TWO PROBLEMS INVOLVING THE NOTION OF PHASE TRANSITION

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ABSTRACT OF THE DISSERTATION

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The first problem we consider is from statistical physics. Write $\mathcal{I} = \mathcal{I}(\Sigma)$ for the set of independent sets of the graph Σ. For Σ finite and $\lambda > 0$, the *hard-core measure* with *activity* λ on $\mathcal{I}$ is given by

$$\mu(I) = \lambda^{|I|}/Z \qquad \forall I \in \mathcal{I},$$

where $Z = \sum\{\lambda^{|I|} : I \in \mathcal{I}\}$ is the appropriate normalizing constant. We say that this measure is *hc(λ)*.

For infinite Σ a measure μ on $\mathcal{I}$ is hc(λ) if for $\mathbf{I}$ chosen from $\mathcal{I}$ according to μ and for all finite $W \subseteq V = V(G)$, the conditional distribution of $\mathbf{I} \cap W$ given $\mathbf{I} \cap (V \setminus W)$ is (μ-a.s.) hc(λ) on the independent sets of $\{w \in W : w \not\sim \mathbf{I} \cap (V \setminus W)\}$. There is always at least one such measure. If there is more than one, the model is said to have a *phase transition.*

Dobrushin [9] (and later, independently, Louth [21]) showed that there is a phase transition in the hard-core model on (the usual nearest neighbor graph on) $\mathbf{Z}^d$ for sufficiently large values of λ (depending on d). In other words, they showed that

$$\lambda(d) := \sup \left\{ \lambda : \begin{array}{c} \text{the hard-core model with activity } \lambda \text{ on} \\ \mathbf{Z}^d \text{ does not have a phase transition} \end{array} \right\} < \infty.$$

Up to now, all known bounds for $\lambda(d)$ increased rapidly with d. However, it has been widely conjectured that $\lambda(d) \to 0$ as $d \to \infty$. This is what we prove.

The second problem we consider comes from discrete probability. It was introduced by Benjamini, Häggström and Mossel [2] (and, in a different context, by Athanasiadis [1]).

Write $\mathcal{F}$ for the set of homomorphisms from the d-dimensional Hamming cube $\{0,1\}^d$ to (the Hamming graph on) $\mathbf{Z}$ which send $\underline{0}$ (the all-zero string) to 0 and $\mathcal{F}_{\leq 5}$ for those which take on five or fewer values. (A *homomorphism* between graphs is an adjacency preserving map between vertex sets.) We show that $|\mathcal{F}| \sim |\mathcal{F}_{\leq 5}| \sim 2e2^{2^{d-1}}$, proving a conjecture of Kahn [15].

This result can be viewed as a "phase transition" statement: with high probability, a randomly chosen $f \in \mathcal{F}$ will be either predominantly 0 on the even vertices of the cube (those vertices whose l_1 distance from $\underline{0}$ is even), with occasional ± 2's, or predominantly 1 (resp. -1) on the odd vertices, with occasional -1's and 3's (resp. -3's and 1's).

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Dedication

To Nastia and Petr

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Chapter 1

Introduction and background

In this chapter, we introduce the two problems with which the thesis is concerned and state the results that we prove. In Chapter 2, we establish the notation and conventions that will be used throughout, list the results from the literature that we will be drawing on, and derive some isoperimetric inequalities and a combinatorial fact with a topological flavor that will be used in subsequent chapters. In Chapter 3 we introduce some notions of "approximation" that will play a role in both of the problems under consideration in the thesis. In Chapter 4, we deal with the problem of phase transition in the hard-core model on $\mathbf{Z}^d$, and in Chapter 5 we consider the problem of homomorphisms from the Hamming cube to $\mathbf{Z}$.

For graph theory basics, see e.g. [5], [8]. For basics of the combinatorics of the Hamming cube, see e.g. [4].

1.1 Phase transition in the hard-core model on $\mathbf{Z}^d$

Write $\mathcal{I}(\Sigma)$ for the collection of independent sets of a graph Σ. (An *independent set* is a set of vertices, no two of which are joined by an edge.)

For Σ finite and $\lambda > 0$, the *hard-core measure* with *activity* (or *fugacity*) λ on $\mathcal{I} = \mathcal{I}(\Sigma)$ (or "on Σ") is given by

$$\mu(I) = \lambda^{|I|}/Z \qquad \text{for} \quad I \in \mathcal{I},$$

where Z is the appropriate normalizing constant (*partition function*),

$$Z = \sum\{\lambda^{|I'|} : I' \in \mathcal{I}\}.$$

(For good introductions to the hard-core model see [3], [13]; see also [11] for more general background. The more usual etiquette here considers probability measures on $\{0,1\}^{V(\Sigma)}$

supported on indicators of independent sets; but the present usage is convenient for us, and we adhere to it throughout.)

In particular $\lambda = 1$ gives uniform distribution. One may also assign different activities λ_v to the different vertices v and take $\mu(I)$ proportional to $\prod_{v\in I} \lambda_v$, but we will not do so here; again see [3], [13].

We will also say the above measure is *hc(λ)*. For infinite Σ a measure μ on $\mathcal{I}(\Sigma)$ is hc(λ) if, for $\mathbf{I}$ chosen according to μ and for each finite $W \subseteq V = V(\Sigma)$, the conditional distribution of $\mathbf{I} \cap W$ given $\mathbf{I} \cap (V \setminus W)$ is μ-a.s. hc(λ) on the independent sets of $\{w \in W : w \not\sim \mathbf{I} \cap (V \setminus W)\}$ (the vertices of W that can still be in $\mathbf{I}$ given $\mathbf{I} \cap (V \setminus W)$). General considerations (see [11]) imply that there is always at least one such μ; if there is more than one, the model is said to have a *phase transition*.

The canonical (and by far most studied) case of the hard-core model is that of (the Hamming, or nearest neighbor graph on) $\mathbf{Z}^d$, with all activities equal (to λ). Here the seminal result is due to Dobrushin [9], who proved that there is a phase transition for sufficiently large λ, depending on d. (Dobrushin's result was rediscovered by Louth [21] in the context of communications networks.)

The λ required in [9] grows as d grows, and attempted improvements have been the subject of considerable effort — if not publication — in both the statistical mechanics and discrete mathematics communities in recent years.

That the required λ increases with d is a little strange, since one expects that as d grows, phase transition should get "easier", in the sense that for a given λ, phase transition in dimension d should imply phase transition in all higher dimensions; but this remains open.

Also open is the existence of a "critical" activity, $\lambda_c = \lambda_c(d)$, such that one has phase transition for $\lambda > \lambda_c(d)$ but not for $\lambda < \lambda_c(d)$. While this seems certain to be true for $\mathbf{Z}^d$, a cautionary note is sounded in [7], where it is shown that there are graphs (even trees) for which there is no such critical activity.

As a temporary substitute one might instead define

$$\lambda(d) := \sup \left\{ \lambda : \begin{array}{c} \text{the hard-core model with activity } \lambda \text{ on} \\ \mathbf{Z}^d \text{ does not have a phase transition} \end{array} \right\} < \infty.$$

So Dobrushin says that $\lambda(d)$ is finite, while "easier as dimension grows" would imply $\lambda(d) = O(1)$. A particular question that has received much of the attention devoted to this problem is whether $\lambda(d) \leq 1$ for large d.

But in fact it has been generally believed (despite some early guesses to the contrary) that $\lambda(d)$ tends to zero. This is what we prove:

Theorem 1.1.1 $\lambda(d) = O(d^{-1/4} \log^{3/4} d)$.

This bound is undoubtedly not best possible; $O(\log d/d)$ and $O(1/d)$ are seemingly natural guesses at the true value of $\lambda(d)$.

Much of the inspiration for our approach to this problem was provided by the beautiful ideas of A. A. Sapozhenko [23], which he used to give, for example, relatively simple derivations of Korshunov's [19] description of the asymptotics for Dedekind's Problem (in [25]), and, in [24], of the asymptotics for the number of independent sets ("codes of distance 2") in the Hamming cube $\{0,1\}^n$ originally established in [20].

Some of our tools also come from [23]: Lemma 3.3.3 is an improved version of one of Sapozhenko's arguments, and our uses of Lemmas 2.4.1-2.4.4 are similar to his.

1.2 Homomorphisms from the Hamming cube to Z

Write Q_d for the d-dimensional Hamming cube (the graph whose vertex set is $\{0,1\}^d$ and in which two vertices are joined by an edge if they differ in exactly one coordinate). Set

$$\mathcal{F} = \{f : V(Q_d) \to \mathbf{Z} : f(\underline{0}) = 0 \text{ and } u \sim v \Rightarrow |f(u) - f(v)| = 1\}$$

(That is, $\mathcal{F}$ is the set of graph homomorphisms from Q_d to $\mathbf{Z}$, normalized to vanish at $\underline{0}$.)

In [2], this set of functions is studied from a probabilistic point of view, a motivating idea being that a typical element of $\mathcal{F}$ should exhibit stronger concentration behavior than an arbitrary element. Put uniform probability measure on $\mathcal{F}$, and define the function R on $\mathcal{F}$ by $R(f) = \{f(v) : v \in V(Q_d)\}$ (R is the *range* of f). In [2] the following conjecture is made about the concentration of $|R|$:

Conjecture 1.2.1 *For each $t > 0$, $\mathbf{P}(|R| > td) \to 0$ as $d \to \infty$.*

In [15], something stronger is proved, and something stronger still conjectured:

Theorem 1.2.2 *There is a constant b such that $\mathbf{P}(|R| > b) = e^{-\Omega(d)}$.*

Conjecture 1.2.3 *$\mathbf{P}(|R| > 5) = e^{-\Omega(d)}$ and $\mathbf{P}(|R| = 5) = \Omega(1)$.*

In Chapter 5, we prove Conjecture 1.2.3 by (asymptotically) counting the number of homomorphisms with various ranges. Specifically, if we set

$$\mathcal{F}_i = \{f \in \mathcal{F} : |R(f)| = i\},$$

we prove

Theorem 1.2.4

$$\begin{aligned} \left| |\mathcal{F}| - 2e2^{2^{d-1}} \right| &= e^{-\Omega(d)} 2^{2^{d-1}} \\ \left| |\mathcal{F}_3| - (2)2^{2^{d-1}} \right| &= e^{-\Omega(d)} 2^{2^{d-1}} \\ \left| |\mathcal{F}_4| - (4\sqrt{e} - 4)2^{2^{d-1}} \right| &= e^{-\Omega(d)} 2^{2^{d-1}} \\ \left| |\mathcal{F}_5| - (2e - 4\sqrt{e} + 2)2^{2^{d-1}} \right| &= e^{-\Omega(d)} 2^{2^{d-1}} \end{aligned} \tag{1.1}$$

Setting $\mathcal{F}_{\leq 5} = \cup_{i \leq 5} \mathcal{F}_i$, we see that Theorem 1.2.4 has the following weaker but more elegantly formulated consequence:

Corollary 1.2.5 $|\mathcal{F}| \sim |\mathcal{F}_{\leq 5}| \sim 2e2^{2^{d-1}}$.

Corollary 1.2.5 makes sense: a little thought suggests that a typical member of $\mathcal{F}$ should be constant on either even or odd vertices of the cube, except for a small set of "blemishes" on which it takes values 2 away from the predominant value, and take just two values on vertices of the other parity.

The problem under discussion is equivalent to the question of the number of rank functions on the Boolean lattice $2^{[d]}$ (here $[d] = \{1, \ldots, d\}$). A *rank function* is an $f : 2^{[d]} \longrightarrow \mathbf{N}$ satisfying $f(\emptyset) = 0$ and $f(A) \leq f(A \cup x) \leq f(A) + 1$ for all $A \in 2^{[d]}$ and $x \in [d]$. An easy lower bound on the number of rank functions is $2^{2^{n-1}}$ (consider those functions which take the value $k/2$ on each element of the kth level of the Boolean lattice for each even k). Athanasiadis [1] conjectured that the total number of rank functions is $2^{2^{n-1}(1+o(1))}$. This conjecture is proved in [16], where it is further conjectured that the number is in fact $O(2^{2^{d-1}})$. Theorem 1.2.4 answers this conjecture in the affirmative; for, as observed in [15], there is a bijection from the set of rank functions to $\mathcal{F}$: identifying a subset A of $[n]$ with a vertex of Q_d in the natural way, the bijection is given by $g \longrightarrow f$ where $f(A) = 2g(A) - |A|$.

As with the problem of phase transition in the hard-core model, our work on this problem is inspired by the papers of Sapozhenko. Our Lemma 3.3.4 is a modification of a lemma in [23], and our overall approach is similar to [24]. Indeed, Theorem 1.2.4 contains the main result of [24] (and [20]), which states that the number of independent sets in Q_d is asymptotically $2\sqrt{e}2^{2^{d-1}}$. The short derivation of this from Theorem 1.2.4 is given in Section 5.7.

Chapter 2

Notation, conventions and preliminary material

In this chapter we establish the notation that we will use throughout the rest of the thesis, gather together all the existing results that we will be drawing on and derive the isoperimetric inequalities in the lattice and the cube that we will need in Chapters 4 and 5. We also present a topological proof of a combinatorial fact that will be used in Chapter 4.

2.1 Notation and conventions

Let Σ be a graph on vertex set $V = V(\Sigma)$. For $u, v \in V$ and $A, C \subseteq V$ we write $u \sim v$ if there is an edge in Σ joining u and v, $\nabla(A)$ for the set of edges having exactly one end in A and (when $A \cap C = \emptyset$) $\nabla(A, C)$ for the set of edges having one end in each of A, C.

Set $N(u) = \{w \in V : w \sim u\}$ ($N(u)$ is the *neighborhood* of u), $N(A) = \cup_{w \in A} N(w)$, $N_C(u) = \{w \in C : w \sim u\}$, $N_C(A) = \cup_{w \in A} N_C(w)$, $d(u) = |N(u)|$ and $d_C(u) = |N_C(u)|$. Write $\rho(u, v)$ for the length of the shortest u-v path in Σ, and set $\rho(u, A) = \min_{w \in A}\{\rho(u, w)\}$ and $\rho(A, C) = \min_{w \in A, w' \in C}\{\rho(w, w')\}$. Set $B(A) = \{v \in V : N(v) \subseteq A\}$.

We define the *boundary* of A by $\partial A = N(A) \setminus A$ and the *internal boundary* of A by $\partial^\star A = \{w \in A : N(w) \not\subseteq A\}$.

We define the *closure* of A to be $[A] = \{v \in V : N(v) \subseteq N(A)\}$ and say that A is *closed* if $[A] = A$.

We say that A is *k-linked* if for every $u, v \in A$ there is a sequence $u = u_0, u_1, \ldots, u_l = v$ in A with $\rho(u_i, u_{i+1}) \leq k$ for $i = 0, \ldots, l-1$. Note that for any k, A is the disjoint union of its maximal k-linked subsets — we call these the *k-components* of A.

For integers $a < b$ we define $[a, b] = \{a, \ldots, b\}$.

We often abuse notation by failing to distinguish between a graph and its set of vertices; so for instance we might use "component" where we should really say "set of vertices of a component."

When the difference makes no difference, we pretend that all large numbers are integers. We use "ln" for the natural logarithm and "log" always means the base 2 logarithm. The implied constants in the O and Ω notation are absolute (independent of d). We always assume that d is large enough to support our assertions. No attempt has been made to optimize constants.

2.2 Notation specific to the lattice

In Chapter 4 we will mainly be concerned with the Hamming (or nearest neighbor) graph on $\mathbf{Z}^d$. This is a $(2d)$-regular, bipartite graph. Write X for the set of odd vertices of $\mathbf{Z}^d$ (those vertices whose distance from the origin $\underline{0}$ is odd) and Y for the set of even vertices.

For $N \in \mathbf{N}$, set $\Lambda_N = \mathbf{Z}^d \cap [-N, N]^d$, and write $\Gamma = \Gamma_N$ for the discrete torus obtained from Λ_N by setting $N = -N$ and identifying vertices accordingly. Write Δ for the image of $\partial^\star \Lambda_N$ under the natural projection $\Lambda_N \mapsto \Gamma$, and $\underline{0}$ for the image of $\underline{0}$ in Γ. As with $\mathbf{Z}^d$, we use X and Y for the sets of odd and even vertices of Γ. This should not cause confusion, as it should always be clear which graph is under consideration.

2.3 Notation specific to the cube

All our work in Chapter 5 will be with Q_d, the d-dimensional Hamming cube. This is a d-regular, bipartite graph. Write V for the vertex set of the cube, $\mathcal{E}$ for the set of even vertices (those whose ℓ_1 distance from $\underline{0}$ is even) and $\mathcal{O}$ for the set of odd vertices. Set $M = 2^{d-1} = |\mathcal{E}| = |\mathcal{O}|$.

For $A \subseteq V$, we say that A is *small* if $|A| < \alpha^d$ for a certain constant $\alpha < 2$ that will be discussed in Section 5.1 (and *large* otherwise), *sparse* if all the 2-components of A are singletons (and *non-sparse* otherwise), and *nice* if A is small, 2-linked and not a singleton. Write $C \prec A$ if C is a 2-component of A and $c(A)$ for the number of 2-components of A. All sets A that we will consider will satisfy either $A \subseteq \mathcal{E}$ or $A \subseteq \mathcal{O}$.

2.4 External ingredients

We list here the main results that we will be drawing on throughout the rest of the thesis.

We begin with a lemma bounding the number of connected subgraphs of a graph. The infinite Δ-branching rooted tree contains precisely $\binom{\Delta n}{n}/((\Delta-1)n+1)$ rooted subtrees with n vertices (see e.g. Exercise 11 (p. 396) of [17]) and this implies that if G is a graph with maximum degree Δ and vertex set $V(G)$ then the number of n-vertex subsets of $V(G)$ which contain a fixed vertex and induce a connected subgraph is at most $(e\Delta)^n$. (This fact is rediscovered in [23].) We will use the following easy corollary.

Lemma 2.4.1 *Let Σ be a graph with vertex set $V(\Sigma)$ and maximum degree Δ. For each fixed k, the number of k-linked subsets of $V(\Sigma)$ of size n which contain a fixed vertex is at most $2^{O(n\log\Delta)}$.*

This follows from the fact that a k-linked subset of Σ is connected in a graph with all degrees $O(\Delta^{k+1})$.

We will need the following fundamental result due to Lovász [22] and Stein [26]. Recall that a *hypergraph* is a collection $\mathcal{H}$ of subsets of a "vertex set" $V(\mathcal{H})$. The *vertex cover number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the least size of a set of vertices meeting all edges of $\mathcal{H}$; the *fractional vertex cover number* $\tau^*(\mathcal{H})$ is

$$\min\left\{\sum t(v) \mid t : V(\mathcal{H}) \to \mathbf{R}^+,\ \sum_{v\in A} t(v) \geq 1\ \forall A \in \mathcal{H}\right\};$$

and the degree of a vertex in $V(\mathcal{H})$ is the number of edges of $\mathcal{H}$ containing it. (See [10] for more hypergraph background.)

