

Assume for each case there is just one node whose predecessor is not among the others. In this case the predecessors are all directed the same way for the nodes on the path of matching edges. Hence all intermediate nodes on the path as well as the last node, whose predecessor is not among the other nodes, are shrunk. Since no triangle is blocking if all three of its nodes are shrunk, the only possible case is a) where w is the node whose predecessor is not among the others. The triangle uvx was blocking a shrinking because the matching edge vy $y \neq u$ could not become a petal. Go to Step 5 with $P_y = (y,v,x,u,v,w)$, $S = (u,v,w,x)$ and $b = w$.

Step 3.9 [w is even and j forms a blocking triangle with one matching edge which does not share an edge with an existing petal].

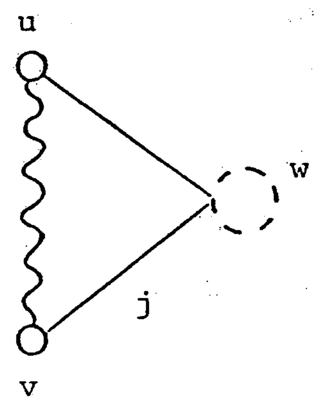


Figure 3.13

Let uvw be the blocking triangle and assume that uv ∈ M and uw ∉ M. Then the node w is a shrunk node and p(w) = u.

Case 3.9a: If u and v are also shrunk nodes, then let S contain all the nodes of G in u,v and w and let b be the node u or v such that p(b) is not among the other two nodes (b may be a root). (If both u and v had this property then these two nodes would have been shrunk together in Step 5.) Go to Step 5.

Case 3.9b: If either u or v is real, call uvw a triangular petal of S. Go to Step 2.

Step 3.10 [w is even and j forms a blocking triangle which has one matching edge and which shares an edge with one or two existing triangular petals].

Let uvw be the blocking triangle containing j. Without loss of generality, assume that w is shrunk and that uv ∈ M. Note that both u and v are even and that either u or v can be shrunk but not both. Let uvx and/or uwz be the existing triangular petals.

<u>Case 3.10a</u>: The node x is a dummy odd node. Therefore, since u and v are even and uv ∈ M, uvx has two matching edges. We have the following possibilities:

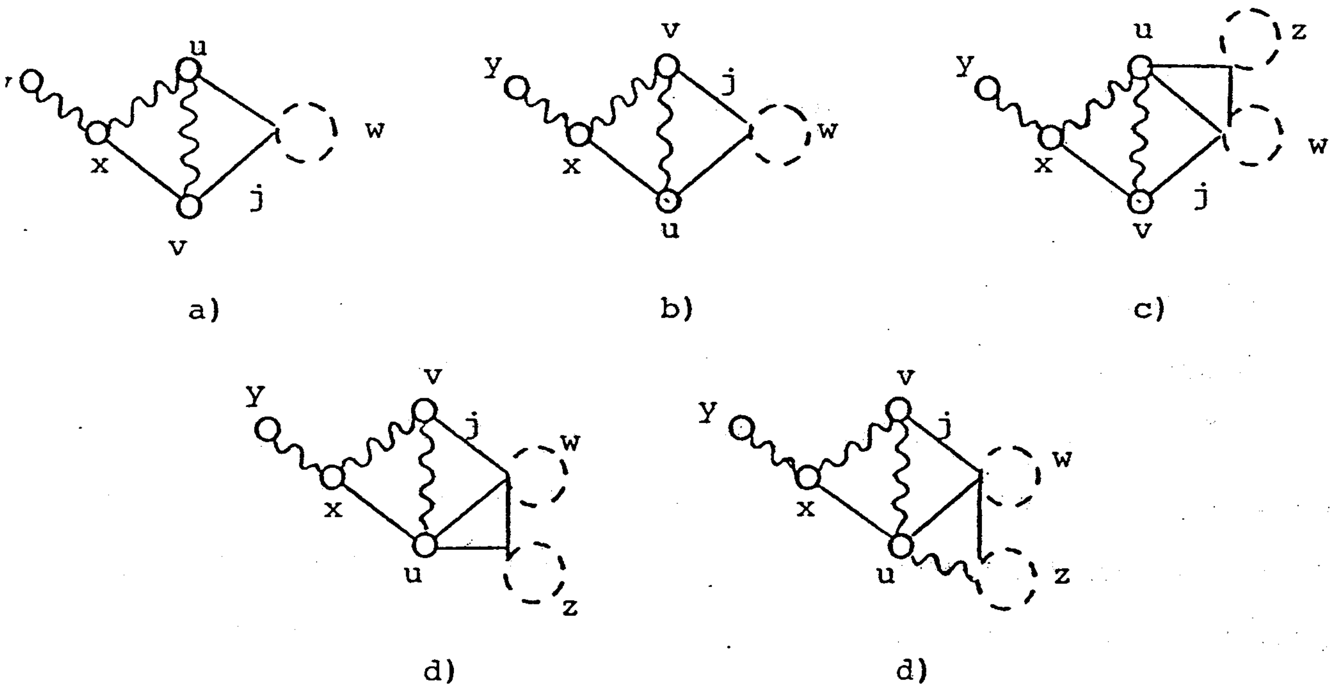


Figure 3.14

Note that, in a) and c), v can also be shrunk and, in b), u can be shrunk. Note also that the node u or v, whichever is adjacent to x via a matching edge, must be real even in all cases.

Therefore the predecessor of this node must be one of the two nodes adjacent to it via a matching edge. In all cases $p(x) = u$. So in cases a) – d) the node u or v, whichever has degree 1 in the matching in the figure, has a predecessor which is not among the other nodes (it may also be a root). In case e), $p(u) = z$, so z has a predecessor which is not among the other nodes (it may also be a root).

In cases a) and b), make x an odd node of $\mathcal{S}$. If y is not in $\mathcal{S}$ or is a dummy odd node of $\mathcal{S}$, make y an even node with $P_y = (y,x,v,u,w,P_v)$ in case a) and $P_y = (y,x,u,v,w,P_u)$ is case b). Make uvw a triangular petal and no longer call uvx a triangular petal. Go to Step 2.

In cases c) - e) make the following assignments:

c) $P_x = (x,u,z,w,P_v)$, $P_y = (y,x,v,u,z,w,P_v)$, $S = (u,v,w,x,z)$, $b = v$;

d) $P_x = (x,v,w,z,P_u)$, $P_y = (y,x,u,v,w,z,P_u)$, $S = (u,v,w,x,z)$, $b = u$;

e) $P_x = (x,v,w,P_z)$, $P_y = (y,x,u,v,w,P_z)$, $S = (u,v,w,x,z)$, $b = z$.

Go to Step 5.

Case 3.10b: The node z is a dummy odd node. Therefore since u and w are even and $uw \notin M$ uwz has one matching edge. We have the following possibilities: