Reverse mathematics and infinite traceable graphs

Peter Cholak
Department of Mathematics, University of Notre Dame
David Galvin
Department of Mathematics, University of Notre Dame
Reed Solomon
Department of Mathematics, University of Connecticut

December 17, 2010

1 Introduction

This paper falls within the general program of investigating the proof theoretic strength (in terms of reverse mathematics) of combinatorial principals which follow from versions of Ramsey's theorem. We examine two statements in graph theory and one statement in lattice theory proved by Galvin, Rival and Sands [1] using Ramsey's theorem for 4-tuples. Our main results are that the statements concerning graph theory are equivalent to Ramsey's theorem for 4-tuples over RCA_0 while the statement concerning lattices is provable in RCA_0. We give the basic definitions for graph theory and lattice theory below, but assume the reader is familiar with the general program of reverse mathematics. The definitions in this section are all given within RCA_0.

If $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, then $[X]^n$ denotes the set of all n-element subsets of X. A *k-coloring* of $[X]^n$ is a function $c : [X]^n \to k$. Ramsey's theorem for n-tuples and k colors (denoted RT^n_k) is the statement that for all k-colorings of $[\mathbb{N}]^n$, there is an infinite set X such that $[X]^n$ is monochromatic. Such a set X is called a *homogeneous set* for the coloring. We let $\mathsf{RT}(4)$ denote the statement $\forall k \mathsf{RT}^4_k$. In terms of reverse mathematics, RT^n_k is equivalent to ACA_0 over RCA_0 for all $n \geq 3$ and $k \geq 2$ and $\mathsf{RT}(4)$ is equivalent to ACA_0 over RCA_0. (See Section III.7 of Simpson [3].)

Before giving the Galvin, Rival and Sands results, we introduce some basic terminology from graph theory in RCA_0. A *graph* G is a pair $(V_G, E_G) = (V, E)$ such that V (the *vertex set*) is a subset of $\mathbb{N}$ and E (the *edge relation*) is a symmetric irreflexive binary relation on V. (Thus our graphs are undirected and have no edges from a vertex to itself.) If $E(x, y)$ holds, then we say there is an *edge* between x and y. When specifying the edge relation on a

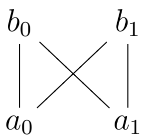
Peter Cholak is partially supported by NSF grants DMS-0800198 and DMS-0652669 and David Galvin is supported in part by National Security Agency grant H98230-10-1-0364.

graph, we assume that whenever we say $E(x, y)$ holds we implicitly declare that $E(y, x)$ holds as well. (That is, we abuse notation by regarding $E(x, y)$ as shorthand for $E(x, y) \wedge E(y, x)$.) When we deal with more than one graph, we denote the vertex set and edge relation of G by V_G and E_G.

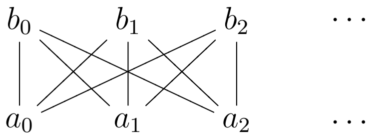
A *n–path* in a graph G is a sequence of distinct vertices $v_0, v_1, \ldots, v_{n-1}$ such that $E(v_i, v_{i+1})$ holds for all $i \leq n-2$. (A *finite path* is an n–path for some $n \in \mathbb{N}$.) Similarly, an *infinite path* is a sequence of distinct vertices $v_0, v_1, \ldots$ (formally, specified by a function $f : \mathbb{N} \to V$) such that $E(v_i, v_{i+1})$ for all $i \in \mathbb{N}$. If a path (finite or infinite) satisfies $E(v_i, v_j)$ if and only if $|i - j| = 1$, then we say the path is *chordless*. That is, a chordless path is a sequence of vertices $v_0, v_1, \ldots$ (possibly finite) in which the only edges are between vertices of the form v_i and v_{i+1}. (We use the terminology of a chordless path from Galvin, Rival and Sands, but such a path is also called an *induced path* in the literature.) An infinite graph G contains *arbitrarily long chordless paths* if for each $n \in \mathbb{N}$, G contains a chordless n–path. Similarly, we say G contains an *infinite chordless path* if G contains an infinite path which is chordless.

An infinite graph $G = (V, E)$ is *traceable* if there is a bijection $T : \mathbb{N} \to V$ (called a *tracing function*) such that for all $i \in \mathbb{N}$, $E(T(i), T(i+1))$. Thus, a traceable graph is one in which there is a path containing all the vertices. (A similar definition can be given when G is finite.)

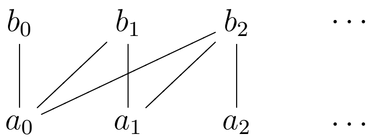
A graph $G = (V, E)$ is *bipartite* if there is a partition $V = V_0 \cup V_1$ such that for each edge $E(x, y)$ there is an $i \in \{0, 1\}$ such that $x \in V_i$ and $y \in V_{1-i}$. We use three specific bipartite graphs in this paper. The first graph is $K_{2,2}$ which has four vertices a_0, a_1, b_0 and b_1 with edges between a_i and b_j for $i, j \leq 1$.



The second graph is the complete countable bipartite graph $K_{\omega,\omega}$. Its vertices are $V = V_0 \cup V_1$ where $V_0 = \{a_n \mid n \in \mathbb{N}\}$ and $V_1 = \{b_n \mid n \in \mathbb{N}\}$ with edges between a_n and b_m for all $n, m \in \mathbb{N}$.



Following the notation of [1], the third graph will be denoted A. Its vertices are $V = V_0 \cup V_1$ where $V_0 = \{a_n \mid n \in \mathbb{N}\}$ and $V_1 = \{b_n \mid n \in \mathbb{N}\}$ with edges between a_n and b_m for all $n \leq m$.



If G and H are graphs, then we say G *contains a subgraph isomorphic to* H (or G *contains a copy of* H, or *there is an embedding of* H *into* G), if there is an injective function $g : V_H \to V_G$

such that for all $x, y \in V_H$, if there is an edge between x and y in H, then there is an edge between $g(x)$ and $g(y)$ in G. (Note that we allow additional edges in G between elements in the range of g.) The two graph theoretic results in [1] are as follows.

Theorem 1.1 (Galvin, Rival and Sands [1]). *Every infinite traceable graph either contains arbitrarily long finite chordless paths or contains a subgraph isomorphic to A.*

Theorem 1.2 (Galvin, Rival and Sands [1]). *Every infinite traceable graph containing no chordless 4–path contains a subgraph isomorphic to $K_{\omega,\omega}$.*

As an application of Theorem 1.1, Galvin, Sands and Rival prove the following lattice theoretic result. (The lattice terminology is defined in Section 3.)

Theorem 1.3 (Galvin, Rival and Sands [1]). *Every finitely generated infinite lattice of length 3 contains arbitrarily long finite fences.*

In Section 2, we show that Theorems 1.1 and 1.2 are equivalent to ACA_0 over RCA_0. In Section 3, we show that Theorem 1.3 is provable in RCA_0. We follow Simpson [3] for the reverse mathematics and we follow Soare [4] for computability theory.

