Problem Solving in Math (Math 43900)

Fall 2019

Instructor: David Galvin

The 2019 William Lowell Putnam Mathematical Competition will take place on Saturday, December 7, 2019. Every year, around 4500 US & Canadian undergraduates from around 550 institutions participate in the competition, which takes place on-campus.

The competition consists of two three-hour sessions (morning and afternoon), with each session having six problems. The problems are hard, not because they are made up of lots of parts, or involve extensive computation, or require very advanced mathematics to solve. They are hard because they each require a moment of cleverness, intuition and ingenuity to reach a solution. Typically, the median score out of 120 (10 possible points per question) is 1! The Putnam Competition may be the most challenging and rewarding tests of mathematical skill that you will ever encounter. See `https://www.maa.org/math-competitions/putnam-competition` for more information, including information about prizes and recognition for high performers.

To help prepare students for the Putnam Competition, the math department runs the 1 credit course Math 43900, which meets Tuesdays, 3.30pm-4.20pm. Each meeting will (usually) be built around a specific theme (pigeon-hole principle, induction & recursion, inequalities, probability, etc.). We'll talk about the general theme, then spend time trying to solve some relevant problems. At the end of each meeting I'll hand out a set of problems on that theme, that you can cut your teeth on. Usually we'll begin the next session with presentations of solutions to some of those problems. On occasional meetings, I might give out a problem set at the beginning, and have everyone pick a problem or two to work on individually for the meeting period (a sort of "mock Putnam").

The grade for the class will be determined solely by active participation in class (participating in class discussions, occasionally presenting problem solutions on the board) and by participation in the 2019 Putnam Competition.

Those who want to get the most out of the Putnam Competition are also encouraged to take part in the Virginia Tech Regional Mathematics Contest (also on-campus) which happens six weeks or so before the Putnam. More information on this competition will be available in early September.

Contents

1 Math 43900 organizational information

To help prepare for the Putnam Competition, the math department runs the 1 credit course Math 43900. If you are signed up for Math 43900, then you are also signed up to participate in the Putnam competition on December 7. One can also participate in the Putnam competition without participating in this course. This year, the details of the course are as follows:

- **Instructor**: David Galvin, 136 Hayes-Healy, `dgalvin1@nd.edu`.
- **Meetings**: Tuesdays, 3.30pm-4.20pm, Hayes-Healy 229, starting August 27, ending December 10 (the Tuesday after the competition).
- **Office hours**: Email me for an appointment.
- **Text**: There is no required text. The following books (one available online through the library, the others on reserve at the Math library) are worth looking at. First, two classics that deal with the art of problem-solving:
  - *Problem-solving through problems* by Larson (QA 43 .L37 1983)
  - *How to solve it* by Pólya (QA 11 .P6 2004).

  Next, a doorstop filled with problems and strategies for the Putnam itself:
  - *Putnam and beyond* by Gelca and Andreescu (online access).

  Finally, three books that exhaustively catalog all Putnam Competitions up to 2000:
  - *The William Lowell Putnam Mathematical Competition 1985-2000: problems, solutions, and commentary* by Kedlaya, Poonen and Vakil (QA 43 .W5425 2002)
  - *The William Lowell Putnam Mathematical Competition problems and solutions: 1938-1964* by Gleason, Greenwood and Kelly (QA 43 .W54)
  - *The William Lowell Putnam Mathematical Competition problems and solutions: 1965-1984* by Alexanderson, Klosinski and Larson (QA 43 .W542 1985).
- **Course website**: `http://www3.nd.edu/~dgalvin1/43900/43900_F19/index.html`. This is where announcements, problem sets, etc., will be posted. Also, I've put here a link to an online archive of the problems & solutions for the Putnam Competitions from 1995 on. (NB — when following this link straight from a pdf file of the course notes, the tilde in front of `dgalvin1` sometimes causes a problem; if so just enter it by hand.)
- **Course organization**: Each meeting will (usually) be built around a specific theme (pigeon-hole principle, induction & recursion, inequalities, probability, etc.). We'll talk about the general theme, then spend time trying to solve some relevant problems. At the end of each meeting I'll hand out a set of problems on that theme, that you can cut your teeth on. Usually we'll begin the next session with presentations of solutions to some of those problems. On occasional meetings, I might give out a problem set at the beginning, and have everyone pick a problem or two to work on individually for the meeting period (a sort of "mock Putnam").

- **Grading and homework**: The grade for the class will be determined solely by active participation in class (participating in class discussions, occasionally presenting problem solutions on the board) and by participation in the 2019 Putnam Competition. I'll give out a problem set at the end of each meeting, usually with many problems; I don't expect anyone to work on all the problems, but I do expect *everyone* to work on at least *some* of the problems each week, and to be prepared to talk about their progress at the beginning of the next meeting.

- **Honor code**: It's perfectly fine to collaborate with your colleagues on problems (although I do encourage you to try many problems on your own, since the Putnam Competition itself is an individual competition). You have all taken the Honor Code pledge, to not participate in or tolerate academic dishonesty. For this course, that means that if you use a source to help you solve a problem (such as discussion with a colleague, or consulting a book or online resource), you should acknowledge that in your write-up or oral presentation.

2 Week one (August 27) — A grab-bag

Most weeks' handouts will be a themed collection of problems — all involving inequalities of one form or another, for example, or all involving linear algebra. The Putnam Competition itself has no such theme, and so every so often we'll have a handout that includes the realistic challenge of figuring out what (possibly different) approach (or approaches) to take for each problem. This introductory problem set is one such.

Look over the problems, pick out some that you feel good about, and tackle them! You'll do best if you engage your conscious brain fully on a single problem, rather than hopping back-and-forth between problems every few minutes (but it's also a good idea to read all the problems before tackling one, to allow your subconscious brain to mull over the whole set).

Remember that problem solving is a full-contact sport: throw everything you know at the problem you are tackling! Sometimes, the solution can come from an unexpected quarter.

The point of Math 43900 is to introduce you to (or reacquaint you with) a variety of tricks and tools that tend to be frequently useful in the solving of competition puzzles; but even before that process starts, there are lots of common-sense things that you can do to make problem-solving fun, productive and rewarding. Whole books are devoted to these strategies (such as Larson's *Problem solving through problems* and Pólya's *How to solve it*).

Without writing a book on the subject, here (adapted from a list by Ravi Vakil) are some slogans to keep in mind when solving problems:

- Try small cases!
- Plug in small numbers!
- Do examples!
- Look for patterns!
- Draw pictures!
- Write lots!
- Talk it out!
- Choose good notation!
- Look for symmetry!
- Break into cases!
- Work backwards!
- Argue by contradiction!
- Consider extreme cases!
- Modify the problem!
- Make a generalization!

- Don't give up after five minutes!
- Don't be afraid of a little algebra!
- Take a break!
- Sleep on it!
- Ask questions!

And above all:

- Enjoy!

We'll elaborate on these slogans as the semester progresses.

2.1 Week 1 problems

1. Let
$$f(n)=\sum_{k=1}^{n}\frac{1}{\sqrt{k}+\sqrt{k+1}}.$$
Evaluate $f(9999)$.

2. Take a walk on the number line, starting at 0, in the following way: start by taking a step of length one, either right or left; then take a step of length two, either right or left, and so on — meaning that in general, the kth you move, it's to take a step of length k, either right or left.

   Show that for each integer m, there is a walk that visits (has a step ending at) m.

3. A locker room has 100 lockers, numbered 1 to 100, all closed. I run through the locker room, and open every locker. Then I run through the room again, and close the lockers numbered 2, 4, 6, etc. (all the even numbered lockers). Next I run through the room, and change the status of the lockers numbered 3, 6, 9, etc. (opening the closed ones, and closing the open ones). I keep going in this manner (on the ith run through the room, I change the status of lockers numbered i, $2i$, $3i$, etc.), until on my 100th run through the room I change the status of locker number 100 only. At the end of all this, which lockers are open?

4. (Maybe harder?) Find all ordered pairs (a, b) of positive integers for which
$$\frac{1}{a}+\frac{1}{b}=\frac{3}{2018}.$$

5. (Definitely harder!) Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that
$$a_n < \frac{a_0+a_1+a_2+\ldots+a_n}{n} \leq a_{n+1}.$$

2.2 Week 1 solutions

1. Let

$$f(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k} + \sqrt{k+1}}.$$

   Evaluate $f(9999)$.

   Solution: This problem was on the 2014 U. Illinois Urbana-Champaign mock Putnam.

   Rationalize!

$$\begin{aligned} f(n) &= \sum_{k=1}^{n} \frac{1}{\sqrt{k} + \sqrt{k+1}} \\ &= \sum_{k=1}^{n} \frac{\sqrt{k} - \sqrt{k+1}}{(\sqrt{k} + \sqrt{k+1})(\sqrt{k} - \sqrt{k+1})} \\ &= \sum_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) \\ &= \sqrt{n+1} - 1, \end{aligned}$$

   the last line by a telescoping sum. So $f(9999) = \sqrt{10000} - 1 = 99$.

2. Take a walk on the number line, starting at 0, in the following way: start by taking a step of length one, either right or left; then take a step of length two, either right or left, and so on — meaning that in general, the kth you move, it's to take a step of length k, either right or left.