Lemma 2.4.2 *For any hypergraph $\mathcal{H}$ with all degrees at most Δ,*

$$\tau(\mathcal{H}) \leq \tau^*(\mathcal{H})(1+\ln\Delta).$$

In this thesis all uses of Lemma 2.4.2 take the following form. For a bipartite graph Σ with bipartition $X \cup Y$, we say that $Y' \subseteq Y$ *covers* X if each $x \in X$ has a neighbor in Y'.

Lemma 2.4.3 *If Σ as above satisfies $|N(x)| \geq a$ for each $x \in X$ and $|N(y)| \leq b$ for each $y \in Y$, then X is covered by some $Y' \subseteq Y$ with $|Y'| \leq (|Y|/a)(1 + \ln b)$.*

(To apply Lemma 2.4.2, take $\mathcal{H} = \{N(x) : x \in X\}$, noting that the constant function $t(y) = 1/a \ \forall y \in Y$ gives $\tau^*(\mathcal{H}) \leq |Y|/a$.)

The next lemma is from [23] (see Lemma 2.1); the reader should have no difficulty supplying a proof.

Lemma 2.4.4 *If Σ is a graph on vertex set $V(\Sigma)$ and $A, C \subseteq V(\Sigma)$ satisfy*

(i) *A is k-linked*

and

(ii) *$\rho(u, C) \leq l$ for each $u \in A$ and $\rho(v, A) \leq l$ for each $v \in C$*

then B is $(k + 2l)$-linked.

The main step from the proof of Theorem 1.2.2 in [15] (obtained via entropy arguments) will also be used here. For $f \in \mathcal{F}$, set $C(f) = \{v \in V(Q_d) : f|_{N(v)} \text{ is constant}\}$. From [15], we get

Lemma 2.4.5 *For $u \sim v$ and $\mathbf{f}$ drawn uniformly from $\mathcal{F}$, $\mathbf{P}(|\{u, v\} \cap C(\mathbf{f})| = 1) = 1 - e^{-\Omega(d)}$.*

We also make use of some known results concerning isoperimetry in the lattice and the cube.

For $x \in \mathbf{Z}^d$ write $|x|$ for the ℓ_1-norm of x, and set $B(r) = \{x \in \mathbf{Z}^d : |x| \leq r\}$, $S(r) = \{x \in \mathbf{Z}^d : |x| = r\}$, $b(r) = |B(r)|$ and $s(r) = |S(r)|$.

Lemma 2.4.6 *Let C be a subset of $\mathbf{Z}^d$ with*

$$|C| = b(r) + \alpha s(r + 1),$$

where $0 \le \alpha < 1$. *Then*

$$|\partial C| \ge (1-\alpha)s(r+1) + \alpha s(r+2).$$

This is an immediate consequence of a corresponding inequality for the torus Γ_N (for even N), given by Bollobás and Leader in [6, Cor. 5]. The case $\alpha = 0$ of Lemma 2.4.6 was proved by Wang and Wang [27].

A *Hamming ball centered at* x_0 in Q_d is any set of vertices B satisfying

$$\{u \in V : \rho(u, x_0) \le k\} \subseteq B \subset \{u \in V : \rho(u, x_0) \le k+1\}$$

for some $k < d$. An *even* (resp. *odd*) *Hamming ball* is a set of vertices of the form $B \cap \mathcal{E}$ (resp. $B \cap \mathcal{O}$) for some Hamming ball B. We use the following result of Körner and Wei [18].

Lemma 2.4.7 *For every* $C \subseteq \mathcal{E}$ *(resp.* $\mathcal{O}$*) and* $D \subseteq V$*, there exists an even (resp. odd) Hamming ball* C' *and a set* D' *such that* $|C'| = |C|$, $|D'| = |D|$ *and* $\rho(C', D') \ge \rho(C, D)$.

2.5 Isoperimetry in the lattice

In this section, we use Lemma 2.4.6 to establish lower bounds for the neighborhood size of a single-parity set in the d-dimensional lattice $\mathbf{Z}^d$. As before, we write X for the set of odd vertices of the lattice.

Lemma 2.5.1 *Suppose* $A \subseteq X$ *is finite. Then (writing* N *for* $N(A)$*)*

$$|A| \le \begin{cases} O(1/d)|N| & \text{if } |A| < d^{O(1)} \\ \left(1 - \Omega\left(|N|^{-1/d}/d\right)\right)|N| & \text{for all } A. \end{cases}$$

Proof: For the first bound, notice that for $r < O(1)$,

$$s(r) = 2^r d^r / r! + O(d^{r-1}),$$

which in view of Lemma 2.4.6 implies that for $C \subseteq \mathbf{Z}^d$ with $|C| < d^{O(1)}$,

$$|\partial C| = \Omega(|C|d).$$

Applying this with $C = A$ immediately gives $|A| \le O(1/d)|N|$.

For the second bound, the main thing we have to show is

Proposition 2.5.2 *With $b(r)$ and $s(r)$ as defined before Lemma 2.4.6,*

$$s(r) = \Omega\left(b(r)^{1-1/d}\right).$$

Notice that this, combined with Lemma 2.4.6, implies that for any $C \subseteq \mathbf{Z}^d$,

$$|\partial C| = \Omega\left(|C|^{(d-1)/d}\right). \tag{2.1}$$

Before proving Proposition 2.5.2, we complete the proof of Lemma 2.5.1. Write N_0 for $\partial^\star(A \cup N)$ $(= \{y \in N : N(y) \not\subseteq A\})$. We consider the possibilities $|N_0| > |A|$ and $|N_0| \leq |A|$ separately, in both cases using the fact that $|N_0| \leq |\nabla(N, X \setminus A)| = d(|N| - |A|)$.

If $|N_0| > |A|$, then $|A| < d(|N| - |A|)$, so that

$$\begin{aligned} |A| &\leq d|N|/(d+1) \\ &\leq \left(1 - \Omega\left(|N|^{-1/d}/d\right)\right)|N|. \end{aligned}$$

If, on the other hand, $|N_0| \leq |A|$, then we have (using (2.1) and the fact that $N_0 = \partial((N \setminus N_0) \cup A))$

$$\begin{aligned} |A| &\leq |N| - |N_0|/d \\ &\leq |N| - \Omega\left(|(N \setminus N_0) \cup A|^{(d-1)/d}\right)/d \\ &\leq |N| - \Omega\left(|N|^{(d-1)/d}/d\right) \\ &= \left(1 - \Omega\left(|N|^{-1/d}/d\right)\right)|N|. \end{aligned}$$

∎

Proposition 2.5.2 is something for which one would hope to just give a reference; but we could not find one, or even give the short proof that seems called for.

Proof of Proposition 2.5.2: Consider the average number of nonzero entries in an element of $S(q)$,

$$t(q) := s(q)^{-1} \sum_{x \in S(q)} |supp(x)|.$$

This is a useful quantity because, setting

$$N(q) = |\{(x, y) \in S(q) \times S(q+1) : x \sim y\}|,$$

we have

$$s(q)(2d - t(q)) = N(q) \le s(q+1)\min\{q+1, d\},$$

implying

$$\frac{s(q)}{s(q+1)} \le \frac{\min\{q+1,d\}}{2d - t(q)}. \tag{2.2}$$

This already implies Proposition 2.5.2 for, say, $r \le .9d$, since in this case we have

$$b(r) \le s(r)\sum_{i=0}^{r}\frac{(r)_i}{(2d-r+i)_i} \le s(r)\sum_{i\ge 0}\left(\frac{r}{2d-r}\right)^i = O(s(r)).$$

For larger r we will have to work harder. Here we first show, for $q = \beta d$ with $\beta > .9$,

$$t(q) < (1 - 1/(20\beta))d. \tag{2.3}$$

Let

$$S(q,t) = \{x \in S(q) : |supp(x)| = t\},$$

$s(q,t) = |S(q,t)|$, and define $B(q,t)$ and $b(q,t)$ similarly. Then

$$f(q,t) := \frac{s(q,t+1)}{s(q,t)} = 2\frac{(d-t)(q-t)}{(t+1)t}.$$

Set $t_0 = t_0(q) = \lceil (1 - 1/(4\beta))d\rceil$. Then $t \ge t_0$ implies

$$\begin{aligned} f(q,t) &\le 2\frac{(1/(4\beta))(\beta - 1 + 1/(4\beta))}{(1-1/(4\beta))^2} \\ &= 2\left(\frac{2\beta-1}{4\beta-1}\right)^2 < \frac{1}{2}. \end{aligned}$$

Thus

$$\begin{aligned} t(q) &= s(q)^{-1}\sum_{t\le q} ts(q,t) \\ &< t_0 + \sum_{i\ge 1} i2^{-i} = t_0 + 2. \end{aligned}$$

This gives (2.3) provided $\beta \le d/15$. For larger β we just use

$$\frac{s(q,d-1)}{s(q,d)} = \frac{d(d-1)}{2(\beta d - d + 1)} > \frac{d-1}{2\beta},$$

whence

$$\begin{aligned} d - t(q) &= s(q)^{-1}\sum (d-i)s(q,i) \ge \sum_{i<d} s(q,i)/(\sum_{i\le d} s(q,i)) \\ &\ge s(q,d-1)/(s(q,d-1)+s(q,d)) \ge (d-1)/(2\beta + d - 1), \end{aligned}$$

which again gives (2.3).

Now let $r = \gamma d \geq .9d$. By (2.3) and (2.2) we have, for $r - i \geq .9d$,

$$s(r-i) \leq s(r) \prod_{j=1}^{i} \frac{d}{d + d^2/(20(r-j))} < s(r)(1 - \Omega(1/\gamma))^i,$$

so

$$b(r) \leq s(r) \sum_{i=0}^{r-.9d} (1 - \Omega(1/\gamma))^i + b(.9d) = O(\gamma s(r)) \tag{2.4}$$

(since we know $b(.9d) = O(s(.9d)) = O(s(r))$).

On the other hand, with $t_0 = t_0(r)$, we have

$$b(r) > b(r, t_0) = 2^{t_0} \binom{d}{t_0} \binom{r}{t_0} > \exp[t_0 \log(r/t_0)],$$

and $b(r)^{1/d} > \exp[(1 - 1/(4\gamma)) \log(r/t_0)] = \Omega(\gamma)$; and this with (2.4) gives the proposition.

∎

2.6 Isoperimetry in the cube

The aim of this section is to put some lower bounds on the neighborhood size of a small set in the Hamming cube Q_d. We begin with

Lemma 2.6.1 *For all $A \subseteq \mathcal{E}$ or $A \subseteq \mathcal{O}$ small, $|A| \leq (1 - \Omega(1))|N(A)|$.*

Proof: By symmetry, we need only prove this when $A \subseteq \mathcal{E}$. Let small $A \subseteq \mathcal{E}$ be given. Applying Lemma 2.4.7 with $C = A$ and $D = V \setminus (A \cup N(A))$, we find that there exists an even Hamming ball A' with $|A'| = |A|$ and $|N(A)| \geq |N(A')|$. So we may assume that A is a small even Hamming ball.

We consider only the case where A is centered at an even vertex, w.l.o.g. $\underline{0}$, the other case being similar. In this case,

$$\{v \in \mathcal{E} : \rho(v, \underline{0}) \leq k\} \subseteq A \subset \{v \in \mathcal{E} : \rho(v, \underline{0}) \leq k + 2\}$$

for some even $k \leq d/2 - \Omega(d)$ (the bound on k coming from the fact that A is small). For each $0 \leq i \leq (k+2)/2$, set $B_i = A \cap \{v : \rho(v, \underline{0}) = 2i\}$, and $N^+(B_i) = N(B_i) \cap$

$\{u : \rho(u, \underline{0}) = 2i+1\}$. It's clear that $N(A) = \cup_{0 \le i \le (k+2)/2} N^+(B_i)$ and that for $i = 0, \ldots, (k+2)/2$

$$\frac{|B_i|}{|N^+(B_i)|} \le \frac{2i+1}{d-2i} \tag{2.5}$$
$$= 1 - \Omega(1), \tag{2.6}$$

from which the lemma follows. The inequality in (2.6) comes from the bound on k. The inequality in (2.5) is actually an equality except when $i = (k+2)/2$, in which case it follows from the observation that each vertex in B_{k+2} has exactly $d-(k+2)$ neighbors in $N^+(B_{k+2})$, and each vertex in $N^+(B_{k+2})$ has at most $(k+2)+1$ neighbors in B_{k+2}.

∎

Lemma 2.6.1 is true for all small A, but can be strengthened considerably when we impose stronger bounds on $|A|$. In this direction, we only need the simple

Lemma 2.6.2 *If $|A| < d^{O(1)}$, then $|A| \le O(1/d)|N(A)|$, and if $|A| \le d/2$, then $|N(A)| \ge d|A| - 2|A|(|A|-1)$.*

Note that the second statement is true for all A, but vacuously so for $|A| > d/2$.

Proof of Lemma 2.6.2: If $|A| < d^{O(1)}$, then we have $k = O(1)$ in the notation of Lemma 2.6.1, and repeating the argument of that lemma we get $|A| \le O(1/d)|N(A)|$.

For the second part, note that each $u \in A$ has d neighbors, of which at least $d - 2(|A|-1)$ must be unique to it, since a pair of vertices in the cube can have at most two common neighbors.

∎

2.7 Topology

In this section we write Ω for the nearest-neighbor graph on $\mathbf{Z}^d$ and for $S \subseteq \mathbf{Z}^d$ we write $\Omega[S]$ for the subgraph of Ω induced by S. Recall that we write X (resp. Y) for the odd (resp. even) vertices of Ω. The purpose of the section is to prove the following combinatorial proposition.

Proposition 2.7.1 *Let $R \cup B$ be a decomposition of $V(\Omega)$ $(= \mathbf{Z}^d)$, with both $\Omega[R]$ and $\Omega[B]$ connected and R finite. Suppose $G := R \cap B$ is contained in Y and is the internal boundary of each of R, B. Then G is 2-linked.*

Remark: We will actually show that G is 2-linked in each of R and B.

We could not see a simple combinatorial proof of this proposition; our proof, which is considerably longer than we would wish and unrelated to the methods in the rest of the thesis, might profitably be skipped on a first reading. We present it here (rather than in Section 4.3, where the proposition will be used) in order not to interrupt the flow of the main argument in Chapter 4. The proof requires a topological detour, based on

Lemma 2.7.2 *If U, V are connected subsets of $X = \mathbf{R}^n$ or S^n, $n > 1$, with $U \cup V = X$, U closed and V compact, then $U \cap V$ is connected.*

(As usual, S^n is the unit sphere $\{x \in \mathbf{R}^{n+1} : \sum x_i^2 = 1\}$. We also write B^{n+1} for the corresponding unit ball.)

The (presumably well-known) proof of Lemma 2.7.2 is given at the end of this section.

Proof of Proposition 2.7.1: With Ω embedded in $\mathbf{R}^d$ in the natural way, we extend R and B to closed connected subsets R^* and B^* of $\mathbf{R}^d$ so that $R^* \cup B^* = \mathbf{R}^d$ and $G^* := R^* \cap B^*$ is path-connected. We then derive the 2-linkedness of G from the path-connectedness of G^*.

We view $\mathbf{R}^d$ as the union of $\mathbf{Z}^d$-translates of $[0,1]^d$ (the *cells* of $\mathbf{R}^d$), and define R^* and B^* cell by cell. Within a cell we proceed by dimension, first defining the extensions for 0-dimensional faces (the vertices of Ω), 1-dimensional faces (the edges of Ω), and 2-dimensional faces, and then continuing inductively. (As usual a *face* of a cell is the intersection of the cell with some supporting hyperplane. Henceforth we use "k-face" for "k-dimensional face".) For the inductive step, we need a topological lemma (Lemma 2.7.5), for the statement of which it's convenient to introduce two local definitions. Let us say that a subset of a topological space is *civilized* if it is closed, has only finitely many components, and each of its components is path-connected.

Definition 2.7.3 *A decomposition $X = R \cup B$ of a topological space X, with $R \cap B = G$, is* nice *if it satisfies:*

(i) *$G = \partial R = \partial B$;*

(ii) *each of R, B, G is civilized; and*

(iii) *each of R, B — and so each component of R and B — is the closure of the union of finitely many open, path-connected sets.*

If $X = R \cup B$ is a nice decomposition, and R', B' are obtained from R, B by adding finitely many points, then we also call the decomposition $X = R' \cup B'$ nice.

(Of course there is some redundancy in conditions (i)-(iii).)

We say that two nice decompositions $X_1 = R_1 \cup B_1$ and $X_2 = R_2 \cup B_2$ are *compatible* if $R_1 \cap X_1 \cap X_2 = R_2 \cap X_1 \cap X_2$ and $B_1 \cap X_1 \cap X_2 = B_2 \cap X_1 \cap X_2$. It is straightforward to check that nice decompositions of different spaces can be combined if they are compatible:

Lemma 2.7.4 *Suppose $X = X_1 \cup \cdots \cup X_m$ with each X_i closed. If $X_i = R_i \cup B_i$ are pairwise compatible, nice decompositions, then $(\cup R_i) \cup (\cup B_i)$ is a nice decomposition of X.*

We now state the topological lemma alluded to above, deferring its proof until after the derivation of Proposition 2.7.1. (Recall B^{n+1} and S^n are the unit ball and sphere in $\mathbf{R}^{n+1}$.)

Lemma 2.7.5 *Assume $n > 1$. If $R \cup B$ is a nice decomposition of S^n, then there is a nice decomposition $R^* \cup B^*$ of B^{n+1}, with $R^* \cap S^n = R$, $B^* \cap S^n = B$, and such that if C is any component of R^* (resp. B^*, G^*), then $C \cap S^n$ is a component of R (resp. B, G).*

(This is easily seen to fail for $n = 1$. It may be worth pointing out that for R and B, condition (iii) of Definition 2.7.3 refers to sets that are open *in* S^n; similarly ∂R and ∂B are boundaries relative to S^n, while ∂R^* and ∂B^* are boundaries relative to B^{n+1}.)

Of course Lemma 2.7.5 still applies if we replace the B^{n+1} by any of its homeomorphic images (and S^n by the corresponding homeomorphic copy); in our case the relevant image will be $[0,1]^d$.

We now fix a cell, and begin defining our extensions. For vertices and edges we do the natural things: $R^* \cap V(\Omega) = R$, $B^* \cap V(\Omega) = B$; and we put (the interior of) an edge in R^* (resp. B^*) iff both its ends are in R^* (resp. B^*), noting that exactly one of these possibilities occurs, since $\nabla(G,G) = \emptyset$.

Next, we deal with 2-dimensional faces. If the vertices of such a face are all in R (resp. B), then put the interior of the face in R^* (resp. B^*). Otherwise, the face has two opposite corner vertices (v_1, v_3, say) in G, with one of its remaining two vertices (v_2) in $R \setminus B$ and the other (v_4) in $B \setminus R$. Put the interior of the convex hull of v_1, v_2, v_3 in R^*, the interior of the convex hull of v_1, v_3, v_4 in B^*, and the interior of the diagonal joining v_1 and v_3 in $R^* \cap B^*$. It is easy to check that these (R^*, B^*)-decompositions of the 2-dimensional faces are nice. (It may be worth observing that a 2-dimensional face contained in R^* may still have one or two of its vertices in B^*, and vice versa.)

We now proceed by induction, assuming the decomposition has been defined on faces of dimension less than $k \in \{3, \ldots, d\}$. Each k-face F is homeomorphic to B^k, and is bounded by the union of finitely many $(k-1)$-dimensional faces. The decomposition of each of these bounding faces is nice, and the decompositions on any two faces are compatible (since we are defining the decomposition from lower dimensions up). So, by Lemma 2.7.4, we have a nice decomposition of the boundary of F. We now apply Lemma 2.7.5 to extend to a nice decomposition of the entire face. Once we have a nice decomposition of each cell, we get the full decomposition $\mathbf{R}^d = R^* \cup B^*$ by combining the decompositions of the cells, again appealing to Lemma 2.7.4 for "nice." (For formal applicability of the lemma, we can use a single $X_i = B_i$ for the union of all cells not meeting R.)

It is clear from the construction that R^* and B^* are closed, R^* is bounded, and $R^* \cup B^* = \mathbf{R}^d$. To see that R^* is connected, notice that by construction, any component of R^* contains an edge of $\Omega[R]$, and that every edge of $\Omega[R]$ is contained in a component of R^*; connectivity of R^* then follows from connectivity of $\Omega[R]$. The same argument shows that B^* is connected.

Lemma 2.7.2 now shows that G^* is connected, which, since G^* is also civilized (since $R^* \cup B^*$ is nice), implies that it is actually path-connected.

It remains to show that the 2-linkedness of G follows from the path-connectedness of G^*. It is enough to show that for each pair of vertices $u, v \in G$, there is a path connecting them in G^* which is supported entirely on the 2-dimensional faces of $\mathbf{R}^d$; for, by the construction of R^* and B^*, such a path is supported on diagonals (of 2-dimensional faces) connecting pairs of vertices from G, and such diagonals correspond to steps of length 2 in Ω. (This also justifies the remark following Proposition 2.7.1.)

So, consider a (u, v)-path P in G^* given by the continuous function $f : [0, 1] \to \mathbf{R}^d$. If P is supported on 2-dimensional faces of $\mathbf{R}^d$, then we are done. Otherwise, let $k > 2$ be the maximum dimension of a face whose interior meets P. It's enough to show that we can replace P by a path meeting the interiors of fewer k-faces than P and no faces of dimension more than k.

To do this, choose a k-face F and component C of $G^* \cap F$ with $C \cap F^0 \cap P \neq \emptyset$ (where F^0 is the interior of F). Let $p = \inf\{x \in [0, 1] : f(x) \in C \cap F^0\}$ and $q = \sup\{x \in [0, 1] : f(x) \in C \cap F^0\}$. Then $f(p), f(q) \in C \cap \partial F$, which, by construction, is path-connected. So we may replace $f([p, q])$ in P by a path contained in ∂F.

∎

Proof of Lemma 2.7.5: To avoid confusion, we now write ∂X, $\partial' X$ and $\partial'' X$ for the boundaries of X relative to, respectively, $\mathbf{R}^{n+1}$, B^{n+1} and S^n.

We may assume neither R nor B contains isolated points: otherwise we can simply delete such points, produce R^* and B^* for the resulting "reduced" R and B, and then add the deleted points of R (B) to R^* (B^*).

We use (R, B)-*component* to mean a component of either R or B, and proceed by induction on the number of (R,B)-components in the decomposition of S^n.

If there is exactly one such component (a component of R, say), then $R = S^n$, and $B = \emptyset$. Setting $R^* = B^{n+1}$ and $B^* = \emptyset$, we get a nice decomposition of B^{n+1} which satisfies the conditions of the lemma.

Otherwise, there must be at least one (R,B)-component T for which $S^n \setminus T^0$ is connected. For suppose $S^n \setminus T^0$ is disconnected for every (R,B)-component T. Choose an (R,B)-component T_0 ($\subseteq R$, say) such that one of the components of $S^n \setminus T_0^0$, C say, contains as few (R, B)-components as possible, and let T_1 be an (R, B)-component of C (i.e. contained in C, noting that each (R, B)-component other than T_0 is either contained in or disjoint from C). Now $S^n \setminus C^0$ is connected in $S^n \setminus T_1^0$, so $S^n \setminus T_1^0$ (which by assumption is not connected) contains a component whose (R,B)-components form a proper subset of the (R,B)-components of C, contradicting the choice of T_0.

Let T, then, be an (R,B)-component with $S^n \setminus T^0$ connected. We may assume that T is a component of R. Applying Lemma 2.7.2 with $X = S^n$, $U = T$ and $V = S^n \setminus T^0$, we find that $\partial'' T$ is connected, so that T meets exactly one component, say C, of B (and $C \supseteq \partial'' T$).

Set $T^* = \{\lambda x : x \in T, \lambda \in [1/2, 1]\}$. This will be one component of R^*. It is easy to see that T^* is closed and path-connected (so civilized), as is $\partial' T^*$, and that $T^* \cap S^n = T$, a component of R.

Now let $(T^*)^0$ be the relative interior of T^* with respect to B^{n+1} (namely, $(T^*)^0 = \{\lambda x : x \in T^0, \lambda \in (1/2, 1]\}$), $P = \partial(B^{n+1} \setminus (T^*)^0)$ ($= (S^n \setminus T^0) \cup \partial' T^*$), and $Q = B^{n+1} \setminus (T^*)^0$. Then (Q, P) is (easily seen to be) homeomorphic to (B^{n+1}, S^n).

Let, further, $R_1 = R \setminus T$, $B_1 = B \cup \partial' T^*$, and $C_1 = C \cup \partial' T^*$. Then

(i) the components of R_1 are precisely the components of R other than T

and

(ii) the components of B_1 are C_1 and the components of B other than C,

and it is easy (if tedious) to deduce that $R_1 \cup B_1$ is a nice decomposition of P.

Our inductive hypothesis thus gives a nice decomposition $R_1^* \cup B_1^*$ of Q, and we obtain the desired decomposition, $R^* \cup B^*$, of B^{n+1} by setting $B^* = B_1$ and $R^* = R_1 \cup T^*$ (again an easy verification using (i) and (ii)).

∎

Proof of Lemma 2.7.2: We first establish a corresponding statement for open sets: *if U, V are connected, open subsets of $X = \mathbf{R}^n$ or S^n, $n > 1$, with $U \cup V = X$, then $U \cap V$ is connected.*

Proof: We use the Mayer-Vietoris sequence. If X is a topological space, and U and V are open subsets of X whose union is X, then this is a long exact sequence of group homomorphisms ending with

$$\cdots \to H_1(X) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0,$$

where H_m is the m^{th} homology group. We apply this with $X = \mathbf{R}^n$ or S^n. Using the facts that $H_m(\mathbf{R}^n) = 0$ whenever $m \geq 1$ and that if O is an open subset of $\mathbf{R}^n$ or S^n, then $H_0(O) \cong \mathbf{Z} \Longleftrightarrow O$ is connected, this long exact sequence becomes

$$0 \to H_0(U \cap V) \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0.$$

From the exactness of this sequence, it follows that $H_0(U \cap V) \cong \mathbf{Z}$, so that $U \cap V$ is connected.

∎

Now let U, V be as in the lemma, and for each $\varepsilon > 0$, set $U_\varepsilon = \{x \in X : d(x, U) < \varepsilon\}$ and $V_\varepsilon = \{x \in X : d(x, V) < \varepsilon\}$. These are open, connected sets whose union is X, so by Lemma 2.7.2, $U_\varepsilon \cap V_\varepsilon$ is connected. Thus $\overline{U_\varepsilon \cap V_\varepsilon}$ is connected; it is also closed and bounded, so compact. So $U \cap V = \cap_{\varepsilon > 0} \overline{U_\varepsilon \cap V_\varepsilon}$ is the intersection of a nested sequence of compact, connected sets and so is itself connected.

∎

Chapter 3

Approximations

In this chapter, we introduce the notions of approximation that will play a role in the two main results of the thesis.

The basic idea is the following. To obtain an upper bound on the size of a set $\mathcal{A}$, we produce a set $\mathcal{C}$ with the properties that $|\mathcal{C}|$ is "small" and that each $\alpha \in \mathcal{A}$ is approximated in an appropriate sense by some $\gamma \in \mathcal{C}$. We may then bound $|\mathcal{A}|$ by, for example, putting a bound b on the number of possible $\alpha \in \mathcal{A}$ that can be approximated by any $\gamma \in \mathcal{C}$ can approximate, so that $|\mathcal{A}| \leq b|\mathcal{C}|$.

This is exactly the approach we take in the problem of counting homomorphisms from the Hamming cube to $\mathbf{Z}$. For the problem of phase transition in the hard-core model, we have to proceed slightly differently, because we are controlling not sizes of sets but weighted sums. But even in this case, we begin exactly as described above by producing a small set $\mathcal{C}$ of approximations to members of $\mathcal{A}$.

In both of our applications, the set $\mathcal{C}$ is itself produced by an approximation process — we first produce a small set $\mathcal{B}$ with the property that each $\alpha \in \mathcal{A}$ is weakly approximated (in an appropriate sense) by some $\beta \in \mathcal{B}$, and then show that for each β there is a small set $\mathcal{B}_\beta$ with the property that for each $\alpha \in \mathcal{A}$ that is weakly approximated by β, there is a $\gamma \in \mathcal{B}_\beta$ which approximates α; we then take $\mathcal{C} = \cup_{\beta \in \mathcal{B}} \mathcal{B}_\beta$.

After setting up the notation and conventions for the chapter in Section 3.1, we introduce, in Section 3.2, the notions of approximation that we will use. We state the main results of the chapter in Section 3.3, and the remaining sections are devoted to the proofs of these results.

3.1 Setup

Throughout this chapter we work in a finite ℓ-regular bipartite graph Σ with bipartition $V = V(\Sigma) = X \cup Y$ and distinguished vertex $v_0 \in X$. Many of our results hold at this level of generality, though we sometimes need to impose further conditions on Σ. Specifically, we will sometimes want Σ to satisfy

$$\forall\ A \subseteq X, |A| \leq \begin{cases} O(1/\ell)|N(A)| & \text{if } |A| < \ell^{O(1)} \\ \left(1 - \Omega\left(|N(A)|^{-2/\ell}/\ell\right)\right)|N(A)| & \text{always} \end{cases} \tag{3.1}$$

and

$$\forall\ v \sim w \in V \text{ and } L \subseteq N(v),\ |N(w) \cap N(L)| \geq |L|. \tag{3.2}$$

Notice that the discrete torus Γ (defined in Section 2.2) satisfies both of these conditions. (Lemma 2.5.1 gives (3.1) for $\Sigma = \Gamma$, and to see (3.2), notice that for $v \sim w$ in Γ, the subgraph of Γ induced by $(N(v) \cup N(w)) \setminus \{v, w\}$ is a matching of all but one vertex of $N(v)$ and all but one vertex of $N(w)$.)

We always take A to be a closed set of a single parity. Given A, set $G = G(A) = N(A)$, $B = B(A)$, $H = H(A) = N(B)$, $G_0 = G_0(A) = \partial^\star(G \cup A)$ and $W = W(A) = G \cup A$. In what follows, G, B, H, G_0 and W are always understood to be $G(A)$, $B(A)$, $H(A)$, $G_0(A)$ and $W(A)$ for whatever A is under discussion. Note that $B \subseteq G$ is a closed set and $H \subseteq A$. Note also that under the assumption that A is closed, A and G determine each other ($G = N(A)$ and $A = \{v \in V : N(v) \subseteq G\}$).

As with Σ, we sometimes wish to impose additional conditions on A, specifically

$$A \text{ is 2-linked,} \tag{3.3}$$

$$v_0 \in A \tag{3.4}$$

and

$$G_0 \text{ is 2-linked} \tag{3.5}$$

We always use a for $|A|$, g for $|G|$, b for $|B|$ and h for $|H|$. Given a, g, b and h we take $t = g - a$ and write

$$\mathcal{G}(t) = \{A \subseteq X : |G| - |A| = t\},$$

$$\mathcal{G}(a,g) = \{A \subseteq X : A \text{ satisfies (3.4) and (3.5)}, |A| = a \text{ and } |G| = g\} \quad (\subseteq \mathcal{G}(t))$$

and

$$\mathcal{H}(g) = \{A \subseteq X : A \text{ satisfies (3.3) and } |G| = g\}.$$

Notice that there is a certain *duality* preserving t: if $A \in \mathcal{G}(t)$ then $Y \setminus G$ belongs to the analogue of $\mathcal{G}(t)$ obtained by reversing the roles of X and Y in Σ — but of course a and g, unlike t, are not usually preserved by this switch.

The following simple observation will be useful. For $A \in \mathcal{G}(t)$

$$|\nabla(W, V \setminus W)| \ (= |\nabla(G, X \setminus A)| = |\nabla(G_0, X \setminus A)|) \ = \ t\ell. \tag{3.6}$$

3.2 Notions of approximation

Before stating the main results of this chapter, we introduce the three notions of approximation that we will be working with. The first notion depends on a parameter $\tau > 0$.

Definition 3.2.1 *A* τ-approximation *for* $A \subseteq X$ *is a pair* $(F', S') \in 2^Y \times 2^X$ *satisfying*

$$F' \subseteq G, \ S' \supseteq A \tag{3.7}$$

and

$$|F' \setminus G|, |S' \setminus A| < \tau. \tag{3.8}$$

Definition 3.2.2 *A* covering approximation *for* $A \subseteq X$ *is a pair* $(F', P') \in 2^Y \times 2^X$ *satisfying*

$$F' \subseteq G, \ P' \subseteq H$$

and

$$N(F') \supseteq A, N(P') \supseteq B.$$

The third notion of approximation depends on a parameter ψ, $0 < \psi < \ell$.

Definition 3.2.3 *A* ψ-approximating pair *for* $A \subseteq X$ *is a pair* $(F, S) \in 2^Y \times 2^X$ *satisfying (3.7) (with* (F, S) *in place of* (F', S')*) as well as*

$$d_F(u) > \ell - \psi \quad \forall u \in S \tag{3.9}$$

and

$$d_{X \setminus S}(v) > \ell - \psi \quad \forall v \in Y \setminus F. \tag{3.10}$$

*For $A \subseteq Y$, we define the notion of a ψ-approximating pair similarly (reversing the roles of X and Y in the above definition). A ψ-*approximating quadruple *for $A \subseteq X$ is a quadruple $(F, S, P, Q) \in 2^Y \times 2^X \times 2^X \times 2^Y$ with (F, S) a ψ-approximating pair for A and (P, Q) a ψ-approximating pair for B.*

Note that if x is in A then all of its neighbors are in G, and if y is in $Y \setminus G$ then all of its neighbors are in $X \setminus A$. If we think of S as "approximate A" and F as "approximate G", (3.9) says that if $x \in X$ is in "approximate A" then almost all of its neighbors are in "approximate G", while (3.10) says that if $y \in Y$ is not in "approximate G" then almost all of its neighbors are not in "approximate A".

Before continuing, we note a property of ψ-approximating pairs that will be of use in Chapter 5.

Lemma 3.2.4 *Given g and ψ, if (F, S) is a ψ-approximating pair for A then*

$$|S| \leq |F| + 2g\psi/(\ell - \psi). \tag{3.11}$$

Proof: Observe that $|\nabla(S, G)|$ is bounded above by $\ell|F| + \psi|G \setminus F|$ and below by $\ell|A| + (\ell - \psi)|S \setminus A| = \ell|S| - \psi|S \setminus A|$, giving

$$|S| \leq |F| + \psi|(G \setminus F) \cup (S \setminus A)|/\ell,$$

and that each $u \in (G \setminus F) \cup (S \setminus A)$ contributes more than $\ell - \psi$ edges to $\nabla(G)$, a set of size $g\ell$, giving

$$|(G \setminus F) \cup (S \setminus A)| \leq 2g\ell/(\ell - \psi).$$

These two observations together give (3.11).

∎

An immediate corollary of this is

Lemma 3.2.5 *Given g, h and ψ, if (F, S, P, Q) is a ψ-approximating quadruple for A then we have (3.11) and*

$$|Q| < |P| + 2h\psi/(\ell - \psi). \tag{3.12}$$

3.3 Statement of the results

The main lemma we need for the application in Chapter 4 applies to any Σ satisfying (3.1) and (3.2):

Lemma 3.3.1 *In any Σ satisfying (3.1) and (3.2), and for any a and g, there is a family $\mathcal{U} = \mathcal{U}(a, g) \subseteq 2^Y \times 2^X$ with*

$$|\mathcal{U}| < \exp[O(t\ell^{-1/2} \log^{3/2} \ell)] \tag{3.13}$$

such that every $A \in \mathcal{G}(a, g)$ has a $\sqrt{\ell}$-approximating pair in $\mathcal{U}$.

This follows from the next two lemmas. (Take $\psi = \sqrt{\ell}$ and $\tau = ct\sqrt{\ell \log \ell}$ (for an appropriate constant c) in Lemma 3.3.3. The expression in the exponent of (3.13) is the maximum of the corresponding expressions from (3.14) and (3.15).) Note that Lemma 3.3.2 requires a good deal of structure on Σ, whereas Lemma 3.3.3 holds in some generality.

Lemma 3.3.2 *For any a, g and $\tau = O(t\sqrt{\ell \log \ell})$, and in any Σ satisfying (3.1) and (3.2), there is a family $\mathcal{S} = \mathcal{S}(a, g, \tau) \subseteq 2^Y \times 2^X$ with*

$$|\mathcal{S}| < \exp[O(t\ell^{-1/2} \log^{3/2} \ell)] \tag{3.14}$$

such that every $A \in \mathcal{G}(a, g)$ has a τ-approximation in $\mathcal{S}$.

Lemma 3.3.3 *Given t, τ, $(F', S') \in 2^Y \times 2^X$ and $0 < \psi < \ell$, there is a family $\mathcal{T} = \mathcal{T}(F', S', \tau, t, \psi) \subseteq 2^Y \times 2^X$ with*

$$|\mathcal{T}| < \exp[O(((\tau/\ell) + (t/\psi)) \log \ell)] \tag{3.15}$$

such that every $A \in \mathcal{G}(t)$ for which (F', S') is a τ-approximation has a ψ-approximating pair in $\mathcal{T}$.

The main lemma we need for the application in Chapter 5 requires no conditions on Σ beyond those mentioned at the beginning of Section 3.1:

Lemma 3.3.4 *For any g, there is a family $\mathcal{X} = \mathcal{X}(g) \subseteq 2^Y \times 2^X \times 2^X \times 2^Y$ with*

$$|\mathcal{X}| \le |Y| 2^{O(g \log \ell / \sqrt{\ell})}$$

such that every $A \in \mathcal{H}(g)$ has a $\sqrt{\ell}$-approximating quadruple in $\mathcal{X}$.

This follows from the next two lemmas. (Take $\psi = \sqrt{\ell}$ in Lemma 3.3.6.)

Lemma 3.3.5 *For any g, there is a family $\mathcal{V} = \mathcal{V}(g) \subseteq 2^Y \times 2^X$ with*

$$|\mathcal{V}| \le |Y| 2^{O(g \log^2 \ell / \ell)}$$

such that each $A \in \mathcal{H}(g)$ has a covering approximation in $\mathcal{V}$.

Lemma 3.3.6 *For any $(F', P') \in 2^Y \times 2^X$, ψ satisfying $\Omega(1) \le \psi \le o(\ell)$ and g there is a family $\mathcal{W} = \mathcal{W}(F', P', g, \psi) \subseteq 2^Y \times 2^X \times 2^X \times 2^Y$ with*

$$|\mathcal{W}| \le 2^{O(g \log \ell / \psi)}$$

such that any $A \in \mathcal{H}(g)$ for which (F', P') is a covering approximation has a ψ-approximating quadruple in $\mathcal{W}$.

In Sections 3.4 and 3.5 we prove Lemmas 3.3.2 and 3.3.5. In Section 3.6 we present an algorithm which is central to the proofs of Lemmas 3.3.3 and 3.3.6; we give these proofs in Sections 3.7 and 3.8. The idea for the algorithm in Section 3.6 is from [23].

3.4 Proof of Lemma 3.3.2: covering the boundary

We say that a set $C \subseteq V$ *separates* $P, Q \subseteq V$ if any path meeting both P and Q also meets C.

We begin the proof of Lemma 3.3.2 by showing that there is a "small" collection of subsets of V, at least one of which separates W $(= G \cup A)$ and $V \setminus W$ for each $A \in \mathcal{G}(a, g)$.

Let

$$G_0' = \{v \in G : d_A(v) \le \ell/2\} \quad (\subseteq G_0),$$

$$B_0' = \{v \in X \setminus A : d_{Y\setminus G}(v) \le \ell/2\} \quad (\subseteq B_0 := (X \setminus A) \cap N(G)),$$

$G_0'' = G_0 \setminus G_0'$ and $B_0'' = B_0 \setminus B_0'$. Then

$$\nabla(G_0'', B_0'') = \emptyset. \tag{3.16}$$

(The more general statement here is: if $v \in G_0$, $w \in B_0$ and $v \sim w$, then (by (3.2) with $L = N(v) \cap A$) $d_G(w) \ge d_A(v)$ $(= \ell - d_{X\setminus A}(v))$, implying $d_{X\setminus A}(v) + d_G(w) \ge \ell$.)

Notice that (3.16) implies

$$G_0' \cup B_0' \text{ separates } W \text{ and } V \setminus W \tag{3.17}$$

(equivalently, $\nabla(W, V \setminus W) \subseteq \nabla(G_0') \cup \nabla(B_0')$).

Lemma 3.4.1 *In any Σ satisfying (3.2), for any $A \in \mathcal{G}(t)$ there exists $U \subseteq N(G_0' \cup B_0')$ satisfying*

$$N(U) \supseteq G_0' \cup B_0' \tag{3.18}$$

and

$$|U| < O(t\sqrt{\log \ell/\ell}). \tag{3.19}$$

Before proving this, we observe that it does accomplish the first goal stated at the beginning of this section (the existence of a small set of separations). For A and U as in Lemma 3.4.1, we have

$$N(U) \text{ separates } W \text{ and } V \setminus W \tag{3.20}$$

(by (3.17) and (3.18)). So we just need to limit the number of possibilities for U when $A \in \mathcal{G}(a, g)$.

To do so, notice that

$$U \text{ is 6-linked.} \tag{3.21}$$

This follows from Lemma 2.4.4 and (3.5), once we observe that $\rho(u, G_0) \le 2\ \forall u \in U$ (since $U \subseteq N(G_0' \cup B_0')$), and that (3.16) and (3.18) imply $\rho(v, U) \le 2\ \forall v \in G_0$.

Before bounding the number of possibilities for U, we show that there is a small (size $O(g\ell^2)$) set of vertices meeting all possible U's. Fix a linear ordering $\ll$ of Y satisfying

$$\rho(v_0, y_1) < \rho(v_0, y_2) \quad \Longrightarrow \quad y_1 \ll y_2,$$

and let C be the initial segment of $\ll$ of size g. We claim that $C \cap G_0 \neq \emptyset$. If $C = G$, this is clear; if not, consider a shortest $y - v_0$ path in Σ for some $y \in C \setminus G$. This path intersects G (since $N(v_0) \subseteq G$). Let y' be the largest (with respect to $\ll$) vertex of G on the path; then $y' \in G_0 \cap C$, establishing our claim. There are at most g possibilities for $y' \in G_0 \cap C$, so at most $g\ell^2$ possibilities for a vertex x' with $\rho(x', y') \leq 2$; and by (3.16) and (3.18) U must contain such an x'.

In view of (3.19) and (3.21), Lemma 2.4.1 then gives a bound of

$$g\ell^2 \exp[O(t\ell^{-1/2} \log^{3/2} \ell)] = \exp[O(t\ell^{-1/2} \log^{3/2} \ell)] \tag{3.22}$$

on the number of possibilities for U. Here we use (3.1) for the equality in (3.22).

Proof of Lemma 3.4.1: By "duality" (see the end Section 3.1) it is enough to show the existence of $S \subseteq N(G_0')$ with

$$N(S) \supseteq G_0' \tag{3.23}$$

and

$$|S| < O(t\sqrt{\log \ell/\ell}\ ). \tag{3.24}$$

Define $Q = \{v \in G_0 : d_A(v) \leq \sqrt{\ell \log \ell}\}$, $K = G_0 \setminus Q$, and $P = N(Q) \cap A$. By (3.2),

$$d_{G_0}(v) \geq \ell - \sqrt{\ell \log \ell} \quad \forall v \in P. \tag{3.25}$$

Let $P' = \{v \in P : d_K(v) \geq \ell/2\}$, $P'' = P \setminus P'$, $Q' = Q \cap N(P')$, $Q'' = Q \setminus Q'$ and $R = \{v \in B_0 \cap N(G_0') : d_{G_0}(v) > \sqrt{\ell \log \ell}\}$.

Now P'' is a cover of Q'' of size $O(t\sqrt{\log \ell/\ell}\ )$, the size bound following from $|Q| \leq t\ell/(\ell - \sqrt{\ell \log \ell}\ ) = O(t)$ (using (3.6)), $d_{P''}(v) \leq d_A(v) \leq \sqrt{\ell \log \ell} \quad \forall v \in Q$, and $d_Q(v) > \ell/2 - \sqrt{\ell \log \ell} \quad \forall v \in P''$ (using (3.25) and the definition of P'').

On the other hand, we can cover $G_0' \setminus Q''$ by a similarly small subset of R, as follows. From (3.2) we have $N(K) \cap N(G_0') \cap B_0 \subseteq R$. This gives $d_R(v) > \ell/2$ for $v \in G_0' \setminus Q$, while for $v \in Q'$,

$$d_R(v) \geq |N(v) \cap N(K)| - |N(v) \cap A| \geq \ell/2 - \sqrt{\ell \log \ell}$$

(the second inequality following from (3.2) and the definitions of Q' and Q). So, noting that $|R| < t\sqrt{\ell/\log \ell}$ (again using (3.6)), Lemma 2.4.3 says that we can cover $G_0' \setminus Q''$

by some $T \subseteq R$ of size at most

$$|R|(1+\log \ell)/(\ell/2 - \sqrt{\ell \log \ell}\ ) < O(t\sqrt{\log \ell/\ell}\ ).$$

(And note $P \subseteq N(G_0')$ since $Q \subseteq G_0'$, and $R \subseteq N(G_0')$ by definition, so $S := P'' \cup T \subseteq N(G_0')$.)

∎

We now complete the proof of Lemma 3.3.2. Given U as above, set $L = N(U)$. Then $|L| = O(t\sqrt{\ell \log \ell})$.

Say a component C of $\Sigma - L$ is *small* if $|C| \leq \ell$ and *large* otherwise. (Note that this definition — which will only be used in this section — is unrelated to the definition of "small" given in Section 2.3.) From (3.1) we have

$$|\nabla(C, L)| = |\nabla(C)| \geq |\partial C| = \Omega(|C|\ell)$$

for small C (actually also for considerably larger C), and

$$|\nabla(C, L)| = \Omega(\ell^2)$$

for large C. But $|\nabla(L)| \leq 2\ell|L| = O(t\ell^{3/2}\sqrt{\log \ell})$, so

$$\text{the number of large components is } O(t\ell^{-1/2}\sqrt{\log \ell}), \tag{3.26}$$

and the number of vertices in small components is $O(t\sqrt{\ell \log \ell})$.

It follows that if A is any (closed) subset of X for which L separates W and $V \setminus W$, then a τ-approximation (F', S') for A with $\tau = O(t\sqrt{\ell \log \ell})$ is given by

$$F' = P \cap Y \quad \text{and} \quad S' = (P \cup Q \cup L) \cap X, \tag{3.27}$$

where P is the union of those large components of $\Sigma - L$ that meet (equivalently, are contained in) W, and Q is the union of (all) the small components. In particular this is true if A is any member of $\mathcal{G}(a, g)$ for which Lemma 3.4.1 applied to A produces U.

By (3.26) the number of possibilities (given L) for (F', S') as in (3.27) is at most $\exp[O(t\ell^{-1/2}\sqrt{\log \ell})]$, and combining this with the bound (3.22) on the number of U's we have Lemma 3.3.2.

3.5 Proof of Lemma 3.3.5

For each $A \in \mathcal{H}(g)$ we obtain a covering approximation for A by taking $F'(A) \subseteq G$ to be a cover of minimum size of A in the graph induced by $G \cup A$ and $P'(A) \subseteq H$ to be a cover of minimum size of B in the graph induced by $H \cup B$. Note that $P'(A) \subseteq N(F'(A))$.

By Lemma 2.4.4, $F'(A)$ is 4-linked (A is 2-linked, $\rho(x, F'(A)) = 1$ for each $x \in A$ and $\rho(y, A) = 1$ for each $y \in F'(A)$). By Lemma 2.4.3, $|F'(A)| \leq g(1+\ln \ell)/\ell = O(g \log \ell/\ell)$ and $|P'(A)| \leq h(1 + \ln \ell)/\ell = O(g \log \ell/\ell)$ (noting that $h \leq g$).

We may therefore take $\mathcal{V}$ to be the set of all pairs $(F', P') \in 2^Y \times 2^X$ with F' 4-linked and $P' \subseteq N(F')$, and F', P' both of size at most $O(g \log \ell/\ell)$. By Lemma 2.4.1, there are at most $|Y|2^{O(g \log^2 \ell/\ell)}$ possibilities for F' (the factor of $|Y|$ is for the choice of a fixed vertex in F'), and, given F', a further

$$\sum_{i \leq O(g \log \ell/\ell)} \binom{|N(F')|}{i} = 2^{O(g \log^2 \ell/\ell)}$$

choices for P'. The lemma follows.

∎

3.6 The degree algorithm

Fix $0 < \xi < \ell$ and $A \subseteq X$ closed. We give an algorithm which, for input $(F', S') \in 2^Y \times 2^X$ satisfying (3.7) produces an output $(F, S) \in 2^Y \times 2^X$ that is a ξ-approximating pair for A. This algorithm will be central to the proofs of Lemmas 3.3.3 and 3.3.6.

Fix a linear ordering $\ll$ of V.

Step 1: If $\{u \in A : d_{G \setminus F'}(u) \geq \xi\} \neq \emptyset$, pick the smallest (with respect to $\ll$) u in this set and update F' by $F' \longleftarrow F' \cup N(u)$. Repeat this until $\{u \in A : d_{G \setminus F'}(u) \geq \xi\} = \emptyset$. Then set $F'' = F'$ and $S'' = S' \setminus \{u \in X : d_{Y \setminus F''}(u) \geq \xi\}$ and go to Step 2.

Step 2: If $\{w \in Y \setminus G : d_{S''}(w) \geq \xi\} \neq \emptyset$, pick the smallest (with respect to $\ll$) w in this set and update S'' by $S'' \longleftarrow S'' \setminus N(w)$. Repeat this until $\{w \in Y \setminus G : d_{S''}(w) \geq \xi\} = \emptyset$. Then set $S = S''$ and $F = F'' \cup \{w \in Y : d_S(w) \geq \xi\}$ and stop.

Claim 3.6.1 *The output of this algorithm is a ξ-approximating pair for A.*

Proof: To see that $F \subseteq G$ and $S \supseteq A$, first observe that $F'' \subseteq G$ (since $F' \subseteq G$, and the vertices added to F' in Step 1 are all in G) and that $S'' \supseteq A$ (or Step 1 would not have terminated). We then have $S \supseteq A$ since Step 2 deletes from S'' only neighbors of $Y \setminus G$, and $F \subseteq G$ since the vertices added to F'' at the end of Step 2 are all in G (or Step 2 would not have terminated).

To verify (3.9) and (3.10), note that $d_{F''}(u) > \ell - \xi \ \forall u \in S''$ by definition, $S \subseteq S''$, and $F \supseteq F''$, so that $d_F(u) > \ell - \xi \ \forall u \in S$; and if $w \in Y \setminus F$ then $d_S(w) < \xi$ (again by definition), so that $d_{X \setminus S}(w) > \ell - \xi \ \forall w \in Y \setminus F$.

∎

In the next two sections, the algorithm described above will be referred to as the *degree algorithm*.

3.7 Proof of Lemma 3.3.3

For each $A \in \mathcal{G}(t)$ for which (F', S') is a τ-approximation we produce a ψ-approximating pair for A by a two-stage procedure. Stage 1 runs the degree algorithm of Section 3.6 with (F', S') as input and with $\xi = \ell/2$. Stage 2 runs it with the output of Stage 1 as the input and with $\xi = \psi$. (Note that by Lemma 3.6.1 the output of Stage 1 is a valid input for Stage 2, and the output of Stage 2 is a ψ-approximating pair for A.)

Claim 3.7.1 *The procedure described above has at most*

$$\exp[O(((\tau/\ell) + (t/\psi)) \log \ell)]$$

outputs as the input runs over those $A \in \mathcal{G}(t)$ for which (F', S') is a τ-approximation.

Taking $\mathcal{T}$ to be the set of all possible outputs of the algorithm, Lemma 3.3.3 follows.

Proof of Claim 3.7.1: Writing $S'_0 = S' \cap N(Y \setminus F')$ and $E'_0 = (Y \setminus F') \cap N(S')$ we have

$$|S'_0|, |E'_0| < \tau + \ell\tau \tag{3.28}$$

(since $S_0' \subseteq (S' \setminus A) \cup N(G \setminus F')$, and similarly for E_0').

The output (F, S) of Stage 1 is determined by the sets of u's used in Step 1 and w's used in Step 2. Since each iteration in Step 1 shrinks $|G \setminus F'|$ by at least ξ, the number of iterations is at most $\tau/\xi = 2\tau/\ell$. Moreover, each u used in Step 1 lies in $N(E_0')$. So the number of possibilities for the set of u's used in Step 1 is less than

$$\sum_{i \leq \tau/\xi} \binom{\ell |E_0'|}{i} < \exp[O((\tau/\ell)\log \ell)]$$

(using (3.28)).

We perform a similar analysis for Step 2. Each iteration in Step 2 reduces $|S'' \setminus A|$ by at least ξ, and each w is drawn from $N(S_0')$. So the number of possibilities for the set of w's used in Step 2 is at most

$$\exp[O((\tau/\ell)\log \ell)].$$

At the end of Step 1 we have $w \in S'' \Rightarrow d_{F''}(w) > \ell - \xi = \ell/2$, which, since $|\nabla(S'' \setminus A, F'')| \leq t\ell$ (see (3.6)), gives $|S'' \setminus A| < 2t$. Step 2 only decreases this, so at the end of Stage 1 we have $|S \setminus A| < 2t$. A similar calculation gives that $|G \setminus F| < 2t$ at this point.

Repeating the analysis above for Stage 2, with (3.28) replaced by

$$|S_0|, |E_0| < 2t(1 + \ell)$$

(where $S_0 = S \cap N(Y \setminus F)$ and $E_0 = (Y \setminus F) \cap N(S)$), we find that the number of possible outputs of Stage 2, for a given output of Stage 1, is at most $\exp[O((t/\psi)\log \ell)]$. So the number of possible outputs of the entire procedure is no more than

$$\exp[O(((\tau/\ell) + (t/\psi))\log \ell)].$$

∎

3.8 Proof of Lemma 3.3.6

For each $A \in \mathcal{H}(g)$ for which (F', P') is a covering approximation, we produce a ψ-approximating quadruple for A by a two-stage procedure. In both stages we take

$\xi = \psi$. Stage 1 runs the degree algorithm with (F', X) as input. Stage 2 runs it with (P', Y) as input and with the roles of X and Y reversed. By Lemma 3.6.1, the quadruple (F, S, P, Q), where (F, S) is the output of Stage 1 and (P, Q) the output of Stage 2, is a ψ-approximating quadruple for A.

Claim 3.8.1 *The procedure described above has at most $2^{O(g \log \ell/\psi)}$ outputs as the input runs over those $A \in \mathcal{H}(g)$ for which (F', P') is a covering approximation.*

Taking $\mathcal{W}$ to be the set of all possible outputs of the algorithm, Lemma 3.3.6 follows.

Proof of Claim 3.8.1: The output of Stage 1 of the algorithm is determined by the set of u's whose neighborhoods are added to F' in Step 1, and the set of w's whose neighborhoods are removed from S'' in Step 2.

Each iteration in Step 1 removes at least ψ vertices from G, so there are at most g/ψ iterations. The u's in Step 1 are all drawn from A and hence $N(F')$, a set of size at most ℓg. So the total number of outputs for Step 1 is at most

$$\sum_{i \le g/\psi} \binom{\ell g}{i} = 2^{O(g \log \ell/\psi)}.$$

We perform a similar analysis on Step 2. Each $u \in S'' \setminus A$ contributes more than $\ell - \psi$ edges to $\nabla(G)$, so initially $|S'' \setminus A| \le g\ell/(\ell - \psi) = O(g)$. Each w used in Step 2 reduces this by at least ψ, so there are at most $O(g/\psi)$ iterations. Each w is drawn from $N(S'')$, a set which is contained in the fourth neighborhood of F' ($S'' \subseteq N(G)$ by construction of S'', $G = N(A)$ and $A \subseteq N(F')$) and so has size at most $\ell^4 g$. So as with Step 1, the total number of outputs for Step 2, and hence for Stage 1, is $2^{O(g \log \ell/\psi)}$.

Noting that $h \le g$, a similar analysis applied to Stage 2 gives that that stage also has at most $2^{O(g \log \ell/\psi)}$ outputs, and the claim follows.

∎

Chapter 4

Phase transition in the hard-core model on $\mathbf{Z}^d$

In this chapter, we will be concerned with the hard-core model on $\mathbf{Z}^d$. See Section 1.1 for an introduction to this model and the problem under consideration. Specifically, we will prove Theorem 1.1.1, which states that the hard-core model on $\mathbf{Z}^d$ with all activities equal to λ exhibits a phase transition for $\lambda = \omega(d^{-1/4}\log^{3/4} d)$.

4.1 Finitizing the problem

The problem of showing existence of a phase transition may be finitized as follows. Let $\Lambda_N = \mathbf{Z}^d \cap [-N, N]^d = X \cup Y$ (with X and Y the sets of odd and even vertices). Let μ_N be the hard-core measure with activity λ on Λ_N, and let μ_N^e be μ_N conditioned on the event $\{\mathbf{I} \supseteq \partial^\star\Lambda_N \cap Y\}$, where $\partial^\star\Lambda_N = [-N, N]^d \setminus [-(N-1), N-1]^d$. (That is, if we set

$$\mathcal{J} = \{I \subseteq \Lambda_N : I \text{ independent}, \partial^\star\Lambda_N \cap Y \subseteq I\},$$

then

$$\mu_N^e(\mathbf{I} = I) \quad \propto \quad w(I) := \lambda^{|I|} \quad \text{for } I \in \mathcal{J}.)$$

Define μ_N^o similarly, with the even boundary condition replaced by an odd boundary condition.

In [3] it is shown (*inter alia*) that the sequences $\{\mu_N^e\}$ and $\{\mu_N^o\}$ converge weakly to hc(λ) limits, called μ^e and μ^o, and that there is a phase transition iff these limits are different. (This is mainly based on the FKG Inequality, and applies to general bipartite graphs Σ, provided we allow $\{\Lambda_N\}$ to be an arbitrary nested sequence with $\cup\Lambda_N = V(\Sigma)$.)

It is thus enough (for showing phase transition) to exhibit any statistic distinguishing

μ^e from μ^o. We will show $\mu^e(\underline{0} \in \mathbf{I}) \neq \mu^o(\underline{0} \in \mathbf{I})$, i.e.

$$\lim_{N\to\infty} \mu_N^e(\underline{0} \in \mathbf{I}) \neq \lim_{N\to\infty} \mu_N^o(\underline{0} \in \mathbf{I}). \tag{4.1}$$

(Of course we are only using the trivial direction of "phase transition iff $\mu^e \neq \mu^o$." It can also be shown that (4.1), too, is equivalent to phase transition.)

To establish (4.1) (assuming at least $\lambda = \Omega(1/d)$, which is easily seen to be necessary for phase transition) it is in turn enough to show that for $v_0 \in \Lambda_N$,

$$\begin{aligned} \mu_N^e(v_0) &= o(1/d) \quad \text{if } v_0 \text{ is odd,} \\ \mu_N^o(v_0) &= o(1/d) \quad \text{if } v_0 \text{ is even.} \end{aligned}$$

For then

$$\begin{aligned} \mu_N^e(\underline{0} \in \mathbf{I}) &= \mu_N^e(N(\underline{0}) \cap \mathbf{I} = \emptyset)\mu_N^e(\underline{0} \in \mathbf{I} | N(\underline{0}) \cap \mathbf{I} = \emptyset) \\ &= (1 - o(1))\lambda/(1+\lambda), \end{aligned}$$

so that $\mu^e(\underline{0} \in \mathbf{I}) = (1 - o(1))\lambda/(1+\lambda)$, whereas $\mu^o(\underline{0} \in \mathbf{I}) = o(1/d)$.

So in particular the next theorem, whose proof is the main business of this chapter, contains Theorem 1.1.1.

Theorem 4.1.1 *For*

$$\lambda = \omega(d^{-1/4} \log^{3/4} d), \tag{4.2}$$

N arbitrary, and v_0 an odd vertex of Λ_N,

$$\mu_N^e(v_0 \in \mathbf{I}) < (1+\lambda)^{-(1-o(1))2d}. \tag{4.3}$$

The same result holds if we reverse the roles of even and odd.

Remark. It is easy to see that

$$\begin{aligned} \mu_N^e(v_0 \in \mathbf{I}) &= \mu_N^e(N(v_0) \cap \mathbf{I} = \emptyset)\mu_N^e(v_0 \in \mathbf{I} | N(v_0) \cap \mathbf{I} = \emptyset) \\ &> (1+\lambda)^{-2d} \frac{\lambda}{1+\lambda}, \end{aligned}$$

so that (4.3) actually gives the asymptotics of $\log \mu_N^e(v_0 \in \mathbf{I})$.

It will be convenient to replace the box Λ_N by the discrete torus $\Gamma = \Gamma_N$ (as described in Section 2.2). Replacing $\partial^{\star}\Lambda_N$ by Δ in the definition of $\mathcal{J}$ (that is, setting

$$\mathcal{J} = \{I \subseteq \Gamma : I \text{ independent}, \Delta \cap Y \subseteq I\})$$

and defining μ_N^e, μ_N^o as before, we may regard Theorem 4.1.1 as referring to Γ, a change which clearly does not affect its meaning. (The torus is a more convenient graph to work with than the box mainly because it is degree regular.)

The proof of Theorem 4.1.1 is a sort of "Peierls argument" (see e.g. [12]): we try to associate with each $I \in \mathcal{J}$ containing v_0 a "contour" — some kind of membrane separating the outer even region from an inner odd region containing v_0 — and then use this to map I to a large set of J's, also from $\mathcal{J}$ but not containing v_0, each obtained from I by some modification of the inner region.

This is no surprise: almost every attempt at settling this problem that we're aware of has attacked it more or less along these lines. (The one exception is the entropy approach of [14], which for now seems unlikely to get us to anything like what is proved here.)

The main difficulty in all these attempts has been getting some kind of control over the set of possible contours. In the next section, we give an informal outline of our approach to this difficulty; this section also serves as an outline of the rest of the chapter.

4.2 Outline of the proof of Theorem 4.1.1

Throughout the rest of the chapter, we assume that λ satisfies (4.2), and take v_0 to be a fixed odd vertex of Γ. We prove only the first part of Theorem 4.1.1; switching "even" and "odd" throughout the argument gives the proof of the second part.

We will show something slightly stronger than (4.3): for $I \in \mathcal{J}$ let $Z = Z(I)$ be the set of vertices that are connected to Δ in $\Gamma - (I \cap X)$, and set $\mathcal{I} = \{I \in \mathcal{J} : v_0 \notin Z(I)\}$. We will show

$$\frac{\sum_{I \in \mathcal{I}} w(I)}{\sum_{J \in \mathcal{J}} w(J)} < (1+\lambda)^{-(1-o(1))2d} \tag{4.4}$$

(Here and throughout the rest of this chapter, we use I for members of $\mathcal{I}$ and J for general members of $\mathcal{J}$). Note that if $v_0 \in \Delta \cup N(\Delta)$ then $\mathcal{I} = \emptyset$, so we may assume that $v_0 \notin \Delta \cup N(\Delta)$.

We prove (4.4) by producing a "flow" $\nu : \mathcal{I} \times \mathcal{J} \to [0,1]$ satisfying

$$\sum_J \nu(I,J) = 1 \quad \forall I \in \mathcal{I} \tag{4.5}$$

and

$$\sum_I \frac{w(I)}{w(J)} \nu(I,J) < (1+\lambda)^{-(1-o(1))2d} \quad \forall J \in \mathcal{J}. \tag{4.6}$$

This gives (4.4):

$$\begin{aligned} \sum_{I \in \mathcal{I}} w(I) &= \sum_{I \in \mathcal{I}} w(I) \sum_{J \in \mathcal{J}} \nu(I,J) \\ &= \sum_{J \in \mathcal{J}} w(J) \sum_{I \in \mathcal{I}} \frac{w(I)}{w(J)} \nu(I,J) \\ &< (1+\lambda)^{-(1-o(1))2d} \sum_{J \in \mathcal{J}} w(J). \end{aligned}$$

Our point of departure for the specification of ν is the observation that each $I \in \mathcal{I}$ is in "even phase" (i.e., consists predominantly of even vertices) near the boundary Δ and in "odd phase" near v_0, so there must be an unoccupied two-layer membrane separating the two phases. We will associate with each I a $W(I) \subseteq V$ which will play the role of the inner odd phase. We refer to $W(I)$ as the "volume" of I, and write $G_0(I)$ for $\partial^\star W(I)$ and refer to it as the "contour" of I. The construction and salient properties of $W(I)$ are given in Section 4.3.

The thickness of the separating membrane gives us room to modify I inside $W(I)$ in such a way that the resulting set is still independent, and can be extended to many different independent sets by the addition of various subsets of $G_0(I)$. In this way we get a one-to-many map

$$\mathcal{I} \ni I \longrightarrow \varphi(I) \subseteq \mathcal{J}.$$

When we define the flow ν, we require that

$$J \notin \varphi(I) \Rightarrow \nu(I,J) = 0. \tag{4.7}$$

To specify the modification of I inside $W(I)$ we first associate with I a direction $j(I)$ from among the $2d$ fundamental directions in the lattice (given by the d standard basis vectors and their negatives). The modification then takes the form of a "shift" inside the volume — a unit distance translation of $I \cap W(I)$ in direction $j(I)$. We then identify a set $G_0^j \subseteq G_0$ of vertices which have no edges to the modified I; $\varphi(I)$ then consists of all those sets obtained by adding subsets of G_0^j to the modified I. We choose the direction of the shift essentially to be one which makes $|G_0^j|$ (and so $|\varphi(I)|$) large, though there are some additional considerations. The precise definition of the shift operation is given in Section 4.4, and in Section 4.5 we specify $j(I)$ and $\varphi(I)$.

An important property of $\varphi(I)$ is that given $J \in \varphi(I)$, $W(I)$ and $j(I)$, I can be uniquely reconstructed. This is significant because for each $J \in \mathcal{J}$ we have, setting $\nu'(I, J) = \nu(I, J)w(I)/w(J)$,

$$\sum_I \nu'(I, J) = \sum_W \sum_j \sum_I \{\nu'(I, J) : W(I) = W, j(I) = j, I \in \varphi^{-1}(J)\},$$

and the reconstructibility of I implies that the inner sum consists of at most one term for each choice of W and j. This suggests that $\sum_I \nu'(I, J)$ may usefully be bounded above by, for each $n \in \mathbf{N}$, bounding the number of volumes W of size n (using, for example, Lemma 2.4.1) and uniformly bounding the quantity $\nu'(I, J)$ on the set of I's whose associated volumes have size n, and then summing the products of these two bounds over all n (we must also sum over all possible directions, but this only adds a relatively insignificant factor of $2d$). This approach (with, for example, $\nu(I, J)$ defined to be $|\varphi(I)|^{-1}$ if $J \in \varphi(I)$ and 0 otherwise) is enough to give phase transition, but only for values of λ that grow exponentially in d.

To get phase transition for smaller values of λ, we have to be more careful both with the way in which we control the volumes and with the definition of ν. We approach the former task via the notion of "approximations" introduced in Chapter 3 — with each volume I we associate a pair $(F(I), S(I))$ which approximates $W(I)$ in an appropriate sense. The two main properties we aim for in the set of possible approximations are that it is "small" and that given $J \in \mathcal{J}$ and a particular approximation (F, S), the set of I's with $I \in \varphi^{-1}(J)$ and $(F, S) = (F(I), S(I))$ contributes little to $\sum_I \nu'(I, J)$. We may then bound $\sum_I \nu'(I,J)$ (for each $J \in \mathcal{J}$) by

$$\sum_I \nu'(I,J) \le \sum_{(F,S)} \sum_I \{\nu'(I,J) : (F,S) = (F(I),S(I)), I \in \varphi^{-1}(J)\}. \tag{4.8}$$

The construction of a ν for which the inner sum is small is made possible by the accuracy of the approximations. Note that there is a trade-off here: requiring greater accuracy in the approximations restricts the number of I's which contribute to the inner sum in (4.8), but forces us to use a larger number of (F,S)'s.

We define the flow ν in Section 4.5. The proof that it behaves as desired is given in Section 4.6, and we are then able to swiftly complete the proof of (4.6) in Section 4.7.

4.3 Volumes

In this section, we describe how to associate with each $I \in \mathcal{I}$ a volume W, which we think of as the odd phase of I near v_0, and a contour G_0 which separates the volume from the outer even phase of I. All but one of the salient properties of these objects are easily derived from the construction in Section 4.3; the one remaining property — the 2-linkedness of G_0 — is a consequence of Proposition 2.7.1.

The following observation is used several times, so we record it as a lemma; its easy proof is left to the reader.

Lemma 4.3.1 *Let Σ be a graph, $S \subseteq V(\Sigma)$, and T (the vertex set of) some component of $\Sigma - (S \setminus \partial^\star S)$. Then $\partial^\star T \subseteq \partial^\star S$.*

Let $I \in \mathcal{I}$, $Z = Z(I)$ be as in Section 4.2, and set $Z_0 = \partial^\star Z$. By the definition of Z, it is clear that $Z_0 \subseteq Y$ and $Z_0 \cap I = \emptyset$. Let W' be the component of v_0 in the graph $\Gamma - (Z \setminus Z_0)$. By Lemma 4.3.1, $\partial^\star W' \subseteq W' \cap Z_0 \subseteq Y$.

Let $W'' = W' \cup \{x \in X | N(x) \subseteq W'\}$. This is clearly connected, with $\partial^\star W'' \subseteq \partial^\star W'$.

Now consider $\Gamma - (W'' \setminus \partial^\star W'')$. This breaks into a number of components, one of which, C say, contains Δ. Again using Lemma 4.3.1, we have $\partial^\star C \subseteq C \cap \partial^\star W''$. Finally, set $W = \Gamma \setminus (C \setminus \partial^\star C)$, $G = W \cap Y$, $A = W \cap X$, and $G_0 = \partial^\star W$. As mentioned in Section 4.2 we refer to $W = W(I)$ as the *volume* of I, and $G_0 = G_0(I)$ as the *contour* of I.

The next proposition collects the relevant properties of these objects. Once we have these properties, we will not be concerned with how G, A etc. were derived from I.

Proposition 4.3.2

$$v_0 \in A \tag{4.9}$$

$$\textit{both } C \textit{ and } W \textit{ are connected;} \tag{4.10}$$

$$G_0 = \partial^\star C; \tag{4.11}$$

$$G = N(A) \quad \textit{and} \quad A = \{x \in X | N(x) \subseteq G\}; \tag{4.12}$$

$$G_0 \textit{ is 2-linked} \tag{4.13}$$

$$G_0 \cap I = \emptyset; \tag{4.14}$$

$$N(G_0) \cap I \subseteq A; \tag{4.15}$$

$$G_0 \subseteq N(A \cap I). \tag{4.16}$$

Proof: Both (4.9) and the connectivity of C are immediate. To see that W is connected, notice that each component of $\Gamma - (W'' \setminus \partial^\star W'')$ must meet $\partial^\star W''$ (or it would be a component of the connected graph Γ). Thus W is the union of the connected set W'' and a number of other connected sets each of which meets W'', so is itself connected. So we have (4.10).

For (4.11): $\partial^\star C \subseteq W \cap Y$ and the connectivity of C give

$$x \in \partial^\star C \Rightarrow \emptyset \neq N(x) \cap C \subseteq C \cap X \subseteq C \setminus W \Rightarrow x \in \partial^\star W,$$

so $\partial^\star C \subseteq \partial^\star W$; and Lemma 4.3.1 and the connectivity of W give the reverse containment.

Connectivity of W and the fact that $G_0 \subseteq Y$ give $G = N(A)$. That $A \subseteq \{x \in X | N(x) \subseteq G\}$ follows from $G = N(A)$ (or just $\partial^\star W \subseteq Y$). For the reverse containment, notice that $x \notin W \Rightarrow N(x) \cap W \subseteq G_0 \subseteq W'$, whereas $N(x) \subseteq W'$ would imply $x \in W'' \subseteq W$; so $x \notin W \Rightarrow N(x) \not\subseteq W$.

The 2-linkedness of G_0 follows from Proposition 2.7.1 (using (4.10) and (4.11)).

For (4.14) recall that $G_0 = \partial^\star C \subseteq \partial^\star W'' \subseteq \partial^\star W' \subseteq Z_0$ and $Z_0 \cap I = \emptyset$.

That $N(G_0) \cap I \subseteq A$ follows from $G_0 \subseteq \partial^\star W'$, since $N(\partial^\star W') \cap I \subseteq A$.

Finally, $v \in G_0 \Rightarrow v \in Z_0 \Rightarrow v \sim I$, so (4.16) follows from (4.15).

∎

4.4 Shifts and φ_j

In this section, we define the shift operation, and take the first step towards defining φ, the one-to-many map from $\mathcal{I}$ to $\mathcal{J}$. We fix $I \in \mathcal{I}$ and take W, G, A and G_0 to be as in Section 4.3.

For $j \in [-d, d] \setminus \{0\}$, define σ_j, the *shift operation in direction* j, by

$$\sigma_j(v) = v + e_j,$$

where e_j is the j^{th} standard basis vector if $j > 0$ and $e_j = -e_{-j}$ if $j < 0$, and set

$$G_0^j = \{v \in G_0 : \sigma_j^{-1}(v) \notin A\} = G_0 \cap \sigma_j(X \setminus A).$$

Proposition 4.4.1 *For each j, the sets $I \setminus W$, $\sigma_j(I \cap W)$ and G_0^j are pairwise disjoint, and their union is an independent set.*

Proof: Trivially, $\sigma_j(I) \cap I = \emptyset$, so in particular $(I \setminus W) \cap \sigma_j(I \cap W) = \emptyset$; $(I \setminus W) \cap G_0^j = \emptyset$ is trivial (because $G_0^j \subseteq W$); and $\sigma_j(I \cap W) \cap G_0^j = \emptyset$ follows from the definition of G_0^j. So the union is disjoint.

Clearly $(I \setminus W)$, $\sigma_j(I \cap W)$ and G_0^j are all independent sets. To show independence of the union, we must show that there are no edges between any two of them. Since $\nabla(I \setminus W, W) = \emptyset$ (by (4.15)) and $\sigma_j(I \cap W) \subseteq W$ (by (4.14)), we have $\nabla((I \setminus W), (\sigma_j(I \cap W) \cup G_0^j)) = \emptyset$.