2 Traceable graphs

We begin this section with a computable combinatorics result which will translate into a result in reverse mathematics. If $G = (V, E)$ is a graph and $x \in V$, then we say x has *infinite degree* if there are infinitely many y such that $E(x, y)$. Let V^∞ denote the set of vertices with infinite degree in G.

Theorem 2.1. *There is an infinite computable graph $G = (V, E)$ such that G has a computable tracing function, G has no chordless 4–paths and*

$$\forall X \left((\exists e\, (W_e^X \text{ is infinite} \wedge W_e^X \subseteq V^\infty)) \rightarrow 0' \leq_T X \right)$$

Proof. We build G in stages using a dump construction to create a computable sequence of nested subgraphs. At stage s, our graph $G_s = (V_s, E_s)$ has $V_s = \{0, 1, \ldots, k_s\}$ for some $k_s \in \mathbb{N}$. If $t > s$, then $k_t > k_s$ and for any $x, y \in V_s$, $E_t(x, y)$ holds if and only if $E_s(x, y)$ holds. Thus, the vertex set $V = \cup_s V_s = \mathbb{N}$ is computable, the edge relation $E = \cup_s E_s$ is computable, and hence G is computable.

At stage s, the vertex set V_s will be further subdivided into nonempty convex blocks $B_{0,s}, B_{1,s}, \ldots, B_{s,s}$. (That is, if $n < p < m$ and $n, m \in B_{j,s}$, then $p \in B_{j,s}$.) Each block $B_{j,s}$ will have a coding vertex $c_{j,s}$, which will be the largest element of the block. Thus, $B_{0,s} = \{x \mid 0 \leq x \leq c_{0,s}\}$ and for $0 < j \leq s$, $B_{j,s} = \{x \mid c_{j-1,s} < x \leq c_{j,s}\}$. For $x, y \leq k_s$, we say x and y are in the same s-block if $\exists j \leq s(x, y \in B_{j,s})$ and we say x and y are in different s-blocks otherwise.

At stage $s+1$, we may collapse a final segment of these blocks by picking a value $0 \leq n \leq s$ and "dumping" all the blocks currently after $B_{n,s}$ into $B_{n,s+1}$, i.e. setting $\cup_{n \leq m \leq s} B_{m,s} \subseteq$

$B_{n,s+1}$. When we do this, we will redefine the coding vertices $c_{m,s+1}$ for $m \geq n$ to be new large numbers. In the end, each coding vertex will have a limit $c_n = \lim_s c_{n,s}$ and each block will have a finite limiting block $B_n = \lim_s B_{n,s}$. The limiting coding vertices will satisfy $c_0 < c_1 < c_2 < \cdots$.

The only vertices with infinite degree will be the limiting c_n coding vertices. Suppose X can enumerate an infinite set of infinite degree vertices. Then X can compute an infinite set of infinite degree vertices in increasing order and hence X can compute a function f such that $f(n) \geq c_n$. Therefore, it suffices to construct G so that any function dominating the sequence $c_0, c_1, \ldots$ can compute $0'$. The obvious way to do this is to make sure that $n \in K$ if and only if $n \in K_{c_n}$. The idea of the construction is to dump later blocks into $B_{n,s}$ if n enters K_s and redefine $c_{n,s+1} \geq s$.

Fix an enumeration K_s of the halting problem K such that exactly one number enters K_s at each stage s. Our construction proceeds in stages as follows. At stage 0, set $V_0 = \{0\}$ (and thus $k_0 = 0$), $E_0 = \emptyset$, $B_{0,0} = \{0\}$ and $c_{0,0} = 0$.

At stage $s+1$, check to see if the number n entering K at stage s is large ($n > s$) or small ($n \leq s$). If a number $n > s$ enters K_s, then define G_{s+1} as follows. Let $V_{s+1} = V_s \cup \{k_s + 1\}$. (Recall that k_s is the largest number in V_s. Thus $k_{s+1} = k_s + 1$.) For each $j \leq s$, leave the blocks $B_{j,s+1} = B_{j,s}$ unchanged and leave the coding vertices $c_{j,s+1} = c_{j,s}$ unchanged. Define a new block $B_{s+1,s+1} = \{k_s + 1\}$ containing the newly added vertex and set its coding vertex $c_{s+1,s+1} = k_s + 1$. Add new edges between $c_{s+1,s+1}$ and the other coding vertices $c_{j,s+1}$ for $j \leq s$ and end the stage. (That is, let E_{s+1} contain E_s plus the edges $E_{s+1}(c_{j,s+1}, c_{s+1,s+1})$ for each $j \leq s$.)

If a number $n \leq s$ enters K at stage s, then define G_{s+1} as follows. Let $u = (s+1) - n$. Expand V_s to V_{s+1} by adding $u + 1$ many new vertices $k_s + 1, k_s + 2, \ldots, k_s + u + 1$. (Thus $k_{s+1} = k_s + u + 1$.) For each $j < n$, leave the blocks $B_{j,s+1} = B_{j,s}$ and the coding vertices $c_{j,s+1} = c_{j,s}$ unchanged. Dump the current later blocks and one additional element $k_s + 1$ into $B_{n,s+1}$, and redefine the coding vertex $c_{n,s+1} = k_s + 1$. That is, set

$$B_{n,s+1} = \bigcup_{n \leq m \leq s} B_{m,s} \cup \{k_s + 1\} \text{ and } c_{n,s+1} = k_s + 1.$$

Use the remaining new elements $k_s + 2, \ldots, k_s + u + 1$ to define new single element blocks $B_{n+1,s+1}, \ldots, B_{s+1,s+1}$ with the single elements as the designated coding vertices. That is, for each $1 \leq v \leq u$ set

$$B_{n+v,s+1} = \{k_s + v + 1\} \text{ and } c_{n+v,s+1} = k_s + v + 1$$

Expand E_s to E_{s+1} by adding new edges between each of the new coding vertices $c_{a,s+1}$ (for $n \leq a \leq s + 1$) and all the other coding vertices $c_{b,s+1}$ (for $0 \leq b \leq s + 1$ with $b \neq a$). Also add edges $E_{s+1}(c_{n,s+1}, x)$ for all $x \in B_{n,s+1}$ with $x \neq c_{n,s+1}$. End the stage.

This completes the construction of G. As indicated above, G is computable because there is an edge between x and y only if there is an edge between them at the first stage s at which $x, y \in G_s$. We check the remaining properties in a series of lemmas.

Lemma 2.2. $\forall s \, \forall j \leq s \, \forall x \in B_{j,s} \, [x \leq c_{j,s} \wedge (x \neq c_{j,s} \rightarrow E_s(x, c_{j,s}))]$.

Proof. This fact by induction on s. For $s = 0$, it holds for $c_{0,0}$ since $B_{0,0}$ is a singleton set. For $s + 1$, we split into cases depending on whether a small number enters K_s. If not, then the property holds for $j < s + 1$ by the induction hypothesis and the fact that the blocks and coding locations indexed by $j < s + 1$ do not change. It holds for $j = s + 1$ since $B_{s+1,s+1}$ is a singleton set.

