$[4]^2$ Tic-Tac-Toe is a Draw

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Abstract

We give a short explicit draw strategy for the second player in four by four tic-tac-toe.

The game of $[4]^2$ tic-tac-toe is played on a 4 by 4 grid. Two players, I (who goes first) and II, alternately occupy previously unoccupied squares of the grid; the winner is the player who first occupies all four squares of any of the ten horizontal, vertical or diagonal lines on the grid. The game is a draw if, after all squares have been occupied, neither player completely occupies a winning line.

An easy but lengthy case analysis shows that II has a draw strategy - he can occupy at least one square of each winning line, irrespective of I's play. C. Y. Lee (see [2], chapter 22) gives a short explicit strategy for II, but the proof that this works is based on the fact that ordinary (three by three) tic-tac-toe is a draw, a result which itself requires a case analysis. Beck [1] gave a one line proof of this result, using the Erdős-Selfridge lemma [3]. In this note, we give another demonstration that avoids a case analysis, using a pairing strategy.

A *pairing strategy* in a tic-tac-toe-like game is a selection of a pair of squares from each winning line, in such a way that the pairs are mutually disjoint. Clearly, if such a selection can be made, then II can occupy at least one square in each winning line. No pairing strategy can exist in $[4]^2$ tic-tac-toe, since the number of winning lines (ten) is more than twice the number of squares (sixteen). But, after I's first move, II can always occupy a square which belongs to three winning lines, at which point there are seven winning lines left that II has not occupied, and fourteen unoccupied squares, giving some hope for a pairing strategy. This hope can be realized, as is seen in the following three pictures:

$$\begin{bmatrix} X & A & B & A \\ D & O & F & G \\ D & F & C & C \\ E & E & B & G \end{bmatrix} \quad \begin{bmatrix} B & X & A & A \\ B & O & C & D \\ E & C & G & E \\ F & F & G & D \end{bmatrix} \quad \begin{bmatrix} A & A & B & C \\ D & O & X & G \\ D & C & F & F \\ E & E & B & G \end{bmatrix}$$

The explanation is as follows: by symmetry, I has three different opening moves, marked by X. In each case, II responds with O, and from then on follows the pairing strategy shown in the picture - after each of I's moves, II takes to other square which has the same letter as the square I has just taken.

References

[1] J. Beck, oral communication.

[2] Conway, Berlecamp and Guy - Winning Ways volume 2.

[3] P. Erdős and J. L. Selfridge, On a Combinatorial Game, *J. Combinatorial Theory Ser. A* **14** (1973), 298-301.