   Show that for each integer m, there is a walk that visits (has a step ending at) m.

   Solution: I found this problem in *A Mathematical Orchard, Problems and Solutions* by Krusemeyer, Gilbert and Larson.

   We can easily get to the number $m = n(n+1)/2$, for any integer n: just do $1+2+3+\ldots+n$. What about a number of the form $n(n+1)/2 - k$, where $1 \leq k \leq n-1$? For k even we can get to this by flipping the $+$ to a $-$ in front of $k/2$. For k odd we can get to this by first adding $n+1$ to the end, then subtracting $n-2$ (net effect: -1), then flipping the $+$ to a $-$ in front of $(k-1)/2$.

   Thus we can reach every number in the interval $[n(n+1)/2-(n-1), n(n+1)/2]$. Using $1+2+3+\ldots+n = n(n+1)/2$ we can easily check that the union of these disjoint intervals, as n increases from 1, covers the positive integers. To get to a negative integer m, just reverse the signs on any scheme that gets to $-m$.

   Another, much simpler way to get to $n > 0$: do $n = (-1+2)+(-3+4)+\ldots+(-(2n-1)+2n)$.

3. A locker room has 100 lockers, numbered 1 to 100, all closed. I run through the locker room, and open every locker. Then I run through the room again, and close the lockers

numbered 2, 4, 6, etc. (all the even numbered lockers). Next I run through the room, and change the status of the lockers numbered 3, 6, 9, etc. (opening the closed ones, and closing the open ones). I keep going in this manner (on the ith run through the room, I change the status of lockers numbered i, $2i$, $3i$, etc.), until on my 100th run through the room I change the status of locker number 100 only. At the end of all this, which lockers are open?

Solution: I learned of this problem from Imre Leader, but I think it has been around for a very long time.

The open lockers in the end are those numbered 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100.

Locker n has its status changed once for each positive divisor of n, and so it is open in the end exactly if n has an odd number of positive divisors. If n has prime factorization $p_1^{a_1} \dots p_k^{a_k}$, then the number of positive divisors is $(a_1+1)\dots(a_k+1)$ (a positive divisor takes the form $p_1^{b_1} \dots p_k^{b_k}$ with $0 \le b_i \le a_i$ for each i; so there are a_i+1 choices for the value of b_i, with choices for different i's being independent). The product $(a_1+1)\dots(a_k+1)$ is odd only if each a_i+1 is odd, so only if each a_i is even. So Locker n has its status changed exactly when all the exponents in the prime factorization of n are even; in other words, exactly when n is a perfect square. So the open lockers are numbered 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100.

4. (Maybe harder?) Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Solution: This is Problem A1 of the 2018 Putnam competition.

The given equation is equivalent to

$$2018(a+b) = 3ab.$$

It possible, but not so pleasant, to work with this equation. It's much easy to work with the equivalent equation

$$(3a-2018)(3b-2018) = 2018^2.$$

Each of the factors on the left is congruent to 1 (mod 3). There are 6 positive factors of $2018^2 = 2^2 \cdot 1009^2$ that are congruent to 1 (mod 3): 1, 2^2, 1009, $2^2 \cdot 1009$, 1009^2, $2^2 \cdot 1009^2$. These lead to the 6 possible pairs: $(a, b) = (673, 1358114)$, $(674, 340033)$, $(1009, 2018)$, $(2018, 1009)$, $(340033, 674)$, and $(1358114, 673)$.

As for negative factors, the ones that are congruent to 1 (mod 3) are $-2, -2 \cdot 1009, -2 \cdot 1009^2$. However, all of these lead to pairs where $a \le 0$ or $b \le 0$.

5. (Definitely harder!) Let $a_0 < a_1 < a_2 \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \dots + a_n}{n} \le a_{n+1}.$$

Solution: This was Question 1 at the 2014 International Mathematical Olympiad. For an extensive discussion of this problem, with lots of different solutions, see the weblog of Fields medalist Tim Gowers, `http://gowers.wordpress.com/2014/07/19/mini-monomath/`).

For convenience of typography set

$$f(n) := \frac{a_0 + a_1 + a_2 + \ldots + a_n}{n}.$$

We begin by showing that there must be at least one $n \geq 1$ such that

$$a_n < f(n) \leq a_{n+1}$$

holds. Indeed, suppose not; then for each $n \geq 1$, we have either

$$f(n) \leq a_n$$

or

$$a_{n+1} < f(n)$$

(both note not both, since $a_n < a_{n+1}$).

We prove, by induction on n, that it's the second of these that must occur, i.e., that

$$a_{n+1} < f(n).$$

The base case $n = 1$ is clear: we cannot have $f(1) \leq a_1$, since this is the same as $a_0 + a_1 \leq a_1$, and we known that $a_0 > 0$; so we must have $a_2 < f(1)$. [Recall that we have made the assume that there is no $n \geq 1$ for which $a_n < f(n) \leq a_{n+1}$.] For $n > 1$, the inductive hypothesis tells us that

$$a_n < f(n-1).$$

Recalling the definition of $f(n-1)$, multiplying across by $n-1$, adding a_n to both sides, and dividing by n, this is the same as

$$a_n < f(n);$$

so we can't have $f(n) \leq a_n$, and so must have $a_{n+1} < f(n)$. This completes the induction.

Summary so far: under the assumption that there is no $n \geq 1$ for which $a_n < f(n) \leq a_{n+1}$, we have concluded that $a_{n+1} \leq f(n)$ for all $n \geq 1$.

We now claim that it follows from this ("this" being "$a_{n+1} \leq f(n)$ for all $n \geq 1$") that

$$a_{n+1} < a_0 + a_1$$

for all $n \geq 1$. Again, we proceed by (this time strong) induction on n, with $n = 1$ immediate from $a_2 < f(1)$. For $n > 1$ we have

$$\begin{aligned}
a_{n+1} &< f(n) \\
&= \frac{a_0 + a_1 + a_2 + \ldots + a_n}{n} \\
&< \frac{a_0 + a_1 + (a_0 + a_1) + \ldots + (a_0 + a_1)}{n} \\
&= \frac{n(a_0 + a_1)}{n} \\
&= a_0 + a_1,
\end{aligned}$$

where in the second inequality we have used the induction hypothesis to bound $a_k < a_0 + a_1$ for each of $k = 2, \ldots, n-1$. This completes the induction.

Summary so far: under the assumption that there is no $n \geq 1$ for which $a_n < f(n) \leq a_{n+1}$, we have concluded that $a_{n+1} \leq a_0 + a_1$ for all $n \geq 1$. But this is a contradiction: since the a_i's are integers and increasing, there must be some $n \geq 1$ for which $a_{n+1} > a_0 + a_1$.

Interim conclusion: there is at least one $n \geq 1$ for which $a_n < f(n) \leq a_{n+1}$.

Now we need to show that there is a unique such n. To do this, suppose that n is such that $a_n < f(n) \leq a_{n+1}$, and that $m > n$. We will prove (by induction on m) that $f(m) \leq a_m$. If we can do this, we will have shown that there can be at most one n such that $a_n < f(n) \leq a_{n+1}$, and combined with our interim conclusion, we would have a unique such n.

The base case for the induction is $m = n+1$; $f(n+1) \leq a_{n+1}$ is just a simple rearrangement of $f(n) \leq a_{n+1}$. We now move onto the induction step. For $m > n+1$, we get to assume $f(m-1) \leq a_{m-1}$. Combining this with $a_{m-1} < a_m$, we get $f(m-1) \leq a_m$, and a simple rearrangement of this leads to $f(m) \leq a_m$, completing the induction.

3 Week two (September 3) — Induction

Often one can tackle a problem that involves one or more parameters by checking what happens with small values of the parameters, noticing a pattern, and then establishing that the pattern holds in general. The most powerful mathematical technique for establishing the correctness of a pattern is *induction*.

Basic induction

Suppose that $P(n)$ is an assertion about the natural number n. *Induction* is essentially the following: if there is some a for which $P(a)$ is true, and if for all $n \geq a$ we have that the truth of $P(n)$ implies the truth of $P(n+1)$, then we can conclude that $P(n)$ is true for all $n \geq a$. This should be fairly obvious: knowing $P(a)$ and the implication "$P(n) \implies P(n+1)$" at $n = a$, we immediately deduce $P(a+1)$. But now knowing $P(a+1)$, the implication "$P(n) \implies P(n+1)$" at $n = a+1$ allows us to deduce $P(a+2)$; and so on. Induction is the mathematical tool that makes the "and so on" above rigourous. Induction works because of the following fundamental fact, often referred to as the *well-ordering principle*:

A non-empty subset of the natural numbers must have a least element.

To see why well-ordering allows induction work, suppose that we know that $P(a)$ is true for some a, and that we can argue that for all $n \geq a$, the truth of $P(n)$ implies the truth of $P(n+1)$. Suppose now (for a contradiction) that there are some $n \geq a$ for which $P(n)$ is **not** true. Let $F = \{n | n \geq a,\ P(n) \text{ not true}\}$. By assumption F is non-empty, so has a least element, n_0 say. We know $n_0 \neq a$, since $P(a)$ **is** true; so $n_0 \geq a+1$. That means that $n_0 - 1 \geq a$, and since $n_0 - 1 \notin F$ (if it was, n_0 would not be the least element) we know $P(n_0 - 1)$ **is** true. But then, by assumption, $P((n_0-1)+1) = P(n_0)$ **is** true, a contradiction!