If $n \leq s$ enters K_s, then the property holds for $j < n$ by the induction hypothesis and the fact that the blocks and coding vertices indexed by $j < n$ do not change. It holds for $j = n$ because $c_{n,s+1} = k_s + 1$ is the largest element of $B_{n,s+1}$ and we add edges at stage $s + 1$ between this coding vertex and all the elements of $B_{n,s+1}$. It holds for $n < j \leq s + 1$ because each block $B_{j,s+1}$ is a singleton set. □

Lemma 2.3. $\forall s\, \forall i \neq j \leq s\, (E_s(c_{i,s}, c_{j,s}))$.

Proof. This lemma follows by induction on s since we add edges between any new or redefined coding vertices and all other coding vertices at each stage. □

Lemma 2.4. $\forall s\, \forall d < k_s\, (E_s(d, d + 1))$.

Proof. This fact follows by induction on s. For $s = 0$, it is vacuously true since $k_0 = 0$. For $s + 1$, we split into cases depending on whether a small number enters K_s. If not, then $k_{s+1} = k_s + 1$. By the induction hypothesis, we have $E_s(d, d + 1)$, and hence $E_{s+1}(d, d + 1)$, for all $d < k_s$. It remains to show that $E_{s+1}(k_s, k_{s+1})$. Since k_s is the greatest element in V_s, it is also the greatest element in $B_{s,s}$. By Lemma 2.2, $c_{s,s} = k_s$ and hence by the construction $c_{s,s+1} = k_s$. Since $c_{s+1,s+1} = k_{s+1} = k_s + 1$ and since we add the edge $E_{s+1}(c_{s+1,s+1}, c_{s,s+1})$, we have the edge $E_{s+1}(k_s, k_{s+1})$ as required.

For the remaining case, suppose $n \leq s$ enters K_s and hence $k_{s+1} = k_s + u + 1$ where $u = (s + 1) - n$. By the induction hypothesis, $E_s(d, d+1)$, and hence $E_{s+1}(d, d+1)$, holds for all $d < k_s$. It remains to show that $E_{s+1}(k_s + v, k_s + v + 1)$ holds for all $v \leq u$. First consider when $v = 0$. By construction, $k_s, k_{s+1} \in B_{n,s+1}$ and $c_{n,s+1} = k_{s+1}$. Since we add edges from $c_{n,s+1}$ to each element of $B_{n,s+1}$, we have $E_{s+1}(k_s, k_{s+1})$ as required. Finally, consider when $v > 0$. In this case, $c_{n+v-1,s+1} = k_s + v$ and $c_{n+v,s+1} = k_s + v + 1$. By construction, we add an edge between these coding vertices at stage $s + 1$ and hence have $E_{s+1}(k_s + v, k_s + v + 1)$ as required. □

By Lemma 2.4, $E(n, n+1)$ holds for all n. Since $T(n) = n$ is a bijection from $\mathbb{N}$ to $V = \mathbb{N}$, $T(n) = n$ is a computable tracing function for G. We next show that G has no chordless 4–paths. It suffices to show that each G_s has no chordless 4–paths. We need two additional technical lemmas before establishing this fact. The first technical lemma says that whenever we have an edge $E_s(x, y)$ with $x < y$, then either x and y are in the same s-block or x is a coding vertex $x = c_{j,s}$ for some $j \leq s$.

Lemma 2.5. $\forall s\, \forall x < y \in G_s\, (E_s(x, y) \rightarrow [\exists j \leq s\, (x, y \in B_{j,s}) \vee \exists j \leq s\, (x = c_{j,s})])$.

Proof. This lemma follows by induction on s. For $s = 0$, it is trivial since $|G_0| = 1$. For $s+1$, assume that $x < y \in G_{s+1}$ and $E_{s+1}(x, y)$. We need to show that either $x, y \in B_{j,s+1}$ for some $j \leq s + 1$ or x has the form $c_{j,s+1}$. We split into cases depending on which (if any) of x and y are in G_s. If $x \notin G_s$, then by construction, $x = c_{j,s+1}$ for some j and we are done.

If $x \in G_s$ and $y \notin G_s$, then y has the form $c_{j,s+1}$ for either $j = s+1$ (if no small number entered K_s) or for some $n \leq j \leq s+1$ (if $n \leq s$ entered K_s). In the former case, by construction $E_{s+1}(x,y)$ implies that $x = c_{j,s+1}$ for some $j \leq s$ and we are done. In the latter case, we split into cases depending on whether $y = c_{n,s+1}$ or $y = c_{j,s+1}$ for $j > n$. If $y = c_{n,s+1}$, then $E_{s+1}(x,y)$ implies that either $x \in B_{n,s+1}$ (and we are done since $y = c_{n,s+1} \in B_{n,s+1}$) or $x = c_{l,s+1}$ for some $l \neq j$ (and we are done). If $y = c_{j,s+1}$ for $j > n$, then $E_{s+1}(x,y)$ implies that $x = c_{l,s+1}$ for some $l \neq j$ (and we are done).

Therefore, we are left with the case when $x, y \in G_s$. Since $x, y \in G_s$ and $E_{s+1}(x,y)$, $E_s(x,y)$ must hold. By the induction hypothesis, either $x, y \in B_{j,s}$ for some $j \leq s$ or x has the form $c_{j,s}$. If x and y are in the same s-block, then by construction they are in the same $(s+1)$-block. (This block may or may not have the same index at stage $s+1$ depending on whether dumping occurred at stage $s+1$.)

Therefore, we are left with the case when $x, y \in G_s$, x and y are not in the same s-block and hence $x = c_{j,s}$ for some $j \leq s$. If $c_{j,s} = c_{j,s+1}$, then $x = c_{j,s+1}$ and we are done. Therefore, assume that $c_{j,s} \neq c_{j,s+1}$. By the construction, this only occurs when a number $n \leq s$ enters K_s and $j \geq n$. In this case, $x = c_{j,s}$ is dumped into $B_{n,s+1}$. Since $x < y$ and $y \in G_s$, y must also be dumped into $B_{n,s+1}$. Hence, we have $x, y \in B_{n,s+1}$ and are done. □

Our second technical lemma says that whenever we have vertices $x < y$ which are connected but in different s-blocks, then x is connected to all of the elements in the s-block containing y.

Lemma 2.6. *The following statement holds for all stages s. Let $x < y \in G_s$ with $E_s(x,y)$ and let $j \leq s$ be such that $y \in B_{j,s}$. If $x \notin B_{j,s}$, then $E_s(x,z)$ holds for all $z \in B_{j,s}$.*

Proof. We prove this lemma by induction on s. If $s = 0$ then the statement holds trivially. For $s+1$, first consider the case when no small number enters K_s. Let j be such that $y \in B_{j,s+1}$ and assume $x \notin B_{j,s+1}$. If $j \neq s+1$, then we are done because $E_s(x,z)$ (and hence $E_{s+1}(x,z)$) holds for all $z \in B_{j,s} = B_{j,s+1}$ by the induction hypothesis. If $j = s+1$, then $y = c_{s+1,s+1}$ and $B_{j,s+1} = \{y\}$, so again we are done.

Second assume that $n \leq s$ enters K_s. As above, let $j \leq s+1$ be such that $y \in B_{j,s+1}$ and assume $x \notin B_{j,s+1}$. If $j < n$, then as above (since $B_{j,s+1} = B_{j,s}$) we are done by the induction hypothesis. If $j > n$, then as above (since $B_{j,s+1} = \{y\}$) we are done trivially. Therefore, assume that $j = n$. In this case, $B_{n,s+1} = \cup_{n \leq l \leq s} B_{l,s} \cup \{c_{n,s+1}\}$. By Lemma 2.5, $x < y$ and $x \notin B_{n,s+1}$ implies that $x = c_{i,s+1}$ for some $i < n$. By construction, $c_{i,s+1} = c_{i,s}$, so $x = c_{i,s}$. Therefore, for all l such that $n \leq l \leq s$, we have $x < c_{l,s}$, $x \notin B_{l,s}$ and $E_s(x, c_{l,s})$ holds. By the induction hypothesis, $E_s(x,z)$ (and hence $E_{s+1}(x,z)$) holds for all $z \in \cup_{n \leq l \leq s} B_{l,s}$. Furthermore, by construction, $E_{s+1}(x, c_{n,s+1})$ holds completing this case. □

Lemma 2.7. $\forall s$ *(G_s has no chordless 4–paths).*

Proof. We proceed by induction on s. For $s = 0$, it follows trivially since $|G_0| = 1$. For $s+1$, split into cases depending on whether a small number enters K_s.

First, assume that no small number enters K_s and assume for a contradiction that there is a chordless 4–path x_0, x_1, x_2, x_3 in G_{s+1}. By definition, we have $E_{s+1}(x_i, x_{i+1})$ for $i < 3$ and no other edges between these nodes (except those induced by symmetry). By the induction hypothesis, at least one x_i must lie outside G_s and hence we have $x_i = k_s + 1 = c_{s+1,s+1}$ for some $i \leq 3$. We break into cases depending on which x_i is equal to $c_{s+1,s+1}$. Notice that if x_0, x_1, x_2, x_3 is a chordless 4–path, then x_3, x_2, x_1, x_0 is also a chordless 4–path. Therefore, by symmetry, it suffices to show that we cannot have $x_0 = c_{s+1,s+1}$ or $x_1 = c_{s+1,s+1}$. (Recall that the elements in a path are required to be distinct. We use this fact repeatedly without mention.)

- If $x_0 = c_{s+1,s+1}$, then by constuction $E_{s+1}(x_0, x_1)$ implies that $x_1 = c_{l,s+1}$ for some $l \leq s$. We break into subcases depending on the form of x_2.
  - Suppose $x_2 < x_1$ and $x_2 \notin B_{l,s+1}$. By Lemma 2.5, $x_2 = c_{j,s+1}$ for some $j < l$ and hence $E_{s+1}(x_0, x_2)$ for a contradiction.
  - Suppose $x_2 \in B_{l,s+1}$ and consider the form of x_3. If $x_3 \in B_{l,s+1}$, then we have $E_{s+1}(x_1, x_3)$ for a contradiction. If $x_3 < x_2$ and $x_3 \notin B_{l,s+1}$, then $x_3 = c_{j,s+1}$ for some $j < l$ and we have $E_{s+1}(x_0, x_3)$ for a contradiction. The remaining case, $x_3 > x_2$ and $x_3 \notin B_{l,s+1}$ is not possible by Lemma 2.5 since $x_2 \in B_{l,s}$ but $x_2 \neq c_{l,s+1}$.
  - Suppose $x_2 > x_1$ (so $x_2 \notin B_{l,s+1}$) and consider the form of x_3. If x_3 is in the same $(s+1)$-block as x_2, then since $E_{s+1}(x_1, x_2)$ holds, we have by Lemma 2.6 that $E_{s+1}(x_1, x_3)$ holds for a contradiction. If $x_3 < x_2$ and is not in the same $(s+1)$-block as x_2, then by Lemma 2.5, $x_3 = c_{i,s+1}$ for some i and hence $E_{s+1}(x_0, x_3)$ holds for a contradiction. If $x_3 > x_2$ and is not in the same $(s+1)$-block as x_2, then by Lemma 2.5, $x_2 = c_{i,s+1}$ for some i and $E_{s+1}(x_0, x_2)$ holds for a contradiction.
- If $x_1 = c_{s+1,s+1}$, then by the construction, $x_0 = c_{l,s+1}$ and $x_2 = c_{m,s+1}$ for some $l \neq m$. By Lemma 2.3, $E_{s+1}(x_0, x_2)$ holds for a contradiction.

Next assume that $n \leq s$ enters K_s and x_0, x_1, x_2, x_3 is a chordless 4–path. By the induction hypothesis, at least one of the x_i is not in G_s and hence must have the form $x_i = c_{j,s+1}$ for some $n \leq j \leq s+1$. If $x_i = c_{j,s+1}$ for $n < j \leq s+1$, then since $B_{j,s+1} = \{c_{j,s+1}\}$, the same argument as in the previous case (when no small number enters K_s) suffices to derive a contradiction. Therefore, we can assume without loss of generality that the chordless path is contained in $B_{0,s+1} \cup \cdots \cup B_{n,s+1}$ and that $x_i = c_{n,s+1}$ for some $i \leq 3$. By symmetry, it suffices to consider the cases when $x_0 = c_{n,s+1}$ and $x_1 = c_{n,s+1}$. (Below, we frequently use without mention that none of the x_i have the form $c_{l,s+1}$ for $l > n$ and that if $x_i \in B_{n,s+1}$ and $x_j \notin B_{n,s+1}$, then $x_j < x_i$.)

- Suppose $x_0 = c_{n,s+1}$ and consider the form of x_1. Since $x_1 < x_0$, either $x_1 \in B_{n,s+1}$ or $x_1 = c_{l,s+1}$ for some $l < n$. Consider these cases separately.
  - Suppose $x_1 \in B_{n,s+1}$ and consider the form of x_2. If $x_2 \in B_{n,s+1}$, then $E_{s+1}(x_0, x_2)$ holds (since $x_0 = c_{n,s+1}$ is connected to all vertices in $B_{n,s+1}$) for a contradiction. If $x_2 \notin B_{n,s+1}$, then $x_2 < x_1$ and hence by Lemma 2.5, $x_2 = c_{l,s+1}$ for some $l < n$. Thus $E_{s+1}(x_0, x_2)$ holds for a contradiction.
  - Suppose $x_1 = c_{l,s+1}$ for some $l < n$ and consider the form of x_2. There are three cases to consider.

  * Assume $x_2 \in B_{l,s+1}$ and consider the form of x_3. If $x_3 \in B_{l,s+1}$, then by Lemma 2.2, $E_{s+1}(x_1, x_3)$ holds for a contradiction. If $x_3 \notin B_{l,s+1}$, then by Lemma 2.5 and the fact that $x_2 \in B_{l,s+1}$ but $x_2 \neq c_{l,s+1}$, we have $x_3 < x_2$ and hence $x_3 = c_{i,s+1}$ for some $i < l$. But then $E_{s+1}(x_0, x_3)$ holds for a contradiction.
  * Assume $x_2 < x_1$ and $x_2 \notin B_{l,s+1}$. By Lemma 2.5, $x_2 = c_{i,s+1}$ for some $i < l$ and hence $E_{s+1}(x_0, x_2)$ holds for a contradiction.
  * Assume $x_2 > x_1$ (so $x_2 \notin B_{l,s+1}$) and consider the form of x_3. If x_3 is the same $(s+1)$-block as x_2, then since $E_{s+1}(x_1, x_2)$ holds, Lemma 2.6 implies $E_{s+1}(x_1, x_3)$ holds for a contradiction. Therefore, x_3 is not in the same $(s+1)$-block as x_2. Therefore, by Lemma 2.5, either x_2 or x_3 has the form $c_{i,s+1}$ for some i. Hence either $E_{s+1}(x_0, x_2)$ holds or $E_{s+1}(x_0, x_3)$ holds, giving a contradiction.

- Suppose $x_1 = c_{n,s+1}$. By the construction, x_1 is connected only to the vertices in $B_{n,s+1}$ and the coding vertices $c_{l,s+1}$. Since $E_{s+1}(x_0, x_1)$ and $E_{s+1}(x_1, x_2)$ hold, either $x_0 \in B_{n,s+1}$ or $x_0 = c_{l,s+1}$ for some $l < n$, and either $x_2 \in B_{n,s+1}$ or $x_2 = c_{i,s+1}$ for some $i < n$. Consider each of the possible combinations separately.

  - Suppose $x_0 = c_{l,s+1}$ and $x_2 = c_{i,s+1}$. In this case, $E_{s+1}(x_0, x_2)$ holds for a contradiction.
  - Suppose $x_0 = c_{l,s+1}$ and $x_2 \in B_{n,s+1}$. Since $x_0 < x_1$, $x_0 \notin B_{n,s+1}$, $E_{s+1}(x_0, x_1)$ holds and $x_1, x_2 \in B_{n,s+1}$, Lemma 2.6 implies that $E_{s+1}(x_0, x_2)$ holds for a contradiction.
  - Suppose $x_0 \in B_{n,s+1}$ and $x_2 = c_{i,s+1}$. Since $x_2 < x_1$, $x_2 \notin B_{n,s+1}$, $E_{s+1}(x_2, x_1)$ holds and $x_0, x_1 \in B_{n,s+1}$, Lemma 2.6 implies that $E_{s+1}(x_2, x_0)$ holds for a contradiction.
  - Suppose $x_0, x_2 \in B_{n,s+1}$. Consider the form of x_3. If $x_3 \in B_{n,s+1}$, then $E_{s+1}(x_1, x_3)$ holds for a contradiction. Therefore, $x_3 \notin B_{n,s+1}$ and $x_3 < x_2$. By Lemma 2.5, $x_3 = c_{j,s+1}$ for some $j < n$. By construction $E_{s+1}(x_1, x_3)$ holds for a contradiction.

□

We have now established that G is a computable graph with a computable tracing function and no chordless 4-paths. It remains to show that if X can enumerate an infinite set of infinite degree vertices, then $0' \leq_T X$.

Lemma 2.8. $\forall k\, (\lim_s c_{k,s} = c_k$ *exists*).

Proof. For any stage $s \geq k$, $c_{k,s+1} \neq c_{k,s}$ only if the block $B_{k,s}$ is dumped at stage $s+1$ into a block $B_{n,s+1}$ with $n \leq k$. Since this happens only if a number $n \leq k$ enters K_s, we have that $c_{k,s}$ can change at most $k+1$ many times after it is first defined. □

From Lemma 2.8 and the construction it is clear that for all indices k and all stages $s \geq k$, $c_{k,s} \leq c_{k,s+1}$. Therefore, each $c_{k,s}$ is increasing in s and stabilizes when it reaches its limit. It is also clear that $c_0 < c_1 < \cdots$ and that $x \leq c_x$ for all x. Finally, since for all stages s, $B_{0,s} = \{x \mid 0 \leq x \leq c_{0,s}\}$ and $B_{j,s} = \{x \mid c_{j-1,s} < x \leq c_{j,s}\}$ for $0 < j \leq s$, we have that each block reaches a limit $B_j = \lim_s B_{j,s}$ and each vertex x sits inside some limiting block. (That is, for each vertex x, there is a stage s and a block B_j such that $x \in B_{j,s} = B_j$.)

Lemma 2.9. *A vertex $x \in G$ has infinite degree if and only if $x = c_k$ for some k.*

Proof. First, note that each vertex c_k has infinite degree since $E(c_k, c_l)$ holds for all $l \neq k$. (More formally, if s and t are stages such that $c_{k,s} = c_k$ and $c_{l,s} = c_l$, then by stage $u = \max\{s, t\}$ we have added an edge $E_u(c_k, c_l)$.)

Second, let x be a vertex such that $x \neq c_k$ for all k. Suppose for a contradiction that x has infinite degree. Fix a stage s and a block such that $x \in B_{j,s} = B_j$. Since $x \neq c_j$ and both $B_{j,s}$ and $c_{j,s}$ have reached limits, it follows that for all stages $t \geq s$, $x \in B_{j,t}$ and $x \neq c_{j,t}$. Since x is assumed to have infinite degree, there must be a vertex $y > c_j$ and a stage $t > s$ such that $E_t(x, y)$ holds. By Lemma 2.2, $y \notin B_{j,t}$ and hence (since $x < y$ and $E_t(x, y)$) by Lemma 2.5, $x = c_{l,t}$ for some l. Since $x \in B_{j,t}$ we must have $x = c_{j,t}$ for a contradiction. □

In addition to having $x \leq c_x$, it is clear that $s \leq k_s$ for all s.

Lemma 2.10. $\forall x\,(x \in K \Leftrightarrow k \in K_{c_x})$.

Proof. Suppose x enters K at stage s. If $s < x$, then $s < c_x$ and hence $x \in K_{c_x}$. If $x \leq s$, then at stage $s + 1$ of the construction, we dump later blocks into $B_{x,s+1}$ and set $c_{x,s+1} = k_s + 1$. Therefore, $c_x > s$ and hence $x \in K_{c_x}$. □

Lemma 2.11. *If X can enumerate an infinite set of infinite degree vertices in G then $0' \leq_T X$.*

Proof. Define a function $f \leq_T X$ by setting $f(0) =$ the first infinite degree vertex enumerated by X and $f(n + 1) =$ the first infinite degree vertex y enumerate by X such that $y > f(n)$. By Lemma 2.9, f has the property that $c_x \leq f(x)$ for all x and hence $x \in K$ if and only if $x \in K_{f(x)}$. □

This completes the proof of Theorem 2.1.

□

Since the graph G constructed in Theorem 2.1 is traceable and has no chordless 4-paths, Theorems 1.1 and 1.2 tell us that G has subgraphs isomorphic to A and to $K_{\omega,\omega}$. However, if f is an embedding of either A or $K_{\omega,\omega}$ into G, then f can enumerate an infinite set of infinite degree vertices in G. Therefore, by Theorem 2.1, $0' \leq_T f$ for any embedding of A or $K_{\omega,\omega}$ into G. Thus we have the following corollary concerning the lack of effectiveness of Theorems 1.1 and 1.2.

Corollary 2.12. *There is a computable graph G with a computable tracing function and no chordless 4-paths such that $0'$ is computable from any embedding of A or $K_{\omega,\omega}$ into G.*

We next translate this result into the language of reverse mathematics.

Theorem 2.13 (RCA_0)**.** *The following are equivalent.*

(1). Theorem 1.1.

(2). Theorem 1.2.

(3). ACA_0.

Proof. The fact that (3) implies (1) and (2) follows immediately from the proofs given in [1]. (Our Theorem 1.1 is Theorem 1 in [1] and our Theorem 1.2 is Theorem 2 in [1].) The proofs translate easily into proofs in $\mathsf{RCA}_0 + \mathsf{RT}(4)$. Since $\mathsf{ACA}_0 \vdash \mathsf{RT}(4)$, this gives the desired implications.

We prove (1) implies (3) and (2) implies (3) with essentially the construction given in the proof of Theorem 2.1. For the remainder of this proof we work in RCA_0. Fix a 1-to-1 function $f : \mathbb{N} \to \mathbb{N}$. It suffices to construct a graph G so that any embedding of A or $K_{\omega,\omega}$ into G yields a Δ^0_1 definition of the range of f.

We build a graph G in stages as in the proof of Theorem 2.1. At stage 0, set $V_0 = \{0\}$ (so $k_0 = 0$), $E_0 = \emptyset$, $B_{0,0} = \{0\}$ and $c_{0,0} = 0$. At stage $s+1$, let $n = f(s)$ and split into cases depending on whether $n > s$ or $n \leq s$.

If $n > s$, then define G_{s+1} as follows. Let $V_{s+1} = V_s \cup \{k_s + 1\}$ and $k_{s+1} = k_s + 1$. For each $j \leq s$, let $B_{j,s+1} = B_{j,s}$ and $c_{j,s+1} = c_{j,s}$. Define $B_{s+1,s+1} = \{k_{s+1}\}$ and $c_{s+1,s+1} = k_{s+1}$. Expand E_s to E_{s+1} by adding edges between $c_{s+1,s+1}$ and each $c_{j,s+1}$ for $j \leq s$.

If $n \leq s$, then let $u = (s+1) - n$ and define G_{s+1} as follows. Let $k_{s+1} = k_s + u + 1$ and define $V_{s+1} = V_s \cup \{x \mid k_s < x \leq k_{s+1}\} = \{x \mid x \leq k_{s+1}\}$. For $j < n$, let $B_{j,s+1} = B_{j,s}$ and $c_{j,s+1} = c_{j,s}$. Set

$$B_{n,s+1} = \bigcup_{n \leq m \leq s} B_{m,s} \cup \{k_s + 1\} \text{ and } c_{n,s+1} = k_s + 1.$$

For $1 \leq v \leq u$, set

$$B_{n+v,s+1} = \{k_s + v + 1\} \text{ and } c_{n+v,s+1} = k_s + v + 1.$$

Expand E_s to E_{s+1} by adding edges between each pair $c_{j,s+1}$ and $c_{i,s+1}$ with $i \neq j \leq s+1$. (If $i, j < n$ then these edges already exist in E_s.)

Let $G = (V, E)$ where $V = \cup_s V_s = \mathbb{N}$ and $E = \cup_s E_s$. Lemmas 2.2, 2.3, 2.4, 2.5 and 2.6 were all proved by Σ^0_1 induction and hence are provable in RCA_0. Therefore, G is traceable and has no chordless 4-paths.

We need an analog of Lemma 2.9. Suppose $x \in G$ and x is placed in G at stage s. By the construction, $x = c_{j,s}$ for some $j \leq s$. If $\forall t > s\,(x = c_{j,t})$, then x has infinite degree because we add an edge between x and each new element added at stage t for all $t > s$. On the other hand, if $\exists t > s(x \neq c_{j,t})$, then by construction x is never equal to a coding vertex $c_{i,u}$ for any $u \geq t$. Since the only edges added at stages $u \geq t$ are between vertices of the form $c_{i,u}$ and $c_{j,u}$, x is never connected by an edge to another vertex after stage t. Therefore, x has finite degree.

We also need an analog of Lemma 2.10. We claim that

$$\forall k\, \forall s \geq k\, (c_{k,s+1} \neq c_{k,s} \leftrightarrow f(s) \leq k).$$

If $f(s) \leq k$, then the dumping in the definition of G_{s+1} causes $c_{k,s}$ to be redefined and hence $c_{k,s+1} \neq c_{k,s}$. On the other hand, if $c_{k,s+1} \neq c_{k,s}$ then dumping must have occurred because

$f(s) = n \leq s$. Furthermore, since $c_{j,s+1} = c_{j,s}$ for all $j < n$, we cannot have $k < n$. Therefore, $f(s) \leq k$.

We are now ready to apply (1) or (2) and extract a definition of the range of f. Since G is traceable and has no chordless 4-paths, by (1) or (2), there is an embedding $g : A \to G$ or $g : K_{\omega,\omega} \to G$. Recall that each of the vertices a_k for $k \in \mathbb{N}$ in A or $K_{\omega,\omega}$ have infinite degree.

Define an auxiliary function $g' : \mathbb{N} \to \mathbb{N}$ by $g'(n) = \max\{g(a_k) \mid k \leq n\}$. (Note that the sets $\{g(a_k) \mid k \leq n\}$ exist by bounded Σ^0_1 comprehension.) We claim that

$$\forall k \, \forall t \geq g'(k) \, (c_{k,g'(k)} = c_{k,t}).$$

Suppose for a contradiction that this property fails for some k and fix $t \geq g'(k)$ such that $c_{k,g'(k)} \neq c_{k,t}$. Let $x = c_{k,g'(k)}$. By our analog of Lemma 2.9, x has finite degree. However, by the definition of $g'(k)$, $x = g(a_i)$ for some $i \leq k$. Thus a_i has infinite degree in A or $K_{\omega,\omega}$ but $g(a_i)$ has finite degree in G, contradicting the fact that g is an embedding.

By our analog of Lemma 2.10, this property implies that

$$\forall t \geq g'(k) \, (f(t) \not\leq k).$$

Thus, k is in the range of f if and only if $\exists x \leq g'(k) \, (f(x) = k)$, completing the proof that (1) and (2) imply (3). □

3 Finitely generated lattices

Our goal for this section is to show that Theorem 1.3 is provable in RCA_0 and hence its proof does not require the use of Theorem 1.1. Before giving the formal lattice theoretic definitions, we prove a finite Ramsey style result, Theorem 3.1 below, that is contained in [1]. The proof of Theorem 3.1 given in [1] explicitly uses Theorem 1.1 (and hence this proof requires ACA_0) although the authors indicate that an alternate proof is available using the Finite Ramsey Theorem. Our proof of Theorem 3.1 formalizes this alternate approach in RCA_0.