This leaves $\nabla(\sigma_j(I \cap W), G_0^j)$. Suppose, for a contradiction, that $y \in G_0^j$ and $\sigma_k(y) \in \sigma_j(I \cap W)$ for some k. Then $z := \sigma_j^{-1}(\sigma_k(y)) \in I \cap W \cap Y \subseteq G \setminus G_0$ (by (4.14)), implying $\sigma_j^{-1}(y) = \sigma_k^{-1}(z) \in A$, contrary to the assumption $y \in G_0^j$. So $\nabla(\sigma_j(I \cap W), G_0^j) = \emptyset$.

∎

Define $\sigma_j^*(I) = (I \setminus W) \cup \sigma_j(I \cap W)$ and

$$\varphi_j(I) = \{J : \sigma_j^*(I) \subseteq J \subseteq \sigma_j^*(I) \cup G_0^j\}.$$

Then Proposition 4.4.1 implies

$$\varphi_j(I) \subseteq \mathcal{J}.$$

Notice also that we recover I from j, J ($\in \varphi_j(I)$) and $W = W(I)$; namely, if we are given W, j, and $J \in \varphi_j(I)$, then

$$I = (J \setminus W) \cup \sigma_j^{-1}(J \cap (W \setminus G_0^j)). \tag{4.17}$$

4.5 Defining the flow ν

If W, G, A and G_0 are produced from $I \in \mathcal{I}$ as in Section 4.3, we write $W(I)$, $G(I)$, $A(I)$ and $G_0(I)$, noting that a given W etc. may correspond to more than one I.

Noting that $A(I)$ determines $W(I)$, $G(I)$, and $G_0(I)$ we set

$$\mathcal{A} = \{A \subseteq X : A = A(I) \text{ for some } I \in \mathcal{I}\}.$$

We always use a for $|A|$ and g for $|G|$ and set $t = g - a$. For each a and g set

$$\mathcal{I}(a, g) = \{I \in \mathcal{I} : |A(I)| = a, |G(I)| = g\}$$

and

$$\mathcal{A}(a, g) = \{A \subseteq X : A = A(I) \text{ for some } I \in \mathcal{I}(a, g)\}.$$

Our tasks are to define ν, for which (4.5) will turn out to be obvious, and establish (4.6).

Let us call $I \in \mathcal{I}$ *small* if $|G(I)| \leq d^3$ (we could get by with $d^{9/4}$; see (4.42)), and *large* otherwise. (Note that this definition — which will only be used in this chapter — is unrelated to any previous definition of "small".)

Most of our work (including everything in Sections 4.6) is geared towards large I (though often valid in general). For most of our discussion we fix a and g, and aim to bound the contribution of $\mathcal{I}(a, g)$ to (4.6). Of course these contributions must eventually be summed, but this turns out not to add anything significant.

For small I — an easy case, as we will see in Section 4.7 — we simply choose $j = j(I)$ to maximize $|G_0^j(I)|$, so that, since

$$\sum_j |G_0^j| = |\nabla(G, X \setminus A)| = 2td, \tag{4.18}$$

we have

$$|G_0^j| \geq t. \tag{4.19}$$

We then set

$$\nu(I, J) = \lambda^{|J|-|I|}(1+\lambda)^{-|G_0^j|} \quad \forall J \in \varphi(I). \tag{4.20}$$

(Note this satisfies (4.5). The separate treatment of small I is unnecessary if we only want phase transition, but is needed for the "correct" bound in (4.3).)

To deal with large I, we appeal to the notions of approximation introduced in Chapter 3. From now until Section 4.7 we fix a and g, set $\psi = \sqrt{2d}$ and always take $I \in \mathcal{I}(a, g)$ and $A \in \mathcal{A}(a, g)$.

We apply Lemma 3.3.1 to produce a set $\mathcal{U} = \mathcal{U}(a, g) \subseteq 2^Y \times 2^X$ with

$$|\mathcal{U}| < \exp[O(td^{-1/2} \log^{3/2} d)] \tag{4.21}$$

and a map $\pi : \mathcal{A}(a, g) \longrightarrow \mathcal{U}$ such that for each $A \in \mathcal{A}(a, g)$, $\pi(A)$ is a ψ-approximating pair for A. (As noted in Section 3.1, the graph Γ satisfies the hypothesis of Lemma 3.3.1 with $\ell = 2d$, and by (4.9), (4.12) and (4.13) we have $\mathcal{A}(a, g) \subseteq \mathcal{G}(a, g)$.)

From now until the end of this chapter, we make the following convention: for whatever G, A, F, S we have under discussion, we set $H = Y \setminus G$, $B = X \setminus A$, $E = Y \setminus F$, $T = X \setminus S$, $B_0 = B \cap N(G)$, $S_0 = S \cap N(E)$, and $E_0 = E \cap N(S)$.

Now consider some $(F, S) \in \mathcal{U}$. Notice that, for any $A \in \mathcal{A}(a, g)$ for which (F, S) is a ψ-approximating pair,

$$Q := S_0 \cup E_0$$

contains all the vertices of Γ whose locations in the partition $\Gamma = G \cup H \cup A \cup B$ are as yet unknown; namely, we have

$$F \subseteq G, \quad T \subseteq B, \quad S \setminus S_0 \subseteq A, \quad E \setminus E_0 \subseteq H$$

(the first two containments are just (3.7); $S \setminus S_0 \subseteq A$ follows from $F \subseteq G$, (4.12) and the definition of S_0, and $E \setminus E_0 \subseteq H$ is similar).

From now on, whenever we are given an (F,S), we take Q to be as defined in the preceding paragraph, and write Γ_Q for the subgraph induced by Q.

We are now in a position to define ν (for large I). We fix $(F,S) \in \mathcal{U}$ and write $I \sim (F,S)$ if $(F,S) = \pi(A(I))$.

To define $\nu(I,\cdot)$ for $I \sim (F,S)$, we first need to choose a direction $j = j(I)$. Fix such an I and let $G = G(I)$, $A = A(I)$, etc. The choice of j will depend only on A. Observe that (using (3.9) and (3.10))

$$\sum_j |\sigma_j(S_0 \cap A) \cap E_0| = |\nabla(S_0 \cap A, G \cap E_0)| < |G \cap E_0|\psi$$

and

$$\sum_j |\sigma_j^{-1}(E_0) \cap (S_0 \setminus A)| = |\nabla(E_0, S_0 \setminus A)| < |S_0 \setminus A|\psi.$$

But (3.9), (3.10) and (3.6) imply $|G \cap E_0| + |S_0 \setminus A| < 2td/(2d-\psi)$, so that

$$\begin{aligned} \sum_j |\sigma_j(S_0) \cap E_0| &= \sum_j (|\sigma_j(S_0 \cap A) \cap E_0| + |\sigma_j^{-1}(E_0) \cap (S_0 \setminus A)|) \\ &< 2td\psi/(2d-\psi). \end{aligned} \tag{4.22}$$

We assert that we can choose j so that

$$|G_0^j| > .8t \tag{4.23}$$

and

$$|\sigma_j(S_0) \cap E_0| < 5|G_0^j|\psi/d. \tag{4.24}$$

To see this, let

$$P = \{j \in [-d,d] \setminus \{0\} : |\sigma_j(S_0) \cap E_0| \geq 5|G_0^j|\psi/d\}.$$

Then (4.22) gives

$$\sum_{j \in P} |G_0^j| \leq \frac{d}{5\psi} \sum |\sigma_j(S_0) \cap E_0| < t\frac{2d^2}{5(2d-\psi)},$$

so (using (4.18))

$$\sum_{j \notin P} |G_0^j| > (1 - d/(5(2d - \psi)))2td.$$

So there exists $j \notin P$ with (say) $|G_0^j| > .8t$, which is what we want.

Having chosen j satisfying (4.23) and (4.24), we turn to defining $\nu(I, \cdot)$. Let

$$C = C^j(I) = G_0^j \cap F \cap \sigma_j(S_0) \quad (= \sigma_j(S_0 \setminus A) \cap F),$$

and

$$D = D^j(I) = G_0^j \cap (\sigma_j(T) \cup (\sigma_j(S_0) \cap E_0)).$$

Then

$$C \cup D \text{ is a partition of } G_0^j. \tag{4.25}$$

Setting $\alpha = \alpha(\lambda) = \lambda/(1+\lambda)^2$ and $\beta = \beta(\lambda) = 1 - \alpha\lambda = (1+2\lambda)/(1+\lambda)^2$, define

$$\nu(I,J) = \begin{cases} (\alpha\lambda)^{|C \cap J|}\beta^{|C \setminus J|}(\lambda/(1+\lambda))^{|D \cap J|}(1+\lambda)^{-|D \setminus J|} & \\ \qquad = \frac{w(J)}{w(I)}\alpha^{|C \cap J|}\beta^{|C \setminus J|}(1+\lambda)^{-|D|} & \text{if } j \in \varphi_j(I) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_J \nu(I,J) = 1 \qquad \forall I \tag{4.26}$$

(because of (4.25)). On the other hand we will show, for any $J \in \mathcal{J}$,

$$\sum_{I \sim (F,S)} \frac{w(I)}{w(J)} \nu(I,J) \le 2d\beta^{t/2}. \tag{4.27}$$

4.6 Proof of (4.27)

We need one easy lemma. Given a bipartite graph Σ on $P \cup R$ and $U \subseteq R$, say that a (vertex) cover $K \cup L \cup M$ of Σ with $K \subseteq P$, $L \subseteq U$ and $M \subseteq R \setminus U$ is *legal* (with respect to U) if it is a minimal cover and

$$K = N(U \setminus L).$$

(Note that minimality implies $K = N(R \setminus (L \cup M))$.)

Lemma 4.6.1 *With notation as above, let $K \cup L \cup M$ be a legal cover with $|K \cup L|$ as small as possible. Then*

(a) $\forall K' \subseteq K \quad |N(K') \cap (U \setminus L)| \geq |K'|$

and

(b) $\forall L' \subseteq L \quad |N(L') \setminus K| \geq |L'|$.

Proof: (a) Given $K' \subseteq K$, let $S = N(K') \cap (U \setminus L)$,

$$K'' = \{v \in K : N(v) \cap U \subseteq S \cup L\} \ \ (\supseteq K'),$$

and $T = N(K'') \cap (R \setminus U)$. Then

(i) $(K \setminus K'') \cup (L \cup S) \cup (M \cup T)$ is a minimal cover

(a straightforward verification using the fact that each vertex of $K \setminus K''$ has a neighbor in $U \setminus (L \cup S)$), and

(ii) $K \setminus K'' = N(U \setminus (L \cup S))$.

Minimality of $|K \cup L|$ thus implies $|K \setminus K''| + |L \cup S| \geq |K| + |L|$, so $|S| \geq |K''| \geq |K'|$.

(b) This is similar. Given $L' \subseteq L$, let $W = N(L') \setminus K$ and

$$L'' = \{u \in L \cup M : N(u) \subseteq K \cup W\} \ \ (\supseteq L').$$

Then

(i) $K \cup W \cup ((L \cup M) \setminus L'')$ is a minimal cover, and

(ii) $K \cup W = N(U \setminus (L \setminus L''))$.

Minimality of $|K \cup L|$ thus implies $|K \cup W| + |L \setminus L''| \geq |K| + |L|$, and $|W| \geq |L''| \geq |L'|$.

∎

Proof of (4.27): Given (F, S), J and j, set

$$\mathcal{I}^\star = \mathcal{I}^\star(F, S, J, j) = \{I \sim (F, S) : j(I) = j, J \in \varphi_j(I)\}.$$

We will show

$$\sum_{I\in\mathcal{I}^\star}\frac{w(I)}{w(J)}\nu(I,J)<\beta^{t/2},$$

which of course gives (4.27).

Set $U=\sigma_j^{-1}(J)\cap S_0$. Suppose $I\in\mathcal{I}^\star$, and set $G=G(I)$, $A=A(I)$, and

$$K=K(I)=G\cap E_0,$$

$$L=L(I)=U\setminus A$$

and

$$M=M(I)=(S_0\setminus U)\setminus A.$$

Then $K\cup L\cup M$ $(=(G\cup B)\cap Q)$ is a minimal cover of Γ_Q. (That it is a cover follows from (4.12); for minimality, notice (e.g.) that each $v\in G\cap E_0$ has a neighbor in A, which must be in S_0 (using $A\subseteq S$ and the definition of S_0).) Moreover, we assert,

$$K=N_{\Gamma_Q}(U\setminus L). \tag{4.28}$$

Proof: We show that each side of (4.28) contains the other. The obvious direction is

$$N_{\Gamma_Q}(U\setminus L)=N_{\Gamma_Q}(U\cap A)\subseteq N(A)\cap E_0=G\cap E_0=K.$$

For the reverse containment, suppose $v\in K$. Since $K\subseteq G_0$, (4.16) says that v has a neighbor $u\in A\cap I$. Then $u\in S_0$ (because $v\in E_0\not\sim S\setminus S_0$), implying $u\in U$ (since $u\in A\cap I\Rightarrow\sigma_j(u)\in J$). And of course $u\notin L$ (since $u\in A$).

∎

Thus $K\cup L\cup M$ is a legal cover of Γ_Q with respect to U in the sense of Lemma 4.6.1.

Now fix $K_0\cup L_0\cup M_0$, a legal cover of Γ_Q with respect to U with $|K_0\cup L_0|$ as small as possible.

Given $I\in\mathcal{I}^\star$, let $K=K(I)$ etc. be as above and set $K'=K_0\setminus K$, $L'=L_0\setminus L$. Then by Lemma 4.6.1,

$$|L|\geq|K'|+|L_0\setminus L'|,\quad |K|\geq|L'|+|K_0\setminus K'|. \tag{4.29}$$

Furthermore, we assert,

$$K = (K_0 \setminus K') \cup N_{\Gamma_Q}(L'). \tag{4.30}$$

The point of this is that it says that (K', L') determines G (so also A), and therefore $I \in \mathcal{I}^\star$ (because of (4.17)).

To see (4.30), just observe that the only point requiring proof is $K \setminus K_0 \subseteq N_{\Gamma_Q}(L_0 \setminus L)$, and that this follows from (4.28) once we notice that $\nabla(K \setminus K_0, U \setminus (L_0 \cup L)) = \emptyset$ (since $K_0 \cup L_0$ covers $\nabla(E_0, U)$).

Now with $C = C^j(I)$, $D = D^j(I)$ as in the discussion preceding (4.25), observe that

$$C \cap J = \sigma_j(L \setminus \sigma_j^{-1}(E_0)) \quad \text{and} \quad C \setminus J = \sigma_j(M \setminus \sigma_j^{-1}(E_0)),$$

and that we may partition D as

$$D = (\sigma_j(T) \cap F) \cup (K \setminus \sigma_j(S_0 \setminus (L \cup M))).$$

Thus, with inequalities justified below,

$$\begin{aligned}
\frac{w(I)}{w(J)}\nu(I,J) &= \alpha^{|\sigma_j(L\setminus\sigma_j^{-1}(E_0))|}\beta^{|\sigma_j(M\setminus\sigma_j^{-1}(E_0))|} \\
&\qquad \cdot (1+\lambda)^{-(|\sigma_j(T)\cap F|+|K\setminus\sigma_j(S_0\setminus(L\cup M))|)} \\
&\leq \alpha^{|L|}\beta^{|M|}(1+\lambda)^{-(|K|+|\sigma_j(T)\cap F|)} \\
&\qquad \cdot \alpha^{-(|\sigma_j(S_0\cap A)\cap K|+|\sigma_j^{-1}(E_0)\cap(S_0\setminus A)|)} && (4.31)\\
&\leq \alpha^{|L|}(1+\lambda)^{-|K|}\beta^{|G_0^j|-(|K|+|L|)}\alpha^{-O(|G_0^j|\psi/\ell)} && (4.32)\\
&\leq \beta^{\delta g/2}\alpha^{|L|}(1+\lambda)^{-|K|}\beta^{-(|K|+|L|)} && (4.33)\\
&= \beta^{\delta g/2}\left(\frac{1+\lambda}{1+2\lambda}\right)^{|K|}\left(\frac{\lambda}{1+2\lambda}\right)^{|L|} \\
&\leq \beta^{\delta g/2}\left(\frac{1+\lambda}{1+2\lambda}\right)^{|L'|+|K_0\setminus K'|}\left(\frac{\lambda}{1+2\lambda}\right)^{|K'|+|L_0\setminus L'|} && (4.34)\\
&= \beta^{\delta g/2}\left(\frac{1+\lambda}{1+2\lambda}\right)^{|K_0|}\left(\frac{\lambda}{1+2\lambda}\right)^{|L_0|}\left(\frac{\lambda}{1+\lambda}\right)^{|K'|-|L'|}.
\end{aligned}$$

(In (4.31) we used $\alpha^{-1} = \max\{\alpha^{-1}, \beta^{-1}, 1+\lambda\}$; in (4.32) we used $G_0^j \subseteq \sigma_j(L \cup M) \cup K \cup (\sigma_j(T) \cap F)$, $(1+\lambda)^{-1} < \beta$ and (4.24); (4.33) is from (4.23), using $(\psi/\ell)\log(1/\alpha) = o(\log(1/\beta))$, which is a consequence of

$$\lambda^2 = \omega((\psi/d)\log(1/\lambda)) \tag{4.35}$$

for small λ, and easily verified when λ is larger; and (4.34) comes from (4.29).)

Thus, recalling — see the remark following (4.30) — that each (K', L') corresponds to at most one $I \in \mathcal{I}^\star$,

$$\begin{aligned}\sum_{I\in\mathcal{I}^\star} \frac{w(I)}{w(J)}\nu(I,J) &\leq \beta^{t/2}\left(\frac{1+\lambda}{1+2\lambda}\right)^{|K_0|}\left(\frac{\lambda}{1+2\lambda}\right)^{|L_0|}\sum_{K'\subseteq K_0}\sum_{L'\subseteq L_0}\left(\frac{\lambda}{1+\lambda}\right)^{|K'|-|L'|}\\ &= \beta^{t/2}.\end{aligned}$$

And then varying (F, S) and j we find (referring to (4.21)) that for $\lambda \leq 2$ (say),

$$\begin{aligned}\sum_{I\in\mathcal{I}(a,g)} \frac{w(I)}{w(J)}\nu(I,J) &= \sum_{(F,S)\in\mathcal{U}}\sum_{j}\sum\{\frac{w(I)}{w(J)}\nu(I,J) : I \in \mathcal{I}^\star(F,S,j,J)\}\\ &\leq 2|\mathcal{U}|d\beta^{t/2}\\ &< 2d\exp[\{O(d^{-1/2}\log^{3/2} d) - \Omega(\lambda^2)\}t]\\ &< \exp[-\Omega(\lambda^2 t)], \qquad (4.36)\end{aligned}$$

while for larger λ,

$$\sum_{I\in\mathcal{I}(a,g)} \frac{w(I)}{w(J)}\nu(I,J) < \lambda^{-\Omega(t)}.$$

4.7 Proof of (4.6)

Now fixing $J \in \mathcal{J}$, we are ready to verify (4.6) (thus completing the proofs of Theorems 4.1.1 and 1.1.1).

