Math 60610, Spring 2009

Final Examination

Due May 7, 2009 at 4pm

Instructions: Each problem should be started on a separate page. Write on one side only. Put your answers together in numerical order, add a cover page with your name and the information that this is the Math 60610 Final Examination for Spring 2009, and staple it all together.

Initially you should think about the questions on your own. After a decent period of thought, collaboration with colleagues is acceptable, but the final write-up of each question should be your own. If you have used ideas from your colleagues, please give them credit. If you use other resources (textbooks, published papers) please cite those as well.

Two questions (3) and 5)) are marked with a +, indicating that I think they might be harder than the others. You should attempt at least one of these, but not necessarily both.

If you would like hints, ask me.

1. Give a combinatorial proof of each of the following identities (that is, show that both sides are counting the same thing).

   (a) $\sum_{k=1}^{n} k.k! = (n+1)! - 1$

   (b) $\sum_{A \subseteq [n],\ B \subseteq [n]} |A \cap B| = n4^{n-1}$

2. The *matching polynomial* of a graph G is the polynomial

$$P(x, G) = \sum_{t \geq 0} m_t(G) x^t$$

   where $m_t(G)$ is the number of matchings in G with t edges (so, for example, $P(x, K_4) = 1 + 6x + 3x^2$ — note that every graph has 1 matching with 0 edges).

   (a) Find a recurrence (with initial conditions) that expresses $P(x, K_n)$ in terms of $P(x, K_{n'})$'s for $n' < n$.

   (b) Use the recurrence to prove that the polynomial $P(x, K_n) = 0$ has only real negative roots.

3. + Put a weight $0 < \lambda_e < 1$ on each edge e of K_n (the weights for different edges may be different). Let $\Lambda = \{\lambda_e : e \in E\}$ be the set of weights. For a matching M of K_n, define the weight of the matching to be $w^\Lambda(M) = \prod_{e \in M} \lambda_e$, and set

$$m_t^\Lambda(n) = \sum_{M \text{ a matching of size } t \text{ in } K_n} w^\Lambda(M).$$

Define the Λ-*weighted matching polynomial* of K_n to be

$$P^\Lambda(x, n) = \sum_{t \geq 0} m_t^\Lambda(n) x^t.$$

(Notice that if $\lambda_e = 1$ for all e, then $w^\Lambda(M) = 1$ for all M and so $m_t^\Lambda(n) = m_t(K_n)$ and $P^\Lambda(x, n) = P(x, K_n)$).

(a) Prove (by induction on n, and using Question 2 as a guide) that for all n and all weight sets Λ, the roots of the polynomial $P^\Lambda(x, n) = 0$ are all real, negative and distinct.

(b) Deduce that for *any* graph G, the roots of $P(x, G) = 0$ are all real. (This wonderful result was originally proved by the physicists Heilman and Lieb.)

4. (a) Show that if G is a graph with m edges, then $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

(b) Give an example where this bound is tight, and an example where it is very far from tight.

(c) + Does the same bound hold for the list-chromatic number $\chi_\ell(G)$? (The + here indicates that I don't at the moment know the answer to this.)

5. + Show that there are constants $c_1, c_2 > 0$ such that for all n

$$c_1 n^{\frac{4}{3}} \leq \mathrm{ex}(n, C_4) \leq c_1 n^{\frac{3}{2}}.$$