Recall that for $m, n, u, k \in \mathbb{N}$, the notation $[0, m] \to (q)^n_k$ means that for any k-coloring of $[Y]^n$, where $Y = [0, m]$, there is a set $X \subseteq Y$ such that $|X| = q$ and the coloring is monochromatic on $[X]^n$. The Finite Ramsey Theorem is the statement

$$\forall n, u, k \, \exists m \, ([0, m] \to (q)^n_k).$$

The least m satisfying $[0, m] \to (q)^n_k$ is called the finite Ramsey number for n-tuples with k many colors and a homogeneous set of size q. Since the Finite Ramsey Theorem is provable in $PA^- + I\Sigma_1$ (see Hájek and Pudlák [2] Chapter II, Theorem 1.10) and $PA^- + I\Sigma_1$ is the first order part of RCA_0 (see Simpson [3] Corollary IX.1.11), it follows that RCA_0 proves the Finite Ramsey Theorem. The finite style Ramsey result from [1] is as follows.

Theorem 3.1 (RCA_0). *For all $n \in \mathbb{N}$, there is an m such that for all finite traceable graphs G with $|G| \geq m$, either G contains a copy of $K_{2,2}$ or G contains a chordless n-path.*

Proof. Let $n' = \max\{n+1, 8\}$ and let m be the finite Ramsey number for 4-tuples with $(n-1)^2 + 1$ many colors and a homogeneous set of size n'. We claim that this m satisfies the theorem.

Let $G = (V, E)$ be a finite traceable graph with $|G| \geq m$. Without loss of generality, we assume that V is an initial segment of $\mathbb{N}$ and that $E(i, i+1)$ holds for all $i < |G|$. If G contains a chordless n-path, then we are done. Hence assume that G does not contain such a chordless n-path.

For each $x < y$ in G, fix a chordless path $x = a_0(x,y) < a_1(x,y) < \cdots < a_{N(x,y)}(x,y) = y$. (By assumption, there is an increasing path from x to y and hence there is a minimal length increasing path from x to y which is necessarily chordless.) Note that $1 \leq N(x,y) \leq n-2$. We use these fixed chordless paths to define a coloring of $[G]^4$ using $(n-1)^2+1$ many colors. For each $0 \leq i, j \leq n-2$ let

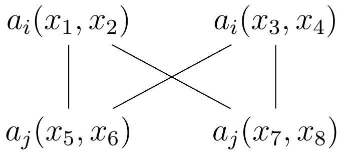
$$K_{i,j} = \{\{x,y,u,v\} \mid x < y < u < v \wedge N(x,y) \geq i \wedge N(u,v) \geq j \wedge E(a_i(x,y), a_j(u,v))\}.$$

That is, an increasing 4-tuple $\langle x, y, u, v \rangle$ is assigned color $K_{i,j}$ if the i-th vertex in the fixed chordless path from x to y is connected to the j-th vertex in the fixed chordless path from u to v. Let

$$K = \{\{x,y,u,v\} \mid x < y < u < v \wedge \forall i \neq j \leq n-1\,(\{x,y,u,v\} \notin K_{i,j})\}$$

be the color of any element of $[G]^4$ not colored by any of the $K_{i,j}$ colors. (Note that some 4-tuples may be assigned more than one color of the form $K_{i,j}$. This does not affect the statement of the finite Ramsey theorem.)

By the finite Ramsey theorem, G must have a homogeneous set of size n' for one of these colors. First consider the case when this homogeneous set is for a color $K_{i,j}$. Since $n' \geq 8$, we have elements $x_1 < x_2 < \cdots < x_8$ in our homogeneous set. By the definition of $K_{i,j}$, we have at least the following edges in G:



Thus, the vertices $a_i(x_1,x_2)$, $a_i(x_3,x_4)$, $a_j(x_5,x_6)$ and $a_j(x_7,x_8)$ form a copy of $K_{2,2}$.

Second, consider the case when this homogeneous set is for the color K. In this case we derive a contradiction by showing that G has a chordless n-path. Since $n' \geq n+1$, we have elements $x_0 < x_1 < \cdots < x_{n-1} < x_n$ in our homogeneous set. Thus, we have a path

$$x_0 = a_0(x_0,x_1) < a_1(x_0,x_1) < \cdots < a_{N(x_0,x_1)}(x_0,x_1) = x_1 = a_0(x_1,x_2) < \cdots$$
$$\cdots < a_{N(x_1,x_2)}(x_1,x_2) = x_2 = a_0(x_2,x_3) < \cdots < a_{N(x_{n-1},x_n)}(x_{n-1},x_n) = x_n.$$

Define a sequence of vertices $y_0 < y_1 < \ldots$ from the vertices of this path as follows: take $y_0 = x_0$ and for $i > 0$ take y_{i+1} to be the greatest vertex w on the path such that $E(y_i, w)$ holds. Continue until either y_{n-1} has been defined or until the vertices of the path have been exhausted. Since $\{x_0, \ldots, x_n\}$ is homogeneous for K, it follows that for each $x_j \leq v \leq x_{j+1}$ on the path ($0 \leq j \leq n-2$), the greatest vertex w on the path such that $E(v, w)$ holds satisfies $w \leq x_{j+2}$, and so for all i we have $y_i \leq x_{i+1}$. This shows that the process of defining the y_i's terminates with the definition of y_{n-1}. By construction, $\{y_0, \ldots, y_{n-1}\}$ is the vertex set of a chordless path. $\square$

It would be of interest to know the minimum $m = m(n)$ such that all finite traceable graphs G with $|G| \geq m$ either contain a copy of $K_{2,2}$ or contain a chordless n-path. Our proof shows that there is a constant $c > 0$ such that for all $n \geq 2$ we have

$$m(n) \leq t_{c\lceil \log n \rceil}(2)$$

where the tower function $t_k(x)$ is defined recursively by $t_1(x) = x$ and $t_k(x) = 2^{t_{k-1}(x)}$ for $k > 1$. (This is an easy calculation based on known bounds for finite Ramsey numbers.) Presumably this is far from the truth, but any substatial improvement would require a new aproach to the proof of Theorem 3.1.

Before proving Theorem 1.3 in RCA_0, we give numerous definitions from lattice theory within RCA_0. A *lattice* is a quadruple $(L, \leq_L, \wedge_L, \vee_L)$ such that $L \subseteq \mathbb{N}$, $\leq_L$ is a binary relation on L satisfying the axioms for a partial order, and $\wedge_L$ and $\vee_L$ are functions from $L \times L$ into L such that for all $x, y \in L$, $x \wedge_L y$ is the greatest lower bound of x and y, and $x \vee_L y$ is the least upper bound of x and y. (Typically we will drop the subscripts on $\leq$, $\wedge$ and $\vee$.) We denote the least element of L (if it exists) by 0_L and we denote the greatest element of L (if it exists) by 1_L.

A *lattice of length 3* is a lattice with a least element and a greatest element such that every element $x \notin \{0_L, 1_L\}$ is either an atom (i.e. there are no elements y such that $0_L < y < x$) or a coatom (i.e. there are no elements y such that $x < y < 1_L$).