Before we begin, we note that Lemma 2.5.1 has the following consequences for t:

$$t = \begin{cases} \Omega(g^{1-1/d}/d) & \text{for all } g \\ g - O(g/d) & \text{if } g = d^{O(1)}. \end{cases} \qquad (4.37)$$

We deal first with large I's (recall I is large if $|G(I)| > d^3$). Here we have already done the work: Assuming first that $\lambda \leq 2$, and with justifications to follow, we have

$$\sum_{I \text{ large}} \frac{w(I)}{w(J)}\nu(I,J) = \sum_{g>d^3}\sum_{t}\sum_{I\in I(a,g)} \frac{w(I)}{w(J)}\nu(I,J)$$

$$
\begin{aligned}
&= \sum_{g>d^3}\sum_t \exp[-\Omega(\lambda^2 t)] && (4.38)\\
&\le \sum_{g>d^3}\sum \{\exp[-\Omega(\lambda^2 t)] : t \ge \Omega(d^{-1}g^{1-1/d})\} && (4.39)\\
&\le \sum_{g>d^3} \exp[-\Omega(\lambda^2(d^{-1}g^{1-1/d}))] && (4.40)\\
&< \exp[-\Omega(\lambda^2 d^{3(1-1/d)-1})] && (4.41)\\
&< \exp[-\omega(\lambda d)]. && (4.42)
\end{aligned}
$$

The main inequality (4.38) is just (4.36), and (4.39) comes from (4.37). In (4.40) we have absorbed a factor λ^{-2} in the exponent. One way to see the inequality in (4.41) is to use

$$(1-\varepsilon)^{g^{1-\delta}} < (1-\varepsilon)^{tK^{1-\delta}} \quad \text{for} \quad t^{1/(1-\delta)}K < g \le (t+1)^{1/(1-\delta)}K$$

with $K = d^3$, $\delta = 1/d$ and $1-\varepsilon = \exp[-\Omega(\lambda^2 d^{-1})]$.

For $\lambda > 2$ a similar analysis gives

$$\sum_{I \text{ large}} \frac{w(I)}{w(J)}\nu(I,J) \le \lambda^{-\Omega(d^2)}. \qquad (4.43)$$

Finally we turn to the easy case of small I. For a (nonempty) $\mathcal{I}(a,g)$ with $g < d^3$, (4.37) gives $a = O(g/d)$, so that, since each $A(I)$ is 2-linked and contains v_0, Lemma 2.4.1 bounds the number of possibilities for $A \in \mathcal{A}(a,g)$ by

$$\exp[O((g/d)\log d)].$$

But we also know (see (4.17)) that, given J and j, $I \in \varphi_j^{-1}(J)$ is determined by $A(I)$, and that (by (4.20), (4.19), and again (4.37))

$$
\begin{aligned}
\frac{w(I)}{w(J)}\nu(I,J) &= (1+\lambda)^{-|G_0^j(I)|}\\
&\le (1+\lambda)^{-t}\\
&= (1+\lambda)^{-(1-O(1/d))g}.
\end{aligned}
$$

So finally, noting that $A(I) \neq \emptyset$ implies $|G(I)| \ge 2d$, we have

$$
\begin{aligned}
\sum_{I\in\mathcal{I}(a,g)} \frac{w(I)}{w(J)}\nu(I,J) &< 2d\exp[O((g/d)\log d)](1+\lambda)^{-(1-O(1/d))g}\\
&< (1+\lambda)^{-(1-o(1))g}
\end{aligned}
$$

and

$$\begin{aligned}
\sum_{I \text{ small}} \frac{w(I)}{w(J)}\nu(I,J) &= \sum_{2d\le g\le d^3}\sum_{a\le g}\sum_{I\in\mathcal{I}(a,g)} \frac{w(I)}{w(J)}\nu(I,J) \\
&< \sum_{2d\le g<d^3} g(1+\lambda)^{-(1-o(1))g} \\
&\le (1+\lambda)^{-(1-o(1))2d};
\end{aligned}$$

and combining this with (4.42) or (4.43) gives (4.6).

Chapter 5

Homomorphisms from the Hamming cube to Z

In this chapter, we consider the problem of (asymptotically) counting homomorphisms from the Hamming cube Q_d to the Hamming graph on $\mathbf{Z}$. (See Section 1.2 for a detailed introduction to the problem.)

In Section 5.1 we use Lemma 2.4.5 to reduce Theorem 1.2.4 to the problem of counting the number of homomorphisms which are predominantly 0 on $\mathcal{E}$. The easy lower bounds on the number of homomorphisms which take on four and five values are given in Section 5.2. In Section 5.3 we examine a general type of sum over small subsets of $\mathcal{E}$ and establish some of its properties. In Section 5.4 we write down an explicit sum of the type examined in Section 5.3 for the number of homomorphisms which are predominantly 0 on $\mathcal{E}$. The rest of the chapter is devoted to estimating this sum. In Section 5.5 we arrive at the heart of the matter, using the results of Chapter 3 to show that the set of "nice" subsets of $\mathcal{E}$ can be "well-approximated" in a precise sense by a "small" collection; this allows us to swiftly complete the proof of Theorem 1.2.4 in Section 5.6. Finally, in Section 5.7 we elaborate on a comment made in the introduction to this problem to the effect that Theorem 1.2.4 contains the asymptotics of the number of independent sets in Q_d originally derived in [24] (and [20]).

5.1 Reduction to mostly constant

We begin the proof of Lemma 1.2.4 by using Lemma 2.4.5 to reduce the problem to that of counting homomorphisms which mainly take a single value on $\mathcal{E}$.

There is an inherent odd-even symmetry in the problem; we now reformulate slightly to make use of this. Write

$$\mathcal{A} = \{f : V \to \mathbf{Z} : u \sim v \Rightarrow |f(u) - f(v)| = 1\}$$

and write $\mathcal{B}$ for the quotient of $\mathcal{A}$ by the equivalence relation

$$f \equiv g \Longleftrightarrow f - g \text{ is constant on } V.$$

For each $f \in \mathcal{A}$ write $[f]$ for the equivalence class of f in $\mathcal{B}$. Noting that R is constant on equivalence classes, we may define

$$\mathcal{B}_i = \{[f] \in \mathcal{B} : |R(f)| = i\}.$$

Clearly $|\mathcal{B}_i| = |\mathcal{F}_i|$ for each i ($\mathcal{F}$ is a complete set of representatives for $\mathcal{B}$).

For $f \in \mathcal{A}$, we say that f is *mostly constant on* $\mathcal{E}$ if there is some c such that $\{v \in \mathcal{E} : f(v) \neq c\}$ is small (see Section 2.1 for the definition of *small*; the constant α in that definition will be specified in the proof of Lemma 5.1.2). We define *mostly constant on* $\mathcal{O}$ analogously. These definitions respect the equivalence relation, so we may define

$$\mathcal{B}^{\mathcal{E}} = \{[f] \in \mathcal{B} : f \text{ is mostly constant on } \mathcal{E}\}.$$

Define $\mathcal{B}^{\mathcal{O}}$ analogously. By symmetry, $|\mathcal{B}^{\mathcal{E}}| = |\mathcal{B}^{\mathcal{O}}|$ (any automorphism of Q_d that sends $\mathcal{E}$ to $\mathcal{O}$ induces a bijection between the two sets).

Lemma 5.1.1

$$|\mathcal{B}^{\mathcal{E}} \cap \mathcal{B}^{\mathcal{O}}| = e^{-\Omega(d)}|\mathcal{B}|.$$

Proof: To specify an $[f] \in \mathcal{B}^{\mathcal{E}} \cap \mathcal{B}^{\mathcal{O}}$ we first specify the predominant values of the representative f on $\mathcal{E}$ and $\mathcal{O}$. W.l.o.g. we may assume that the predominant value on $\mathcal{E}$ is 0, and so the predominant value on $\mathcal{O}$ is one of ± 1. We then specify the small sets from $\mathcal{E}$ and $\mathcal{O}$ on which f does not take the predominant values, and finally the values of f on these small sets. Noting that once $f(v)$ has been specified for any $v \in V$ there are most $2d+1$ values that f can take on any other vertex and that 2^M is a trivial lower bound on $|\mathcal{B}|$, we get

$$\begin{aligned} |\mathcal{B}^{\mathcal{E}} \cap \mathcal{B}^{\mathcal{O}}| &\leq 2 \sum_{i,j \leq \alpha^d} \binom{M}{i}\binom{M}{j}(2d+1)^{i+j} \\ &\leq e^{-\Omega(d)}|\mathcal{B}|. \end{aligned}$$

∎

Lemma 5.1.2

$$|\mathcal{B}| = (2 \pm e^{-\Omega(d)})|\mathcal{B}_{\mathcal{E}}|.$$

Proof: For $f \in \mathcal{A}$, set $C(f) = \{v \in V : f|_{N(v)} \text{ is constant}\}$ (extending the definition given in Section 2.4). We choose a uniform member $[\mathbf{f}]$ of $\mathcal{B}$ by choosing $\mathbf{f}$ uniformly from $\mathcal{F}$. For $[\mathbf{f}]$ and $u, v \in V$, let Q_u be the event $\{u \in C(\mathbf{f})\}$, $Q_{u\overline{v}} = Q_u \cap Q_{\overline{v}}$ and $Q_{\overline{uv}} = Q_{\overline{u}} \cap Q_{\overline{v}}$. Write $K_u = K_u(\mathbf{f})$ for the set of vertices that can be reached from u in $C(\mathbf{f})$ via steps of size exactly 2, and let Q^*_{uv} be the event $\{v \in K_u\}$. (Note that if $f, g \in \mathcal{A}$ are equivalent then $C(f) = C(g)$, so all these events are well defined.)

Let u and v be two vertices of the same parity. We claim that $Q_{\overline{uv}} \cup Q^*_{uv}$ occurs with probability $1 - e^{-\Omega(d)}$. For, let $ua_1a_2 \ldots a_{2k-1}v$ be a u-v path of length at most d (the diameter of Q_d). Writing a_0 for u and a_{2k} for v, we have

$$Q_{\overline{uv}} \cup Q^*_{uv} \supseteq \cap_{i=0}^{2k-1}(Q_{a_i\overline{a_{i+1}}} \cup Q_{\overline{a_i}a_{i+1}}).$$

By Lemma 2.4.5, $\mathbf{P}(Q_{a_i\overline{a_{i+1}}} \cup Q_{\overline{a_i}a_{i+1}}) = 1 - e^{-\Omega(d)}$ for each i. Hence $\mathbf{P}(Q_{\overline{uv}} \cup Q^*_{uv}) \geq 1 - de^{-\Omega(d)} = 1 - e^{-\Omega(d)}$, as claimed.

We therefore have, for fixed $u \in V$ and any v of the same parity as u, $\mathbf{P}(Q^*_{uv}|Q_u) > 1 - c^{-d}$, where $c > 1$ is fixed. So, conditioning on Q_u, we have

$$\mathbf{E}(|\{v : \rho(u,v) \ \textit{even}, v \notin K_u\}|) \leq (2/c)^d,$$

so that, by Markov's Inequality (with the constant c' chosen so that $2/c < c' < 2$),

$$\mathbf{P}(|K_u| < M - (c')^d|Q_u) \leq (2/cc')^d = e^{-\Omega(d)}. \tag{5.1}$$

If $u \notin C(\mathbf{f})$, then $K_u(\mathbf{f}) = \emptyset$, so that $\mathbf{P}(|K_u| < M - (c')^d|Q_{\overline{u}}) = 1$. By symmetry, $\mathbf{P}(Q_{u\overline{v}})$ is the same for every adjacent u and v, and this together with Lemma 2.4.5 gives $1/2 + e^{-\Omega(d)} > \mathbf{P}(Q_u), \mathbf{P}(Q_{\overline{u}}) > 1/2 - e^{-\Omega(d)}$. Combining these observations with (5.1), we get

$$\mathbf{P}(|K_u| < M - (c')^d) \leq 1/2 + e^{-\Omega(d)}.$$

Noting that $\mathbf{f}$ is constant on the neighborhood of K_u, this says (taking u to be any vertex in $\mathcal{O}$) that there is a constant $\beta < 2$ such that

$$\mathbf{P}(\mathbf{f} \text{ is constant on a subset of } \mathcal{E} \text{ of size at least } M - \beta^d) > 1/2 - e^{-\Omega(d)}.$$

Taking $\alpha = \beta$ in the definition of small, this says

$$|\mathcal{B}^{\mathcal{E}}| \geq (1/2 - e^{-\Omega(d)})|\mathcal{B}|.$$

The lemma now follows from Lemma 5.1.1.

∎

It is now convenient to choose as a complete set of representatives for $\mathcal{B}^{\mathcal{E}}$ the collection

$$\mathcal{F}^{\mathcal{E}} = \{f \in \mathcal{A} : \mathcal{E} \setminus f^{-1}(0) \text{ is small}\}.$$

Set

$$\mathcal{F}_i^{\mathcal{E}} = \{f \in \mathcal{F}^{\mathcal{E}} : |R(f)| = i\}.$$

Noting that $|\mathcal{F}_3^{\mathcal{E}}| \geq 2^M$, we see that Theorem 1.2.4 will now follow from

Theorem 5.1.3

$$\begin{aligned} |\mathcal{F}^{\mathcal{E}}| &\leq (e + e^{-\Omega(d)})2^M && (5.2) \\ |\mathcal{F}_4^{\mathcal{E}}| &\geq (2\sqrt{e} - 2 - e^{-\Omega(d)})2^M && (5.3) \\ |\mathcal{F}_5^{\mathcal{E}}| &\geq (e - 2\sqrt{e} + 1 - e^{-\Omega(d)})2^M. && (5.4) \end{aligned}$$

It is this that we proceed to prove.

5.2 Lower bounds on $|\mathcal{F}_4^{\mathcal{E}}|$ and $|\mathcal{F}_5^{\mathcal{E}}|$

The aim of this section is to prove (5.3) and (5.4).

With each sparse $A \subseteq \mathcal{E}$ of size at least 2 we associate a subset $\mathcal{F}_5^{\mathcal{E}}(A) \subseteq \mathcal{F}_5^{\mathcal{E}}$ of size

$$(2^{|A|} - 2)2^{M-d|A|} = 2^M M^{-|A|}(1 - 2^{-|A|+1})$$

consisting of those $f \in \mathcal{F}_5^{\mathcal{E}}$ for which $R(f) = [-2, 2]$ and $f^{-1}(\{\pm 2\}) = A$ (on A, choose values for f from $\{\pm 2\}$, choosing at least one 2 and at least one -2; on $\mathcal{E} \setminus A$ give f value 0; and on $\mathcal{O} \setminus N(A)$ choose values from $\{\pm 1\}$, all choices made independently). Then $\mathcal{F}_5^{\mathcal{E}}(A) \cap \mathcal{F}_5^{\mathcal{E}}(B) = \emptyset$ whenever $A \neq B$. Noting that there are at least $\binom{M}{k} - Md^2\binom{M-2}{k-2}$

sparse subsets of $\mathcal{E}$ of size k, and that for $k \leq d$, this number is $(1 - e^{-\Omega(d)})\binom{M}{k}$, we can lower bound $|\mathcal{F}_5^{\mathcal{E}}|$ by

$$\begin{aligned}
\left|\mathcal{F}_5^{\mathcal{E}}\right| &\geq 2^M \sum_{k\geq 2} |\{A \subseteq \mathcal{E} : A \text{ sparse}, |A| = k\}| M^{-k}(1 - 2^{-k+1}) \\
&\geq 2^M(1 - e^{-\Omega(d)}) \sum_{k=2}^{d} \binom{M}{k} M^{-k}(1 - 2^{-k+1}) \\
&\geq 2^M(1 - e^{-\Omega(d)}) \sum_{k=2}^{d} (1/k!)(1 - 2^{-k+1}) \\
&\geq 2^M(1 - e^{-\Omega(d)})((e - 2) - 2(\sqrt{e} - 3/2)) \\
&\geq 2^M(e - 2\sqrt{e} + 1 - e^{-\Omega(d)}),
\end{aligned}$$

so we have (5.4).

We do something similar for (5.3). With each nonempty, sparse $A \subseteq \mathcal{E}$ we associate a subset $\mathcal{F}_4^{\mathcal{E}}(A) \subseteq \mathcal{F}_4^{\mathcal{E}}$ of size

$$2^{1+M-d|A|} = 2^M M^{-|A|} 2^{-|A|+1}$$

consisting of those $f \in \mathcal{F}_4^{\mathcal{E}}$ for which either $R(f) = [-2, 1]$ or $R(f) = [-1, 2]$ and $f^{-1}(\{\pm 2\}) = A$ (choose a value from ± 2 for f to take on A; on $\mathcal{E} \setminus A$ give f value 0; and choose values from ± 1 on $\mathcal{O} \setminus N(A)$, all choices made independently). So we have

$$\begin{aligned}
\left|\mathcal{F}_4^{\mathcal{E}}\right| &\geq 2^M \sum_{k\geq 1} |\{A \subseteq \mathcal{E} : A \text{ sparse}, |A| = k\}| M^{-k} 2^{-k+1} \\
&\geq 2^M(2\sqrt{e} - 2 - e^{-\Omega(d)}).
\end{aligned}$$

5.3 Sums over small subsets of $\mathcal{E}$

In this section, we examine a certain kind of sum that will arise when we try to write down an explicit expression for $|\mathcal{F}^{\mathcal{E}}|$. Specifically, we prove

Lemma 5.3.1 *Suppose that $g : 2^{\mathcal{E}} \to \mathbf{R}^+$ satisfies*

$$g(A) = \prod \{g(A_i) : A_i \prec A\}, \tag{5.5}$$

$$g(\{y\}) = c2^{-d} \quad \forall y \in \mathcal{E} \text{ for some constant } c > 0 \tag{5.6}$$

and

$$\sum_{A \text{ nice}} g(A) = e^{-\Omega(d)}. \tag{5.7}$$

Then

$$\left| \sum_{A \subseteq D,\ A \text{ small}} g(A) - (1 + c2^{-d})^{|D|} \right| = e^{-\Omega(d)} \quad \forall D \subseteq \mathcal{E}.$$

Proof: All summations below are restricted to subsets of D. We begin by observing that $(1 + c2^{-d})^{|D|} = \sum_A c^{|A|}2^{-d|A|}$ and that if A is sparse then $g(A) = c^{|A|}2^{-d|A|}$, so that

$$\left| \sum_{A \text{ small}} g(A) - (1 + c2^{-d})^{|D|} \right| \leq \sum{}' g(A) + \sum{}'' c^{|A|}2^{-d|A|} + \sum{}''' c^{|A|}2^{-d|A|}, \tag{5.8}$$

where $\sum'$ is over A small and non-sparse, $\sum''$ is over A large and $\sum'''$ is over A non-sparse.