Example: Prove that a set of size $n \geq 1$ has 2^n subsets (including the empty set and the set itself).

Solution: Let $P(n)$ be the statement "a set of size n has 2^n subsets". We prove that $P(n)$ is true for all $n \geq 1$ by induction. We first establish a *base case*. When $n = 1$, the generic set under consideration is $\{x\}$, which has $2 = 2^1$ subsets ($\{x\}$ and $\emptyset$); so $P(1)$ is true.

Next we establish the *inductive step*. Suppose that for some $n \geq 1$, $P(n)$ is true. Consider $P(n+1)$. The generic set under consideration now is $\{x_1, \ldots, x_n, x_{n+1}\}$. We can construct a subset of $\{x_1, \ldots, x_n, x_{n+1}\}$ by first forming a subset of $\{x_1, \ldots, x_n\}$, and then either adding the element x_{n+1} to this subset, or not. This tells us that the number of subsets of $\{x_1, \ldots, x_n, x_{n+1}\}$ is 2 times the number of subsets of $\{x_1, \ldots, x_n, \}$. Since $P(n)$ is assumed true, we know that $\{x_1, \ldots, x_n, \}$ has 2^n subsets (this step is usually referred to as applying the *inductive hypothesis*); so $\{x_1, \ldots, x_n, x_{n+1}\}$ has $2 \times 2^n = 2^{n+1}$ subsets. This shows that the truth of $P(n)$ implies that of $P(n+1)$, and the proof by induction is complete.

Strong Induction

Induction is a great tool because it gives you somewhere to start from in an argument. And sometimes, the more you start with, the further you'll go. That's why the principle of *Strong*

Induction is worth keeping in mind: if there is some a for which $P(a)$ is true, and if for each $n > a$ we have that the truth of $P(m)$ for all m, $a \leq m < n$, implies the truth of $P(n)$, then we can conclude that $P(n)$ is true for all $n \geq a$.

The proof that this works is almost the same as the proof that induction works. What's good about strong induction is that when you are at the part of the argument where you have to show that the truth of $P(n+1)$ from some assumptions about earlier assertions, you now have a lot more to work with: each of $P(a), P(a+1), \ldots, P(n-1)$, rather than just $P(n)$ alone. Sometimes this is helpful, and sometimes it's absolutely necessary.

Example: Prove that every integer $n \geq 2$ can be written as $n = p_1 \ldots p_\ell$ where the p_i's are (not necessarily distinct) prime numbers.

Solution: Let $P(n)$ be the statement: "n can be written as $n = p_1 \ldots p_\ell$ where the p_i's are (not necessarily distinct) prime numbers". We'll prove that $P(n)$ is true for all $n \geq 2$ by strong induction.

$P(2)$ is true, since $2 = 2$ works.

Now consider $P(n)$ for some $n > 2$. We want to show how the (simultaneous) truth of $P(2), \ldots, P(n-1)$ implies the truth of $P(n)$. If n is prime, then $n = n$ works to show that $P(n)$ holds. If n is not a prime, then its composite, so $n = ab$ for some numbers a, b with $2 \leq a < n$ and $2 \leq b < n$. We're allowed to assume that $P(a)$ and $P(b)$ are true, that is, that $a = p_1 \ldots p_\ell$ where the p_i's are (not necessarily distinct) prime numbers, and that $a = q_1 \ldots q_m$ where the q_i's are (not necessarily distinct) prime numbers. It follows that

$$n = ab = p_1 \ldots p_\ell q_1 \ldots q_m.$$

This is a product of (not necessarily distinct) prime numbers, and so $P(n)$ is true.

So, by strong induction, we conclude that $P(n)$ is true for all $n \geq 2$.

Notice that we would have gotten exactly nowhere with this argument if, in trying to prove $P(n)$, all we had been allowed to assume was $P(n-1)$.

Recurrences

Sometimes we are either given a sequence of numbers via a recurrence relation, or we can argue that there is such relation that governs the evolution of the sequence. A sequence $(b_n)_{n \geq a}$ is defined via a *recurrence relation* if some initial values, $b_a, b_{a+1}, \ldots, b_k$ say, are given, and then a rule is given that allows, for each $n > k$, b_n to be computed as long as we know the values $b_a, b_{a+1}, \ldots, b_{n-1}$.

Sequences defined by a recurrence relation, and proofs by induction, go hand-in-glove. We may have more to say about recurrence relations later in the semester, but for now, we'll confine ourselves to an illustrative example.

Example: Let a_n be the number of different ways of covering a 1 by n strip with 1 by 1 and 1 by 3 tiles. Prove that $a_n < (1.5)^n$.

Solution: We start by figuring out how to calculate a_n via a recurrence. Some initial values of a_n are easy to compute: for example, $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. For $n \geq 4$, we can tile the 1 by n strip EITHER by first tiling the initial 1 by 1 strip with a 1 by 1 tile, and then finishing by tiling the remaining 1 by $n-1$ strip in any of the a_{n-1} admissible ways; OR by first tiling the initial 1 by 3 strip with a 1 by 3 tile, and then finishing by tiling the remaining 1 by $n-3$ strip in any of the a_{n-3} admissible ways. It follows that for $n \geq 4$ we have $a_n = a_{n-1} + a_{n-3}$. So a_n (for $n \geq 1$) is determined by the recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3, \text{ and} \\ a_{n-1} + a_{n-3} & \text{if } n \geq 4. \end{cases}$$

Notice that this gives us enough information to calculate a_n for all $n \geq 1$: for example, $a_4 = a_3 + a_1 = 3$, $a_5 = a_4 + a_2 = 4$, and $a_6 = a_5 + a_3 = 6$.

Now we prove, by strong induction, that $a_n < 1.5^n$. That $a_1 = 1 < 1.5^1$, $a_2 = 1 < (1.5)^2$ and $a_3 = 2 < (1.5)^3$ is obvious. For $n \geq 4$, we have

$$\begin{aligned} a_n &= a_{n-1} + a_{n-3} \\ &< (1.5)^{n-1} + (1.5)^{n-3} \\ &= (1.5)^n \left(\frac{2}{3} + \left(\frac{2}{3} \right)^3 \right) \\ &= (1.5)^n \left(\frac{26}{27} \right) \\ &< (1.5)^n, \end{aligned}$$

(the second line using the inductive hypothesis) and we are done by induction.

Notice that we really needed strong induction here, and we really needed all three of the base cases $n = 1, 2, 3$ (think about what would happen if we tried to use regular induction, or what would happen if we only verified $n = 1$ as a base case); notice also that an induction argument can be written quite concisely, while still being fully correct, without fussing too much about "$P(n)$".

Presenting a proof by induction

A proof by induction (any sort of proof, indeed), should be presented in complete sentences. If you read the proof aloud, giving the mathematical symbols their usual english-language names, it should form a coherent narrative. And remember that for a write-up of the solution to a problem, the goal is not to convince yourself that you have solved the problem; it is to convince someone else, who doesn't get to ask you for clarification as they read your solution.

Here is a template for the presentation of an induction proof.

Claim 3.1 *For every natural number n,*

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Proof: We proceed by induction on n.[1]

[1] Here you might chose to say specifically that you are proving the predicate $P(n)$: "$1+2+3+\ldots+n = \frac{n(n+1)}{2}$" for $n \in \mathbb{N}$; this is usually not necessary for proving a simple statement, but it can be very useful, when proving a more complex statement, especially one involving multiple variables, to introduce explicit notation for the predicate.

Base case: The base case $n = 1$ is obvious.[2]

Induction step: Let n be an arbitrary natural number. Assume

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}{}^{3}.$$

From this we get

$$\begin{aligned}
1 + 2 + 3 + \ldots + n + (n+1) &= (1 + 2 + 3 + \ldots + n) + (n+1) \\
&= \frac{n(n+1)}{2} + n + 1 \\
&= \frac{n^2 + n + 2n + 2}{2} \\
&= \frac{n^2 + 3n + 2}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}.
\end{aligned}$$

The equality of the first and last expressions in this chain is the case $n + 1$ of the assertion[4], so we have verified the induction step.

By induction the assertion is true for all n. $\square$

In proving an identity — an equality between two expressions, both depending on some variable(s) — by induction, it is often very helpful to start with one side of the $n + 1$ case of the identity, and manipulate it via a sequence of equalities in a way that introduces one side of the n case of the identity into the mix; this can then be replaced with the *other* side of the n case, and then the whole thing might be massage-able into the other side of the $n + 1$ identity. That's exactly how we proceeded above.

Nested induction

Note: The last few examples in this section are taken from notes written by Amites Sarkar (and the last example is copied almost verbatim from his notes).

Sometimes the induction step requires more than just simple algebra; it may itself require an application of induction!

Example: Show that for all $n \geq 1$, the number $2 \times 7^n + 3 \times 5^n - 5$ is divisible by 24.

Solution: We proceed by (an *outer*) induction on n, with the base case $n = 1$ straightforward.

For the induction step, suppose that 24 divides $2 \times 7^n + 3 \times 5^n - 5$, and consider the number

$$2 \times 7^{n+1} + 3 \times 5^{n+1} - 5.$$

[2] It's ok to say this, if the base case really is obvious!

[3] Or, if you have named the predicate $P(n)$, "assume $P(n)$".

[4] Or, "is $P(n + 1)$

We wish to show that this number is divisible by 24. Now

$$\begin{aligned} 2 \times 7^{n+1} + 3 \times 5^{n+1} - 5 &= 14 \times 7^n + 15 \times 5^n - 5 \\ &= 5\,(2 \times 7^n + 3 \times 5^n - 5) + 4 \times 7^n + 20. \end{aligned}$$

By induction we know that 24 divides $5\,(2 \times 7^n + 3 \times 5^n - 5)$, show we need to show that also 24 divides $4 \times 7^n + 20$, which is the same as showing that 6 divides $7^n + 5$. We prove *this* by (an *inner*) induction on n, with the base case $n = 1$ easy. For the induction step, note that

$$7^{n+1} + 5 = 7(7^n + 5) - 30.$$

By the (inner) induction hypothesis, 6 divides $7(7^n + 5)$, and since also 6 divides 30 we get that 6 divides $7^{n+1} + 5$, completing the (inner) induction, and so also completing the (outer) induction.

In this example we had an induction within an induction, which of course is perfectly OK — the "inner" induction makes no reference to the "outer" one.

Induction beyond $\mathbb{N}$

Induction can work to prove families of statements indexed by sets other than $\mathbb{N}$, as long as the indexing set has the well-ordering property (non-empty subsets always have least elements). Rather than giving a general statement, we present an illustrative example.

Example: Suppose we have a function f defined on pairs of positive integers, given by $f(1,1) = 2$ and

$$\begin{aligned} f(m+1,n) &= f(m,n) + 2(m+n) \\ f(m,n+1) &= f(m,n) + 2(m+n-1) \end{aligned}$$

for all pairs (m,n) of positive integers other than $(1,1)$.

It's not actually clear that this is a well defined function! For example, we could try to calculate

$$f(2,2) = f(1{+}1,2) = f(1,2){+}2(1{+}2) = f(1,1{+}1){+}6 = f(1,1){+}2(1{+}1{-}1){+}6 = 2{+}2{+}6 = 10$$

or

$$f(2,2) = f(2,1{+}1) = f(2,1){+}2(2{+}1{-}1) = f(1{+}1,1){+}4 = f(1,1){+}2(1{+}1){+}4 = 2{+}4{+}4 = 10;$$

but was it just a coincidence that these two paths from $(2,2)$ to $(1,1)$ actually gave the same answer? If we had defined a function f' by $f'(1,1) = 2$ and

$$\begin{aligned} f'(m+1,n) &= f'(m,n) + 5(m+n) \\ f'(m,n+1) &= f'(m,n) + 2(m+n-1) \end{aligned}$$

for all pairs (m,n) of positive integers other than $(1,1)$, then we would immediately have run into trouble: viewing $f'(2,2)$ as $f'(1+1,2)$, we get

$$f'(2,2) = f'(1{+}1,2) = f'(1,2){+}5(1{+}2) = f'(1,1{+}1){+}15 = f'(1,1){+}2(1{+}1{-}1){+}15 = 2{+}2{+}15 = 19$$

while viewing $f'(2,2)$ as $f'(2,1+1)$, we get

$$f'(2,2) = f'(2,1+1) = f'(2,1)+2(2+1-1) = f'(1+1,1)+4 = f'(1,1)+5(1+1)+4 = 2+10+4 = 16.$$

So this slightly modified function f' is not, in fact, well defined.

Nevertheless, a lot of experimentation suggests that no matter which path you take from (m,n) to $(1,1)$, the recurrence always yields the same answer for $f(m,n)$, and in fact yields that answer

$$f(m,n) = (m+n)^2 - (m+n) - 2n + 2.$$

How might we prove this by induction?

One approach is first to fix $n = 1$. There is only one way to go from $(m,1)$ to $(1,1)$ (keep using the first clause of the recurrence), so clearly $f(m,1)$ is well defined. We can prove, by induction on m, that

$$f(m,1) = (m+1)^2 - (m+1) - 2$$

(implicit in this statement is that $f(m,1)$ is well-defined).

Now we can try to prove the following statement, by induction on n:

$$P(n) : \text{for all } m,\ f(m,n) = (m+n)^2 - (m+n) - 2n + 2.$$

The base case $n = 1$ has already been dealt with. For the induction step, we get to assume that for some $n \geq 1$, $f(m,n)$ is know to be well-defined and equal to $(m+n)^2-(m+n)-2n+2$ for all m (or, if we were doing strong induction, $f(m,k)$ is know to be well-defined and equal to $(m+k)^2 - (m+k) - 2k + 2$ for all m and all $k \leq n$). We then try to establish that $f(m,n+1)$ is well defined and equal to $(m+(n+1))^2-(m+(n+1))-2(n+1)+2$ for all m.

This we do — by induction on m. The base case $m = 1$ is easy (and unambiguous), because there is only one thing we can do, namely apply the second clause in the definition of f. For $m > 1$, we either

- apply the first clause of the definition, in which case we quickly get an expression for $f(m,n+1)$ because (by this second induction) $f(m-1,n+1)$ is know to be well-defined, and its value is known, or

- apply the second clause of the definition, in which case we quickly get an expression for $f(m,n+1)$ because (by the first induction) $f(m,n-1)$ is know to be well-defined, and its value is known.

As long as these two values for $f(m,n+1)$ agree, and agree with $(m+(n+1))^2 - (m+(n+1)) - 2(n+1) + 2$, the induction step works.

(Notice that I'm leaving the algebraic details to the reader!)

Strengthening the induction hypothesis

If you try to prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

using induction on n, you run into trouble — it just doesn't work. (Try it and see). But if we you try to prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

again using induction on n, this time it *does* work (again, try it and see). What's going on? Why is it easier to prove a stronger statement? The answer lies in the fact that, in the inductive step, while the conclusion has gotten stronger, *so has the induction hypothesis*. We have more to prove, but also more to prove it with. The stronger statement is better matched to the inductive step.

Many famous theorems in mathematics have been proven by finding an appropriate strengthening, and proving the stronger statement by induction. In the example above, the strengthening was not obvious. Now here is an example where the strengthening comes more naturally.

With F_n denoting the nth Fibonacci number (so that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, suppose we have to prove that

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

for $n \geq 1$. Let's call that statement $P(n)$. Assuming it, and trying to prove $P(n+1)$, we end up needing to show that

$$2F_nF_{n+1} + F_{n+1}^2 = F_{2n+2}$$

again for $n \geq 1$. Let's call *that* statement $Q(n)$. In trying to prove $Q(n)$ by induction, we end up needing to show that

$$F_{n+1}^2 + F_{n+2}^2 = F_{2n+3}.$$

But this is just $P(n+1)$. Our inner induction loop has broken out and somehow become entangled with the outer induction. Is this a disaster? No! We just have to strengthen the original statement to the statement $R(n)$:

$$F_n^2 + F_{n+1}^2 = F_{2n+1} \quad \text{and} \quad 2F_nF_{n+1} + F_{n+1}^2 = F_{2n+2}.$$

Since $P(n)$ and $Q(n)$ together imply $P(n+1)$, and $Q(n)$ and $P(n+1)$ together imply $Q(n+1)$ (both of these need to be checked by the reader!), if $R(n)$ is the statement that both $P(n)$ and $Q(n)$ are true, then $R(n)$ implies $R(n+1)$. So we are done, once we know that $R(1)$ holds (and it does).

3.1 Week 2 problems

1. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point (so $f(1) = 2$, $f(2) = 4$ and $f(3) = 7$). Find a formula for $f(n)$, and prove that it is correct.

2. Let $x_1, \ldots, x_n$ be n positive numbers satisfying

   $$x_1 + x_2 + \cdots + x_n = \frac{1}{2}.$$

   Prove that

   $$\frac{(1-x_1)}{(1+x_1)} \cdot \frac{(1-x_2)}{(1+x_2)} \cdot \cdots \cdot \frac{(1-x_n)}{(1+x_n)} \geq \frac{1}{3}.$$

3. Define a sequence $(a_n)_{n\geq 1}$ by

$$a_1 = 1, \quad a_{2n} = a_n, \quad \text{and} \quad a_{2n+1} = a_n + 1.$$

Prove that a_n counts the number of 1's in the binary representation of n.

4. At time 0, a particle resides at the point 0 on the real line. Just before 1 second passes, it divides into 2 particles that fly in opposite directions and stop at distance 1 from the original particle. Just before 2 seconds pass, each of these particles again divides into 2 particles flying in opposite directions and stopping at distance 1 from the point of division, and so on, every second. Whenever two particles meet they annihilate each other (leaving nothing behind). How many particles will there be at time 2050?

5. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

6. The numbers 1 through $2n$ are partitioned into two sets A and B of size n, in an arbitrary manner. The elements $a_1, \ldots, a_n$ of A are sorted in increasing order, that is, $a_1 < a_2 < \ldots < a_n$, while the elements $b_1, \ldots, b_n$ of B are sorted in decreasing order, that is, $b_1 > b_2 > \ldots > b_n$. Find (with proof!) the value of the sum

$$\sum_{i=1}^{n} |a_i - b_i|.$$

7. On an infinite sheet of white graph paper (a paper with a square grid), n squares are colored black. At moments $t = 1, 2, \ldots$, squares are recolored according to the following rule: each square gets the color occurring at least twice in the triple formed by this square, its top neighbor, and its right neighbor.

   (a) Prove that after the moment $t = n$, all squares are white.

   (b) Can you find, for infinitely many n, an initial configuration of n squares such that *before* the moment $t = n$ there are still some squares colored black?

8. Players $1, 2, 3, \ldots n$ are seated around a table, and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

9. Show that the sequence

$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7-\sqrt{7}}},\ \sqrt{7-\sqrt{7-\sqrt{7-\sqrt{7}}}}, \ldots$$

converges, and find its limit.

3.2 Week 2 solutions

1. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point (so $f(1) = 2$, $f(2) = 4$ and $f(3) = 7$). Find a formula for $f(n)$, and prove that it is correct.

   Solution: This is a folklore problem.

   We claim that $f(n) = (n^2 + n + 2)/2$.

   To prove this, suppose you have n lines drawn already, so $f(n)$ regions. The $(n + 1)$st line, not being parallel to any other will, will meet all n, and all at different places (since no three lines meet in a point). Without loss of generality we can assume that the $(n + 1)$st line is the x-axis, and along the line we can mark, in order, the n meeting points with other lines, $-\infty < x_1 < x_2 < \ldots < x_n < \infty$. The segment on the $(n + 1)$st line from $-\infty$ to x_1 form the boundary of two regions (above and below it) that were previously one region; so this segment adds one region to the total. Similarly all the other segments add one region. There are $n + 1$ segments in all, so we get the relation

   $$f(n + 1) = f(n) + n + 1 \quad ((\text{for } n \geq 1)), \quad f(1) = 2.$$

   Computing the first few values, it seems clear that $f(n)$ grows quadratically, and that in fact $f(n) = (n^2 + n + 2)/2$. We prove this by induction on n, with $P(n)$ the statement "$f(n) = (n^2 + n + 2)/2$". $P(1)$ asserts "$f(1) = (1^2 + 1 + 2)/2 = 2$, which is true. Suppose $P(n)$ is true for some $n \geq 1$. Let's look at $P(n + 1)$, which is the assertion "$f(n + 1) = ((n + 1)^2 + (n + 1) + 2)/2 = (n^2 + 3n + 4)/2$". Since $P(n)$ is assumed true, we know

   $$f(n) = \frac{n^2 + n + 2}{2}.$$

   We also know $f(n + 1) = f(n) + n + 1$, so

   $$f(n + 1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2},$$

   and so indeed $P(n + 1)$ is true. That $P(n)$ is true for all $n \geq 1$, i.e., that $f(n) = (n^2 + n + 2)/2$ for all $n \geq 1$, has now been proved by induction.

2. Let $x_1, \ldots, x_n$ be n positive numbers satisfying

   $$x_1 + x_2 + \cdots + x_n = \frac{1}{2}.$$

   Prove that

   $$\frac{(1 - x_1)}{(1 + x_1)} \cdot \frac{(1 - x_2)}{(1 + x_2)} \cdot \cdots \cdot \frac{(1 - x_n)}{(1 + x_n)} \geq \frac{1}{3}.$$

   Solution: I found this problem on a Putnam prep class handout prepared by Amites Sarkar, Western Washington University.

   We prove the result be induction on n, with the base case $n = 1$ very easy (the inequality holds with equality in this case).

For the induction step, assume that for some $n \geq 1$, whenever $x_1, \ldots, x_n$ are n positive numbers satisfying

$$x_1 + x_2 + \cdots + x_n = \frac{1}{2}$$

then

$$\frac{(1-x_1)}{(1+x_1)} \cdot \frac{(1-x_2)}{(1+x_2)} \cdot \cdots \cdot \frac{(1-x_n)}{(1+x_n)} \geq \frac{1}{3}.$$

Let $x_1, \ldots, x_n, x_{n+1}$ be $n+1$ positive numbers satisfying

$$x_1 + x_2 + \cdots + x_n + x_{n+1} = \frac{1}{2}.$$

Setting $x'_i = x_i$ for $i = 1, \ldots, n-1$, and $x'_n = x_n + x_{n+1}$, we apply the induction hypothesis to the numbers $x'_1, \ldots, x'_n$ (which are positive and sum to $1/2$) to conclude

$$\frac{(1-x'_1)}{(1+x'_1)} \cdot \frac{(1-x'_2)}{(1+x'_2)} \cdot \cdots \cdot \frac{(1-x'_n)}{(1+x'_n)} \geq \frac{1}{3}. \quad (\star)$$

We claim that

$$\frac{(1-x_n)}{(1+x_n)} \cdot \frac{(1-x_{n+1})}{(1+x_{n+1})} \geq \frac{(1-x_n-x_{n+1})}{(1+x_n+x_{n+1})}. \quad (\star\star)$$

If $(\star\star)$ is true, then

$$\begin{aligned}
\frac{(1-x_1)}{(1+x_1)} \cdot \cdots \cdot \frac{(1-x_{n-1})}{(1+x_{n-1})} \cdot \frac{(1-x_n)}{(1+x_n)} \cdot \frac{(1-x_{n+1})}{(1+x_{n+1})} &\geq \frac{(1-x_1)}{(1+x_1)} \cdot \cdots \cdot \frac{(1-x_{n-1})}{(1+x_{n-1})} \cdot \frac{(1-x_n-x_{n+1})}{(1+x_n+x_{n+1})} \\
&= \frac{(1-x'_1)}{(1+x'_1)} \cdot \cdots \cdot \frac{(1-x'_n)}{(1+x'_n)} \\
&\geq \frac{1}{3},
\end{aligned}$$

the last inequality by $(\star)$.

So what remains is to prove $(\star\star)$; but after a little algebra this reduces to

$$2x_n^2 x_{n+1} + 2x_n x_{n+1}^2 \geq 0,$$

which is clearly true.

3. Define a sequence $(a_n)_{n\geq 1}$ by

$$a_1 = 1, \quad a_{2n} = a_n, \quad \text{and} \quad a_{2n+1} = a_n + 1.$$

Prove that a_n counts the number of 1's in the binary representation of n.

Solution: I found this problem on Stanford's Putnam prep site, where it is sourced to the book "The Art and Craft of Problem Solving" by P. Zeitz.

Let $f(n)$ count the number of 1's in the binary representation of n. We first show that

$$f(1) = 1, \quad f(2n) = f(n), \quad \text{and} \quad f(2n+1) = f(n) + 1.$$

This is easy: Clearly $f(1) = 1$; if the binary representation of n is $a_1a_2\dots a_k$, then the binary representation of $2n$ is $a_1a_2\dots a_k0$, so $f(2n) = f(n)$; if the binary representation of n is $a_1a_2\dots a_k$, then the binary representation of $2n+1$ is $a_1a_2\dots a_k1$, so $f(2n+1) = f(n) + 1$.

Since $f(n)$ satisfies the same initial conditions as a_n, and the same recurrence, it seems clear that $f(n) = a_n$. Formally, we prove the statement "$f(n) = a_n$" for all $n \geq 1$ by by strong induction. For $n = 1$ it's clear. For $n > 1$, if $n = 2m$ is even then we have

$$f(n) = f(m) = a_m = a_n,$$

the first equality by what we've proved about f, the second by (strong) induction, and the third by hypothesis on a. Similarly if $n = 2m + 1$ is odd then we have

$$f(n) = f(m) + 1 = a_m + 1 = a_n,$$

and we are done.

Remark: The above induction prove works (suitably modified) to establish rigorously the evident but important fact that if two sequences are defined recursively, with the same initial conditions and same recurrence relations, then they are in fact the same sequence.

4. At time 0, a particle resides at the point 0 on the real line. Just before 1 second passes, it divides into 2 particles that fly in opposite directions and stop at distance 1 from the original particle. Just before 2 seconds pass, each of these particles again divides into 2 particles flying in opposite directions and stopping at distance 1 from the point of division, and so on, every second. Whenever two particles meet they annihilate each other (leaving nothing behind). How many particles will there be at time 2050?

   Solution: I took this problem from Matousek and Nesetril, *Invitation to Discrete Mathematics*, Section 1.3 exercise 9 (of 2nd edition).

   We prove the following by induction on $m \geq 1$: at time $2^m - 1$ (at the moment when $2^m - 1$ seconds have passed), there is a single particle at every odd number from $-2^m + 1$ to $2^m + 1$; AND, at no earlier time has there a particle as far away from 0 as $\pm(2^m - 1)$.

   Once we have this, we easily see that at time $2^{11} = 2048$ there is a single particle at each of $\pm 2^{11}$ (all other newly created particles have annihilated each other), so that at time $2^{11} + 1$ there are four particles, at $-2^{11} - 1$, $-2^{11} + 1$, $2^{11} - 1$ and $2^{11} + 1$, and at time $2^{11} + 2 = 2050$ there are again four particles, at $-2^{11} - 2$, $-2^{11} + 2$, $2^{11} - 2$ and $2^{11} + 2$ (two particles created at -2^{11} annihilate each other, and two created at 2^{11} also annihilate each other).

   The base case for the induction is clear. For the induction step, assume that for some $m \geq 1$, at time $2^m - 1$ there is a single particle at every odd number from $-2^m + 1$ to $2^m + 1$, and at no earlier time has there a particle as far away from 0 as $\pm(2^m - 1)$. It follows that at time 2^m, there will be particles at $\pm 2^m$, and no others (all other newly created particles have annihilated each other).

Consider the evolution of the particles at $\pm 2^m$. Scaling the whole system to recenter at 2^m, after a further $2^m - 1$ seconds the particle at 2^m will have given rise to particles at every odd number from $(-2^m + 1) + 2^m$ to $(2^m - 1) + 2^m$, and this particle will not have created any particle that goes beyond 1 (to the left) or $2^{m+1} - 1$ (to the right). [This is the induction hypothesis]. Similarly, the particle at -2^m will have given rise to particles at every odd number from $(-2^m + 1) - 2^m$ to $(2^m - 1) - 2^m$, and this particle will not have created any particle that goes beyond -1 (to the right) or $-2^{m+1} + 1$ (to the left).

These two applications of the induction hypothesis proceeded independently of each other; but in fact since the evolution of the particle at 2^m stays to the right of 0 and the evolution of the particle at -2^m stays to the left of 0, the two can be run simultaneously without interacting.

It follows that at time $2^{m+1} - 1$ there is a single particle at every odd number from $-2^{m+1} + 1$ to $2^{m+1} + 1$, and at no earlier time has there a particle as far away from 0 as $\pm(2^{m+1} - 1)$. This completes the induction.

Note: The number of particles at time m is exactly the number of odd entries in the mth row of Pascal's triangle, i.e., the number of numbers k, $0 \leq k \leq m$ such that $\binom{m}{k}$ is odd; and in fact, after a suitable (and fairly obvious) translation, the locations of the particles at time m correspond to the specific k for which $\binom{m}{k}$ is odd. All this can be proven by induction. See, for example, `http://www.alunw.freeuk.com/pascal.html` for a discussion of this.

5. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Solution: This was problem B2 of the 1985 Putnam competition.

The answer is 101^{99}. It is a fairly easy induction that $f_n(x) = x(x+n)^{n-1}$ (left to reader). Once this relation is established, the result follows immediately.

6. The numbers 1 through $2n$ are partitioned into two sets A and B of size n, in an arbitrary manner. The elements $a_1, \ldots, a_n$ of A are sorted in increasing order, that is, $a_1 < a_2 < \ldots < a_n$, while the elements $b_1, \ldots, b_n$ of B are sorted in decreasing order, that is, $b_1 > b_2 > \ldots > b_n$. Find (with proof!) the value of the sum

$$\sum_{i=1}^{n} |a_i - b_i|.$$

Solution: I got this problem from the UMass Putnam preparation class.

The wording of the question strongly suggests that the answer is independent of the choice of A and B, so we should start with a particularly nice choice of A and B, see what answer we get, conjecture that this is *always* the answer, and then try to prove the conjecture.

Letting $A = \{1, 2, 3, \ldots, n\}$ and $B = \{2n, 2n-1, \ldots, n+1\}$, we find that

$$\sum_{i=1}^{n} |a_i - b_i| = (2n-1) + (2n-3) + \ldots + 1 = n^2$$

(it is an easy induction that the sum of the first n odd positive integers is n^2).

So, we try to prove by induction that $\sum_{i=1}^{n} |a_i - b_i| = n^2$. The base case $n = 1$ is trivial.

For $n \geq 2$, we can consider two cases. Case 1 is when 1 and $2n$ end up in different partition classes. Let's start by considering $1 \in A$, $2n \in B$. In this case, $|a_1 - b_1|$ will contribute $2n - 1$ to the sum. What about the remaining terms? Notice that $A \setminus \{1\}$ and $B \setminus \{2n\}$ form a partition $\{a_2, \ldots, a_n\} \cup \{b_2, \ldots, b_n\}$ of $\{2, \ldots, 2n-1\}$, with the a's increasing and the b's decreasing. Setting $a'_1 = a_2 - 1$, $a'_2 = a_3 - 1$, etc., up to $a'_{n-1} = a_n - 1$, and also setting $b'_1 = b_2 - 1$, $b'_2 = b_3 - 1$, etc., up to $b'_{n-1} = b_n - 1$, we get that A' and B' form a partition $\{a'_1, \ldots, a'_{n-1}\} \cup \{b'_1, \ldots, b'_{n-1}\}$ of $\{1, \ldots, 2n-2\}$, with the a''s increasing and the b''s decreasing. By induction,

$$\sum_{i=1}^{n-1} |a'_i - b'_i| = (n-1)^2,$$

and so, since $|a'_i - b'_i| = |a_{i+1} - b_{i+1}|$ for each i,

$$\sum_{i=1}^{n} |a_i - b_i| = (2n-1) + \sum_{i=2}^{n} |a_{i+1} - b_{i+1}| = (2n-1) + \sum_{i=1}^{n-1} |a'_i - b'_i| = (2n-1) + (n-1)^2 = n^2.$$

If $2n \in A$, $1 \in B$, an almost identical argument gives the same result.

Case 2 is where 1 and $2n$ end up in the same partition class. We start with the case where $1, 2n \in A$. Let x be such that $1, 2, \ldots, x \in A$, but $x + 1 \in B$. Then

$$\sum_{i=1}^{n} |a_i - b_i| = \sum_{i=1}^{x} (b_i - i) + \sum_{i=x+1}^{n-1} |a_i - b_i| + (2n - (x+1)).$$

(Note that this is valid even if $x = n - 1$, the largest it can possibly be; the second sum in this case is empty and so 0). If we modify A and B to form $A' = \{a'_1, \ldots, a'_n\}$, $B' = \{b'_1, \ldots, b'_n\}$ by swapping 1 and $x + 1$, then

$$\begin{aligned}
\sum_{i=1}^{n} |a'_i - b'_i| &= \sum_{i=1}^{x} (b_i - (i+1)) + \sum_{i=x+1}^{n-1} |a_i - b_i| + (2n-1) \\
&= \sum_{i=1}^{x} (b_i - i) + \sum_{i=x+1}^{n-1} |a_i - b_i| + (2n - (x+1)) \\
&= \sum_{i=1}^{n} |a_i - b_i|.
\end{aligned}$$

But now (with A' and B') we are back in case 1, so

$$\sum_{i=1}^{n} |a_i - b_i| = \sum_{i=1}^{n} |a'_i - b'_i| = n^2.$$

A very similar reduction works if $1, 2n \in B$.

7. On an infinite sheet of white graph paper (a paper with a square grid), n squares are colored black. At moments $t = 1, 2, \ldots$, squares are recolored according to the following rule: each square gets the color occurring at least twice in the triple formed by this square, its top neighbor, and its right neighbor. Prove that after the moment $t = n$, all squares are white.

   Solution: I got this problem from Matousek and Nesetril, *Invitation to Discrete Mathematics*, Section 1.3 exercise 8 (of 2nd edition).

   (Sketch) Strong induction on n. Let R be the lowest row initially containing a black square, and let C be the leftmost such column. By the inductive hypothesis, after moment $n-1$ all squares above R are white, and also all squares to the right of C. The only possible remaining black square at the intersection of R and C disappears at moment n.

8. Players $1, 2, 3, \ldots n$ are seated around a table, and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

   Solution: This was problem A2 on the Putnam competition, 1997. The solution below is taken verbatim from Kiran Kedlaya's Putnam archive `https://kskedlaya.org/putnam-archive/`.

   We show more precisely that the game terminates with one player holding all of the pennies if and only if $n = 2^m + 1$ or $n = 2^m + 2$ for some m. First suppose we are in the following situation for some $k \geq 2$. (Note: for us, a "move" consists of two turns, starting with a one-penny pass.)

   - Except for the player to move, each player has k pennies;
   - The player to move has at least k pennies.

   We claim then that the game terminates if and only if the number of players is a power of 2. First suppose the number of players is even; then after m complete rounds, every other player, starting with the player who moved first, will have m more pennies than initially, and the others will all have 0. Thus we are reduced to the situation with half as many players; by this process, we eventually reduce to the case where the number of players is odd. However, if there is more than one player, after two complete rounds everyone has as many pennies as they did before (here we need $m \geq 2$), so the game fails to terminate. This verifies the claim.

Returning to the original game, note that after one complete round, $\lfloor \frac{n-1}{2} \rfloor$ players remain, each with 2 pennies except for the player to move, who has either 3 or 4 pennies. Thus by the above argument, the game terminates if and only if $\lfloor \frac{n-1}{2} \rfloor$ is a power of 2, that is, if and only if $n = 2^m + 1$ or $n = 2^m + 2$ for some m.

9. Show that the sequence

$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7-\sqrt{7}}},\ \sqrt{7-\sqrt{7-\sqrt{7-\sqrt{7}}}},\ldots$$

converges, and find its limit.

Solution: I had intended this to be problem A6 from the 1953 Putnam competition, but I transcribed it in error.

We claim that the limit is $\ell = \frac{-1+\sqrt{50}}{2}$.

The sequence we are working with can be defined recursively by $a_1 = \sqrt{7}$ and, for $n \geq 1$, $a_{n+1} = \sqrt{7 - a_n}$.

We first observe that this is a sequence of real numbers (i.e., that we never take the square root of a negative number). We'll prove this by induction. Specifically, we prove, by induction on n, that for all n, $0 \leq a_n \leq \sqrt{7}$. The base case is trivial. For the induction step, assume that for some $n \geq 1$ we have $0 \leq a_n \leq \sqrt{7}$. Then $7 - \sqrt{7} \leq 7 - a_n \leq 7$, so $0 \leq \sqrt{7-\sqrt{7}} \leq a_{n+1} \leq \sqrt{7}$.

If the sequence tends to a limit, say ℓ, then this limit must also satisfy $\ell \leq \sqrt{7}$ (by properties of limits). Then, because the function $f(x) = \sqrt{7-x}$ is continuous at and near ℓ, it follows from $a_n \to \ell$ that $f(a_n) \to \sqrt{7-\ell}$. But $f(a_n) = a_{n+1}$, so from $f(a_n) \to \sqrt{7-\ell}$ we deduce that $a_{n+1} \to \sqrt{7-\ell}$. Now $a_{n+1} \to \ell$, so we conclude that

$$\ell = \sqrt{7-\ell}, \quad (\star)$$

from which we get $\ell = \frac{-1+\sqrt{50}}{2}$ (the only non-negative solution to $(\star)$).

It remains to show that a_n converges to a limit. We can show this (sketch) by:

- showing that $a_1, a_3, a_5, \ldots$ is monotone decreasing (an induction argument) and tends to limit $\frac{-1+\sqrt{50}}{2}$, and
- showing that $a_2, a_4, a_6, \ldots$ is monotone increasing (an induction argument) and tends to limit $\frac{-1+\sqrt{50}}{2}$.

Messy details omitted...

4 Week three (September 8) — Pigeon-hole principle

"If $n+1$ pigeons settle themselves into a roost that has only n pigeonholes, then there must be at least one pigeonhole that has at least two pigeons."

This very simple principle, sometimes called the *box principle*, and sometimes *Dirichlet's box principle*, can be very powerful.

The proof is trivial: number the pigeonholes 1 through n, and consider the case where a_i pigeons land in hole i. If each $a_i \leq 1$, then $\sum_{i=1}^n a_i \leq n$, contradicting the fact that (since there are $n+1$ pigeons in all) $\sum_{i=1}^n a_i = n+1$.

Since it's a simple principle, to get some power out of it it has to be applied cleverly (in the examples, there will be at least one such clever application). Applying the principle requires identifying what the pigeons should be, and what the pigeonholes should be; sometimes this is far from obvious.

The pigeonhole principle has many obvious generalizations. I'll just state one of them: "if more than mn pigeons settle themselves into a roost that has no more than n pigeonholes, then there must be at least one pigeonhole that has at least $m+1$ pigeons".

Example: 10 points are placed randomly in a 1 by 1 square. Show that there must be some pair of points that are within distance $\sqrt{2}/3$ of each other.

Solution: Divide the square into 9 smaller squares, each of dimension 1/3 by 1/3. These are the pigeonholes. The ten randomly chosen points are the pigeons. By the pigeonhole principle, at least one of the 1/3 by 1/3 squares must have at least two of the ten points in it. The maximum distance between two points in a 1/3 by 1/3 square is the distance between two opposite corners. By Pythagoras this is $\sqrt{(1/3)^2+(1/3)^2} = \sqrt{2}/3$, and we are done.

Example: Show that there are two people in New York City who have the exactly same number of hairs on their head.

Solution: Trivial, because *surely* there are at least two baldies in NYC! But even if we weren't sure of that: a quick websearch shows that a typical human head has around 150,000 hairs, and it is then certainly reasonable to assume that no one has more than 5,000,000 hairs on their head. Set up 5,000,001 pigeonholes, numbered 0 through 5,000,000, and place a resident of NYC (a "pigeon") into bin i if (s)he has i hairs on her head. Another websearch shows that the population of NYC is around 8,300,000, so there are more pigeons than pigeonholes, and some pigeonhole must have multiple pigeons in it.

Example: Show that every sequence of $nm+1$ real numbers must contain EITHER a decreasing subsequence of length $n+1$ OR an increasing subsequence of length $m+1$. (In a sequence $a_1, a_2, \ldots$, an *increasing subsequence* is a subsequence $a_{i_1}, a_{i_2}, \ldots$ [with $i_1 < i_2 < \ldots$] satisfying $a_{i_1} \leq a_{i_2} \leq \ldots$, and a *decreasing subsequence* is defined analogously).

Solution: Let the sequence be $a_1, \ldots, a_{nm+1}$. For each k, $1 \leq k \leq nm+1$, let $f(k)$ be the length of the longest decreasing subsequence that starts with a_k, and let $g(k)$ be the length of the longest increasing subsequence that starts with a_k. Notice that $f(k), g(k) \geq 1$ always.

If there is a k with either $f(k) \geq n+1$ or $g(k) \geq m+1$, we are done. If not, then for every k we have $1 \leq f(k) \leq n$ and $1 \leq g(k) \leq m$. Set up nm pigeonholes, with each pigeonhole labeled by a different pair (i, j), $1 \leq i \leq n$, $1 \leq j \leq m$ (there are exactly nm such pairs).

For each k, $1 \leq k \leq nm+1$, put a_k in pigeonhole (i,j) iff $f(k) = i$ and $g(k) = j$. There are $nm+1$ pigeonholes, so one pigeonhole, say hole (r,s), has at least two pigeons in it.

In other words, there are two terms of the sequence, say a_p and a_q (where without loss of generality $p < q$), with $f(p) = f(q) = r$ and $g(p) = g(q) = s$.

Suppose $a_p \geq a_q$. Then we can find a decreasing subsequence of length $r+1$ starting from a_p, by starting a_p, a_q, and then proceeding with any decreasing subsequence of length r that starts with a_q (one such exists, since $f(q) = r$). But that says that $f(p) \geq r+1$, contradicting $f(p) = r$.

On the other hand, suppose $a_p \leq a_q$. Then we can find an increasing subsequence of length $s+1$ starting from a_p, by starting a_p, a_q, and then proceeding with any increasing subsequence of length s that starts with a_q (one such exists, since $g(q) = s$). But that says that $g(p) \geq s+1$, contradicting $g(p) = s$.

So, whether $a_p \geq a_q$ or $a_p \leq a_q$, we get a contradiction, and we CANNOT ever be in the case where there is NO k with either $f(k) \geq n+1$ or $g(k) \geq m+1$. This completes the proof.

Remark: This beautiful result was discovered by P. Erdös and G. Szekeres in 1935; the incredibly clever application of pigeonholes was given by A. Seidenberg in 1959.

4.1 Week three problems

1. (a) Given m integers $a_1, \dots, a_m$, prove that some nonempty subset of them has sum divisible by m.

   (b) (A stronger statement than the previous part) Given m integers $a_1, \dots, a_m$, show that there is a consecutive subsequence whose sum is divisible by m. (A consecutive subsequence means a subsequence

$$a_i, a_{i+1}, a_{i+2}, \dots a_{i+j-1}$$

   of length j, where j could be as small as one.)

2. (a) 51 different integers are chosen between 1 and 100, inclusive. Show that some two of them are coprime (have no prime factor in common).

   (b) 51 different integers are chosen between 1 and 100, inclusive. Show that there are some two of them such that one divides the other.

3. Prove that from a set of ten distinct two-digit numbers, it is possible to select two nonempty disjoint subsets whose members have the same sum.

4. Let A and B be 2 by 2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

5. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

6. Inside a circle of radius 4 are 45 points. Show that you can find two of these points at most distance $\sqrt{2}$ apart.

7. Show that among any 4^{n-1} people, there are either some n of them who mutually know each other, or some n who mutually don't know each other. (The relation "knowing" is assumed to be symmetric — if I know you, you know me, and vice-versa.)

8. The Fibonacci numbers are defined by the recurrence $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Show that the Fibonacci sequence is periodic modulo any positive integer. (I.e, show that for each $k \geq 1$, the sequence whose nth term is the remainder of f_n on division by k is a periodic sequence).

4.2 Week three solutions

1. (a) Given m integers $a_1, \ldots, a_m$, prove that some nonempty subset of them has sum divisible by m.

   Solution: This is a classic (as is the second part). I won't give a solution, since the solution I give for the second part covers this first part as well.

   (b) (A stronger statement than the previous part) Given m integers $a_1, \ldots, a_m$, show that there is a consecutive subsequence whose sum is divisible by m. (A consecutive subsequence means a subsequence

   $$a_i, a_{i+1}, a_{i+2}, \ldots a_{i+j-1}$$

   of length j, where j could be as small as one.)

   Solution: Look at the m numbers a_1, $a_1 + a_2$, etc., up to $a_1 + \ldots + a_m$. If any one of these is divisible by m, we are done. If not, then these m numbers have between them at most $m-1$ remainders on division by m (1 through $m-1$), so by pigeon-hole principle, some two of them must have the same remainder on division by m.