Lemma 3.2 (RCA_0)**.** *Let L be a lattice of length 3. There do not exist atoms $x \neq y$ and coatoms $u \neq v$ such that $x <_L u$, $x <_L v$, $y <_L u$ and $y <_L v$.*

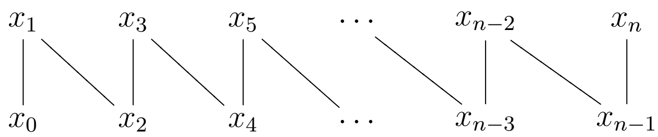
Proof. Suppose for a contradiction there are such elements. Since atoms are incomparable $x <_L x \vee y$ and since coatoms are incomparable $x \vee y <_L u$. Therefore, $0_L <_L x <_L x \vee y <_L u <_L 1_L$, contradicting the definition of length 3. $\square$

Let L be a lattice. For each finite subset $\{g_0, g_1, \ldots, g_k\}$ of elements of L, we define an increasing sequence of finite subsets $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ of L by

$$F_1 = \{g_0, g_1, \ldots, g_k\}$$
$$F_{n+1} = \{x \wedge y \mid x, y \in F_n\} \cup \{x \vee y \mid x, y \in F_n\}.$$

(More formally, we define a sequence of finite set codes for these sets. Although we can form this sequence of finite sets, we cannot in general form their union in RCA_0 as that uses Σ^0_1 comprehension.) We say L *is finitely generated* if there exists a finite set $\{g_0, \ldots, g_k\}$ such that $\forall x \in L\, \exists n\, (x \in F_n)$.

A lattice L *contains arbitrarily long finite fences* if for every odd n, there is a sequence of elements $x_0, x_1, \ldots, x_n$ of L such that the Hasse diagram for these elements looks like



That is, $x_0 <_L x_1$, for each even i with $0 < i < n$, $x_i <_L x_{i-1}$ and $x_i <_L x_{i+1}$, and no other comparability relations hold between these elements.

We can now formalize the proof of Theorem 1.3 (restated below) in RCA_0. The classical part of this proof is a straightforward formalization of the proof given in [1] with an application of Theorem 3.1 in place of an application of Theorem 1.1.

Theorem 3.3 (RCA_0)**.** *Every finitely generated infinite lattice of length 3 contains arbitrarily long finite fences.*

Proof. Because this theorem is a Π^1_1 statement and RCA_0 is conservative over WKL_0 for Π^1_1 statements, it suffices to give a proof in WKL_0. Therefore, we work in WKL_0.

Let L be an infinite lattice of length 3 which is finite generated by $\{g_0, \ldots, g_k\}$. Define the finite subsets $F_0 \subseteq F_1 \subseteq \cdots$ as above. We say that an element $x \in L$ has rank 0 if $x \in F_0$ (i.e. x is a generator of L). We say x has rank $n+1$ if $x \in F_{n+1} \setminus F_n$. Note that every element has a rank and there are only finitely many elements of each rank. Therefore, since L is infinite, for every $n \in \mathbb{N}$, there is an element of rank n. Furthermore, there is a function $r(x)$ giving the rank of each element and there is a function $m(n)$ such that

$$\forall x \in L \, \forall n \in \mathbb{N} \, (r(x) = n \rightarrow x \leq m(n)).$$

Form a tree $T \subseteq (L \setminus \{0_L, 1_L\})^{<\mathbb{N}}$ as follows. The sequence $\langle x_0, x_1, \ldots, x_n \rangle \in T$ if and only if for every $i \leq n$, $r(x_i) = i$, and for every $0 < i \leq n$, there is an $a \in L$ with $r(a) < i$ such that $x_i = x_{i-1} \vee a$ or $x_i = x_{i-1} \wedge a$. T has the following properties.

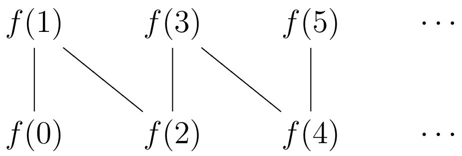
(P1) If $\langle x_0, \ldots, x_n \rangle \in T$, then the x_i are distinct and for all $i < n$, x_i is comparable with x_{i+1}. This property follows since $r(x_i) = i$, $r(x_{i+1}) = i + 1$ and $x_{i+1} = x_i \wedge a$ or $x_{i+1} = x_i \vee a$ for some $a \in L$. (Because L has length 3, the x_i elements alternate between atoms and coatoms.)

(P2) For every $x \in L \setminus \{0_L, 1_L\}$, there is a $\sigma \in T$ such that $\sigma * x \in T$. This property follows by induction on the rank of x. If the $r(x) = 0$, then $\sigma = \emptyset$. If $r(x) = n + 1$, then $x \in F_{n+1} \setminus F_n$. Without loss of generality suppose $x = y \wedge z$ where $y, z \in F_n$. Then $r(y), r(z) \leq n$ and either $r(y) = n$ or $r(z) = n$. Suppose $r(y) = n$. By the induction hypothesis, there is a $\tau \in T$ such that $\tau * y \in T$. Let $\sigma = \tau * y$ and it follows from the definition of T that $\sigma * x \in T$.

(P3) T is infinite. This property follows from (P2) and the fact that L has elements of rank n for each $n \in \mathbb{N}$.

(P4) The branching in T is bounded by the function $m(n)$ in the sense that if $\sigma \in T$ then for all $i < |\sigma|$, $\sigma(i) \leq m(i)$. This property follows from the definition of T.

Since T is an infinite tree with bounded branching, WKL_0 proves that T has an infinite path $f : \mathbb{N} \to L \setminus \{0_L, 1_L\}$. (See Lemma IV.1.4 in Simpson [3].) By (P1), f is 1-to-1. Furthermore the range of f exists since $x \in \text{range}(f)$ if and only if $f(n) = x$ where $n = r(x)$. If $f(0)$ is an atom, then the Hasse diagram of the range of f contains at least the following comparability relations



and may contain additional comparability relations. If $f(0)$ is a coatom, then we obtain the dual of this picture. To avoid breaking into simple dual cases, we will assume $f(0)$ is an atom for the remainder of the proof.

Define a graph $G = (V, E)$ with $V = \text{range}(f)$ and $E(f(n), f(m))$ holds if and only if $f(n)$ and $f(m)$ are comparable in L. G looks like the Hasse diagram above with possibly additional edges (since the lattice elements in this diagram could have additional comparability relations). However, each $f(2n)$ is an atom and each $f(2n+1)$ is a coatom. Therefore, by Lemma 3.2, G does not contain a copy of $K_{2,2}$. By (P1), f is a tracing function for G, so G is an infinite traceable graph that does not contain a copy of $K_{2,2}$. Therefore, by Theorem 3.1, G contains arbitrarily long finite chordless paths. Since finite chordless paths in G are finite fences when viewed in L, L contains arbitrarily long finite fences.

$\square$

References

[1] F. Galvin, I. Rival and B. Sands, "A Ramsey-type theorem for traceable graphs," *Journal of Combinatorial Theory Series B* **33**, 1982, 7–16.

[2] P. Hájek and P. Pudlák, *Metamathematics of first–order arithmetic*, Springer–Verlag, 1993.

[3] S.G. Simpson, *Subsystems of second order arithmetic*, Springer–Verlag, Heidelberg, 1999.

[4] R.I. Soare, *Recursively enumerable sets and degrees*, Springer–Verlag, Heidelberg, 1987.