We bound each of the terms on the right-hand side of (5.8). For the first we have

$$\begin{aligned} \sum{}' g(A) &\leq \sum \left\{ g(A')g(A' \setminus A) : A' \text{ nice, } A \text{ small, } A' \prec A \right\} \\ &\leq \sum_{A' \text{ nice}} g(A') \sum_{A \text{ small}} g(A) \\ &= e^{-\Omega(d)} \sum_{A \text{ small}} g(A). \end{aligned} \tag{5.9}$$

For the second we have

$$\begin{aligned} \sum{}'' c^{|A|}2^{-d|A|} &\leq \sum_{|A| \geq d} c^{|A|}2^{-d|A|} \\ &\leq \sum_{i=d}^{|D|} \binom{|D|}{i} (c2^{-d})^i \\ &\leq \sum_{i \geq d} c^i / i! \\ &= e^{-\Omega(d)}. \end{aligned} \tag{5.10}$$

Finally, for the third we have

$$\begin{aligned} \sum{}''' c^{|A|}2^{-d|A|} &\leq \sum_{x,x' \in D,\ \rho(x,x')=2} c^2 2^{-2d} \sum_A c^{|A|}2^{-d|A|} \\ &\leq |D| c^2 d^2 2^{-2d} (1 + c2^{-d})^{|D|} \\ &= e^{-\Omega(d)}. \end{aligned} \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we get

$$\left|\sum_{A \text{ small}} g(A) - (1+c2^{-d})^{|D|}\right| = e^{-\Omega(d)}\left(\sum_{A \text{ small}} g(A)+1\right) \tag{5.12}$$

$$= e^{-\Omega(d)}. \tag{5.13}$$

(We get (5.13) from (5.12) because the latter implies that $\sum_{A \text{ small}} g(A)$ is bounded.)

∎

The most important g that we will be considering is

$$g(A) = 2^{-|N(A)|+|B(A)|}.$$

It's easy to see that this satisfies (5.5) and (5.6) (with $c = 1$). It is far from obvious that it satisfies (5.7); Sections 5.5 and 5.6 are devoted to the proof of this fact, which we state now for use in Section 5.4.

Theorem 5.3.2

$$\sum_{A \subseteq \mathcal{E} \textit{ nice}} 2^{-|N(A)|+|B(A)|} = e^{-\Omega(d)}.$$

5.4 Proof of (5.2)

In this section, we write an explicit sum of the type introduced in Section 5.3 for $|\mathcal{F}^{\mathcal{E}}|$ and use Lemma 5.3.1 to estimate it, modulo Theorem 5.3.2. This will give (5.2).

For each small $A \subseteq \mathcal{E}$, set

$$\mathcal{F}^{\mathcal{E}}(A) = \{f \in \mathcal{F}^{\mathcal{E}} : f^{-1}(0) = \mathcal{E} \setminus A\}.$$

We specify $f \in \mathcal{F}^{\mathcal{E}}(A)$ by the following procedure. First, noting that f must be either always positive or always negative on a 2-component of A, we specify a sign ($\pm$) for each such 2-component. Next, we specify a nested sequence

$$A = C_2 \supseteq C_4 \supseteq \ldots \supseteq C_{2[d/2]}.$$

For each $i = 1, \ldots, [d/2]$, $C_{2i} = \{u \in \mathcal{E} : |f(u)| \geq 2i\}$. Because the diameter of Q_d is d, we have $|f(u)| \leq 2[d/2]\ \forall u \in \mathcal{E}$, so this second step completes the specification of f on $\mathcal{E}$. Note that not every sequence of C_{2i}'s gives rise to a legitimate $f \in \mathcal{F}^{\mathcal{E}}$.

To specify f on $\mathcal{O}$, first specify a value from ± 1 on each vertex of $\mathcal{O} \setminus N(A)$, and then, for each $i = 1, \ldots, [d/2]$, specify a value from $2i \pm 1$ for $|f(u)|$ for each $u \in B(C_{2i}) \setminus N(C_{2i+2})$ (note that the sign of $f(u)$ for such u has been determined by the specification of signs on A). To see that this completes the specification of f on $\mathcal{O}$, note that we have a choice for the value of $|f|$ at $u \in N(A)$ iff f is constant on $N(u)$ iff $u \in B(C_{2i}) \setminus N(C_{2i+2})$ for some $1 \leq i \leq [d/2]$ (setting $C_{2[d/2]+2} = \emptyset$), and that in this case we can choose from two possible values, $2i \pm 1$.

So, noting that $N(C_{2i+2}) \subseteq B(C_{2i})$ for each $i = 1, \ldots, [d/2]$, we have

$$|\mathcal{F}^{\mathcal{E}}(A)| = 2^{c(A)+M-|N(A)|+|B(A)|} \sum \prod_{i=2}^{[d/2]} 2^{-|N(C_{2i})|+|B(C_{2i})|}$$

where the sum — here and in the next line — is over all legitimate choices of $C_2 \supseteq \ldots \supseteq C_{2[d/2]}$. Setting

$$h(A) = 2^{c(A)-|N(A)|+|B(A)|} \sum \prod_{i=2}^{[d/2]} 2^{-|N(C_{2i})|+|B(C_{2i})|}$$

we get

$$|\mathcal{F}^{\mathcal{E}}| = 2^M \sum_{A \subseteq \mathcal{E} \text{ small}} h(A).$$

It is easy to check that h satisfies (5.5) and (5.6) (with $c = 2$). To see that it satisfies (5.7), note that for each $A \subseteq \mathcal{E}$ small, each C_{2i} is a small subset of A, and so we can crudely upper bound $h(A)$ by

$$\begin{aligned} h(A) &\leq 2^{c(A)-|N(A)|+|B(A)|} \left(\sum_{C \subseteq A \text{ small}} 2^{-|N(C)|+|B(C)|} \right)^{[d/2]} \\ &\leq 2^{c(A)-|N(A)|+|B(A)|} \left((1+2^{-d})^{\alpha^d} + e^{-\Omega(d)} \right)^{[d/2]} \qquad (5.14) \\ &\leq (1+o(1))\, 2^{c(A)-|N(A)|+|B(A)|}. \end{aligned}$$

The inequality in (5.14) is obtained by applying Lemma 5.3.1 and Theorem 5.3.2, and (5.7) for h now follows directly from Theorem 5.3.2.

We can now easily establish (5.2), thus completing the proofs of Theorems 5.1.3 and 1.2.4. Applying Lemma 5.3.1, we have (where $\sum'$ is over $A \subseteq \mathcal{E}$ small)

$$\begin{aligned} \left| |\mathcal{F}^{\mathcal{E}}| - e2^M \right| &\leq 2^M \left(\left| {\sum}' h(A) - (1-2^{-d+1})^{|\mathcal{E}|} \right| + \left| (1-2^{-d+1})^{|\mathcal{E}|} - e \right| \right) \\ &= e^{-\Omega(d)} 2^M. \end{aligned}$$

5.5 Using approximation

We now begin the proof of Theorem 5.3.2. The approach will be to partition the set of A's over which we are summing according to the sizes of $[A]$, $N(A)$, $B(A)$ and $N(B(A))$ (note that the summand in Theorem 5.3.2 is constant on each partition class). The bulk of the work will be in bounding the sizes of the partition classes.

Given $A \subseteq \mathcal{E}$, set $G = G(A) = N(A)$, $B = B(A)$ and $H = H(A) = N(B)$. In what follows, G, B and H are always understood to be $G(A)$, $B(A)$ and $H(A)$ for whatever A is under discussion. Note that $B \subseteq G$ is a closed set, and $H \subseteq A$.

Given a, g, b and h, set

$$\mathcal{H} = \mathcal{H}(a, g, b, h) = \{A \subseteq \mathcal{E} \text{ 2-linked: } |[A]| = a, |G| = g, |B| = b \text{ and } |H| = h\}.$$

The aim of this section is to prove

Lemma 5.5.1 *For each a, g, b and h,*

$$|\mathcal{H}| < M2^{g-b-\Omega(g/\log d)}.$$

Proof: By Lemma 3.3.4 there is a family $\mathcal{X} \in 2^{\mathcal{O}} \times 2^{\mathcal{E}} \times 2^{\mathcal{E}} \times 2^{\mathcal{O}}$ with

$$|\mathcal{X}| \leq 2^{O(g \log d/\sqrt{d})}$$

such that for each $A \in \mathcal{H}$ there is a $\sqrt{d}$-approximating quadruple (F, S, P, Q) for $[A]$ in $\mathcal{X}$ (here we are using the fact that if A is 2-linked then $[A]$ is also). By Lemma 3.2.5 each such (F, S, P, Q) satisfies

$$|S| \leq |F| + O(g/\sqrt{d}) \tag{5.15}$$

and

$$|Q| \leq |P| + O(h/\sqrt{d}). \tag{5.16}$$

Lemma 5.5.1 now follows from

Lemma 5.5.2 *For each $(F, S, P, Q) \in 2^{\mathcal{O}} \times 2^{\mathcal{E}} \times 2^{\mathcal{E}} \times 2^{\mathcal{O}}$ satisfying (5.15) and (5.16), there are at most $2^{g-\Omega(g/\log d)-b}$ A's in $\mathcal{H}$ satisfying*

$$F \subseteq G, S \supseteq [A], P \subseteq H \text{ and } Q \supseteq B. \tag{5.17}$$

Proof: By Lemma 2.6.1, there is an absolute constant $\gamma > 0$ (independent of a, g, b and h) such that

$$g - a > \gamma g \qquad \text{and} \qquad h - b > \gamma h. \tag{5.18}$$

Say that Q is *small* if $|Q| < b + \gamma h/(4 \log d)$, and *large* otherwise, and that S is *small* if $|S| < g - \gamma g/(4 \log d)$ and *large* otherwise. (Note that this definition — which will only be used in this section — is unrelated to any previous definition of "small".)

We can obtain all $A \in \mathcal{H}(a, g, b, h)$ satisfying (5.17) as follows.

We begin by identifying a subset D of A which can be specified relatively "cheaply": if Q is small, we pick $B \subseteq Q$ with $|B| = b$ and take $D = N(B)$; if Q is large, we simply take $D = P$ (recalling that $P \subseteq H \subseteq A$).

If S is small, we complete the specification of A by choosing $A \setminus D \subseteq S \setminus D$. If S is large, we first complete the specification of G by choosing $G \setminus F \subseteq N(S) \setminus F$. Note that in this case, (5.15) implies

$$|G \setminus F| < \gamma g/(3 \log d). \tag{5.19}$$

We then complete the specification of A by choosing $A \setminus D \subseteq [A] \setminus D$ (noting that we do know $[A] \setminus D$ at this point).

This procedure produces all possible A's (and more). Before bounding the number of outputs, we gather together some useful observations.

From (5.15) and (5.16) we have

$$|S| = O(g) \qquad \text{and} \qquad |Q| = O(h). \tag{5.20}$$

If Q is small then there are at most

$$\sum_{i \leq \gamma h/(4 \log d)} \binom{|Q|}{|Q| - i} \leq 2^{\gamma h/3} \tag{5.21}$$

possibilities for D, and in this case $|D| = h$; while if Q is large there is just one possibility for D, and in this case (using (5.16))

$$\begin{aligned} |D| = |P| &> |Q| - \Omega(h/\sqrt{d}) \\ &> b + \gamma h/(4 \log d) - \Omega(h/\sqrt{d}) \\ &\geq b + \gamma h/(5 \log d). \end{aligned} \tag{5.22}$$

If S is large then (since $|N(S) \setminus F| \leq d|S| \leq O(dg)$; see (5.20)) the number of possibilities for $G \setminus F$ is at most

$$\sum_{i<\gamma g/(3\log d)} \binom{O(gd)}{i} \leq 2^{\gamma g/2}. \tag{5.23}$$

We now bound the number of outputs of the procedure, considering separately the four cases determined by whether S and Q are large or small.

If S and Q are both small then the number of possibilities for A is at most

$$2^{\gamma h/3+g-\gamma g/(4\log d)-h} < 2^{g-\gamma g/(4\log d)-b-2\gamma h/3} \tag{5.24}$$

(using (5.21) and the first part of (5.18)). If S is small and Q is large then the total is at most

$$2^{g-\gamma g/(4\log d)-b-\gamma h/(5\log d)} \tag{5.25}$$

(using (5.22)).

If Q is small then $|[A] \setminus D| = a - h$, so that if S is large (and Q small) then the number of possibilities for A is at most

$$2^{\gamma h/3+\gamma g/2+a-h} < 2^{g-\gamma g/2-b-2\gamma h/3} \tag{5.26}$$

(here using (5.21), (5.23) and both parts of (5.18)). Finally, if Q is large then $|[A] \setminus D| \leq a - b - \gamma h/(5\log d)$ (see (5.22)), so that if S and Q are both large the number of possibilities for A is at most

$$2^{\gamma g/2+a-b-\gamma h/(5\log d)} < 2^{g-\gamma g/2-b-\gamma h/(5\log d)}. \tag{5.27}$$

Noting that $h \leq g$, the lemma follows from (5.24), (5.25), (5.26) and (5.27).

∎

5.6 Proof of Theorem 5.3.2

We say that a nice $A \subseteq \mathcal{E}$ is *of type I* if $|[A]| < d/2$, *of type II* if $d/2 \leq |[A]| < d^2$ and *of type III* otherwise. We consider the portions of the sum in Theorem 5.3.2 corresponding to type I, II and III A's separately.

If A is of type I, then $|A| < d/2$ and by Lemma 2.6.2, $|N(A)| \geq d|A| - 2|A|(|A|-1)$. Note also that in this case, $B(A) = \emptyset$. By Lemma 2.4.1, for each $2 \leq i < d/2$, there are at most $M2^{O(i \log d)} < 2^{d+O(i \log d)}$ 2-linked subsets of $\mathcal{E}$ of size i. So

$$\begin{aligned} \sum_{A \text{ of type I}} 2^{-|N(A)|+|B(A)|} &\leq \sum_{i=2}^{d/2} 2^{d+O(i\log d)-di+2i(i-1)} \\ &= e^{-\Omega(d)}. \end{aligned} \tag{5.28}$$

We do something similar if A is of type II. Here Lemma 2.6.2 gives $|N(A)| \geq \Omega(d)|A|$ and $|B(A)| \leq O(1/d)|A|$ (recalling that $N(B) \subseteq A$), and so

$$\begin{aligned} \sum_{A \text{ of type II}} 2^{-|N(A)|+|B(A)|} &\leq \sum_{i=d/2}^{d^2} 2^{d+O(i\log d)-\Omega(d)i+O(1/d)i} \\ &= e^{-\Omega(d)}. \end{aligned} \tag{5.29}$$

We partition the set of A's of type III according to the sizes of $[A]$, $N(A), B(A)$ and $H(A)(= N(B(A)))$ and use Lemma 5.5.1 to bound the sizes of the partition classes. In this case we have $|N(A)| \geq d^2$. So (summing only over those values of a, g, b and h for which $\mathcal{H}(a,g,b,h) \neq \emptyset$ and $g \geq d^2$, and with the inequalities justified below)

$$\begin{aligned} \sum_{A \text{ of type III}} 2^{-|N(A)|+|B(A)|} &= \sum_{a,g,b,h} |\mathcal{H}(a,g,b,h)| 2^{-g+b} \\ &\leq M \sum_{a,g,b,h} 2^{-\Omega(g/\log d)} && (5.30) \\ &< M^4 \sum_{g \geq d^2} 2^{-\Omega(g/\log d)} && (5.31) \\ &\leq \left(M^4/(1-2^{-\Omega(1/\log d)})^2\right) 2^{-\Omega(d^2/\log d)} \\ &= e^{-\Omega(d)}. && (5.32) \end{aligned}$$

Here (5.30) is from Lemma 5.5.1 and in (5.31) we use the fact that there are fewer than M choices for each of a, b and h.

Combining (5.28), (5.29) and (5.32) we have Theorem 5.3.2.

5.7 Independent sets in the cube

As mentioned in the introduction, we may quickly derive from Theorem 1.2.4 the asymptotics of the number of independent sets in Q_d established in [20] and [24]. This reflects the fact that our approach is very similar to that of [24]. The main difference is our use of Lemma 2.4.5. The analog of Theorem 5.3.2 in [24] is

$$\sum\left\{2^{-|N(A)|} : A \subseteq \mathcal{E}, A \text{ 2-linked}, |A| > 1, |[A]| < M/2\right\} = e^{-\Omega(d)};$$

Lemma 2.4.5 allows us to replace $M/2$ by $(2-\Omega(1))^d$ here, which considerably simplifies the proof of Lemma 3.3.5.

Write $\mathcal{I} = \mathcal{I}(Q_d)$ for the set of independent sets in Q_d, and for each $a < b \in \mathbf{Z}$ set

$$\mathcal{F}_{[a,b]} = \{f \in \mathcal{F} : R(f) \subseteq \{a, \ldots, b\}\}.$$

There is a bijection from $\mathcal{F}_{[-1,2]} \cup \mathcal{F}_{[0,3]}$ to $\mathcal{I}$ given by

$$f \longrightarrow \begin{cases} f^{-1}(\{-1,2\}) & \text{if } f \in \mathcal{F}_{[-1,2]} \\ f^{-1}(\{0,3\}) & \text{if } f \in \mathcal{F}_{[0,3]} \end{cases}.$$

By inclusion-exclusion and symmetry, we have

$$\begin{aligned} |\mathcal{F}_4| &= |\mathcal{F}_{[-3,0]}| + |\mathcal{F}_{[-2,1]}| + |\mathcal{F}_{[-1,2]}| + |\mathcal{F}_{[0,3]}| \\ &\quad -2(|\mathcal{F}_{[-2,0]}| + |\mathcal{F}_{[-1,1]}| + |\mathcal{F}_{[0,2]}|) + |\mathcal{F}_{[-1,0]}| + |\mathcal{F}_{[0,1]}| \\ &= 2(|\mathcal{F}_{[-1,2]}| + |\mathcal{F}_{[0,3]}|) - 2|\mathcal{F}_{\leq 3}| - 2|\mathcal{F}_{[-1,0]}| \\ &= 2|\mathcal{I}| - 2|\mathcal{F}_{\leq 3}| - 2 \end{aligned}$$

and so by Theorem 1.2.4

$$\left||\mathcal{I}| - 2\sqrt{e}2^M\right| = e^{-\Omega(d)}2^M.$$

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