   Say those two are $a_1 + \ldots + a_k$ and $a_1 + \ldots + a_k + \ldots + a_\ell$ for some $\ell > k$. Then the difference of these two, $a_{k+1} + \ldots + a_\ell$, is divisible by m.

2. (a) 51 different integers are chosen between 1 and 100, inclusive. Show that some two of them are coprime (have no prime factor in common).

   Solution: The two parts to this problem were favorites of Paul Erdős. The first is often called "Posá's soup problem"; see `http://www.math.uwaterloo.ca/navigation/ideas/articles/honsberger/index.shtml` for an explanation.

   Among 51 numbers chosen from between 1 and 100, two must be consecutive, and so coprime (use pigeon-hole principle with 50 pigeon-holes labelled "1, 2", "3, 4", etc., up to "99, 100").

   (b) 51 different integers are chosen between 1 and 100, inclusive. Show that there are some two of them such that one divides the other.

   Solution: Every positive whole number can be expressed uniquely as $n = m2^k$ where m is odd and k is a non-negative whole number. Create 50 pigeon-holes labelled "1", "3", etc., up to "99". Place number n in pigeon-hole labelled "m" if $n = m2^k$ for some non-negative whole number k. By the pigeon-hole principle, there is some odd m such that there are two distinct numbers n_1, n_2 among the 51 with $n_1 = m2^{k_1}$ and $n_2 = m2^{k_2}$. The smaller of these divides the larger.

3. Prove that from a set of ten distinct two-digit numbers, it is possible to select two nonempty disjoint subsets whose members have the same sum.

   Solution: This was problem 1 on the IMO in 1972.

   There are $2^{10} - 1 = 1023$ non-empty subsets. The smallest sum that any of these sets can have is 11, and the largest is $99+98+97+96+95+94+93+92+91+90 = 945$. So there are only 935 possible sums among 1023 non-empty subsets; by PHP some two, A and B, must have the same sum. These sets might not be disjoint, but $A' := A \setminus (A \cap B)$ and $B' := B \setminus (A \cap B)$ are disjoint sets. Both are non-empty.

> (Justification: it could only happen that A' is empty if $A \subseteq B$; but since A and B are not the same, we would then have $B = A \cup C$ for some non-empty C, so the sum of the elements in B would be greater than that in A, a contradiction. And similarly we can't have B empty.)

Also, since we have removed the same set of elements from both A and B to get A' and B', and the sum of the elements of A is the same as that of B, it follows that the sum of the elements of A' is the same as that of B'.

4. Let A and B be 2 by 2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

   Solution: This was Putnam A4, 1994.

   The following solution is from Kiran Kedlya:

   First recall that an integer matrix A has an integer inverse if and only if $\det(A) = \pm 1$.

   > **Proof**: clearly if A^{-1} has integer entries, then $1 = \det(AA^{-1}) = \det(A)\det(A^{-1})$ so $\det(A)$ divides 1. Conversely, if $\det(A) = \pm 1$ then A^{-1} equals $1/\det(A)$ times the signed cofactor matrix of A, which has integer entries.

   So let $f(n) = \det(A) + nB$. Clearly f is a quadratic polynomial in n with integer coefficients (just write it out in terms of the entries); our claim is that if $f(i) \in \{1, -1\}$ for $i = 0, 1, 2, 3, 4$, then $f(n) = \pm 1$ for all n.

   Note that since f has integer coefficients, $x - y$ divides $f(x) - f(y)$ for any integers x and y.

   > **Proof**: If $f(x) = a_n x^n + \cdots + a_0$, then $f(x) - f(y) = a_n(x^n - y^n) + a_{n-1}(x^{n-1} - y^{n-1}) + \ldots$, and each term is divisible by $x - y$.

   But if x and y both belong to $\{0, 1, 2, 3, 4\}$ and $|x - y| > 3$, then $|f(x) - f(y)| \leq |f(x)| + |f(y)| = 2$, so $f(x) - f(y)$ must be 0. Using this, we conclude $f(3) = f(0) = f(4) = f(1)$ (apply what we just said to the two sides of each equality). Thus the quadratic polynomial $f(n) - f(1)$ has four zeroes; that's too many, so it must be the zero polynomial, and $f(n) = f(1) = \pm 1$ for all x.

   Aside: $A, \ldots, A + 3B$ is not enough: consider

   $$A = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is

written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Solution: This was problem A1 from the 2013 Putnam competition.

A key fact to know about the icosahedron is how many faces meet at each vertex. Suppose it is x. The the number of pairs (v, F) where v is a vertex and F is a face that has v as a vertex is $12x$ (each of 12 vertices contributes x to the number); but it is also 3×20 (each of 20 faces contributes 3 to the number). So $x = 5$.

Suppose that there are no two faces that share a vertex and have the same integer written on them. Then, for each vertex, the smallest possible sum of the five numbers on the faces meeting that vertex is $0 + 1 + 2 + 3 + 4 = 10$. Consider the double sum

$$\sum_{\text{vertices } x} \sum_{\text{faces } F \text{ touching } x} \text{number on } F.$$

We have just argued that for each x, the inner sum is at least 10, so the double sum is at least 120. But because each face is a triangle, the double sum counts each number exactly three times, and hence the sum is $3 \times 39 = 117$. This is a contradiction; hence, there are no two faces that share a vertex and have the same integer written on them.

6. Inside a circle of radius 4 are 45 points. Show that you can find two of these points at most distance $\sqrt{2}$ apart.

Solution: From Andrei Jorza.

I can do this easily with "45" replaced with "50". Put the circle on the Cartesian plane, centered at $(0, 0)$. Either on or inside the circle there are 49 points with integer co-ordinates

> (Justification: there are 81 points with integer co-ordinates on the square that encloses the circle, with vertices at $(\pm 8, \pm 8)$. Of those 81, 32 are on the boundary of the square, and of those 32 evidently all but $(0, \pm 4)$ and $(\pm 4, 0)$ are outside the circle. Of the remaining 49 interior integer points, the four $(\pm 3, \pm 3)$ are outside the circle, since their distance from the center is $\sqrt{3^3 + 3^2} > 4$, and the rest are easily seen to by inside the circle. This means that 49 of the integer points ither on or inside the circle.)

Draw a disc of radius $\sqrt{2}/2$ around each of these integer points. The union of these 49 discs is easily seen to include the whole of the radius 4 circle and its interior (this is because the diagonal of each integer square in the grid is $\sqrt{2} = 2(\sqrt{2}/2)$). So among 50 points chosen inside the circle, two must land inside a single circle of radius $\sqrt{2}/2$, and so be distance at most $\sqrt{2}$ apart.

I'm not sure how to easily bring "50" down to "45". One possibility is to put the circle at the center of the triangular grid where all triangles have all side lengths $\sqrt{6}/2$. This length is exactly chosen to ensure that the whole large circle is covered by circles of radius $\sqrt{2}/2$ whose centers are at the corners of the triangles that cover the circle. One then needs to count how many such circles there are; if it is fewer than 44, we are done.

Update: Here's a solution presented in class, by Chang Zhou:

Suppose that it is possible to put down 45 points in the circle of radius 4, with no two of them at distance $\sqrt{2}$ apart or nearer. Draw a circle of radius $\sqrt{2}/2$, so area $pi/2$, around each point. These circles are disjoint (if any two of them overlapped, the two centers would be within $\sqrt{2}$ of each other). So they cover an area of exactly $(45/2)\pi$.

But, these 45 circles all fit inside a circle of radius $4+(\sqrt{2}/2)$ (the extra $\sqrt{2}/2$ is to allow for the possibility that some of the 45 points may be on or near the boundary of the circle of radius 4). This large circle has area $(4+(\sqrt{2}/2))^2\pi = (16.5+4\sqrt{2})\pi < 22.2\pi$ (using $\sqrt{2} < 1.42$).

It is impossible for disjoint circles with total area 22.5π to fit inside a circle of area less than 22.2π. This contradiction proves that if 45 points are chosen in the circle of radius 4, some two of them must be within $\sqrt{2}$ of each other.

7. Show that among any 4^{n-1} people, there are either some n of them who mutually know each other, or some n who mutually don't know each other. (The relation "knowing" is assumed to be symmetric — if I know you, you know me, and vice-versa.)

Solution: This is (a version of) *Ramsey's theorem*, a central theorem in combinatorics.

Note that $4^{n-1} = 2^{2n-2}$. Pick a person a_1 arbitrarily. Of the remaining $2^{2n-2}-1$ people, it must be the case that either a_1 knows at least 2^{2n-3} of them, or doesn't know at least 2^{2n-3} of them (this is pigeon-hole principle, essentially; if she knows fewer than 2^{2n-3}, and doesn't know fewer than 2^{2n-3}, this accounts for fewer than $2^{2n-2}-1$ people).

If a_1 knows at least 2^{2n-3} people, then label a_1 with a "K", and select arbitrarily a subset of the people she knows of size 2^{2n-3}. Remove all other people from consideration. If she doesn't know at least 2^{2n-3} people, then label her with a "D", and select arbitrarily a subset of the people she doesn't know of size 2^{2n-3}. Remove all other people from consideration.

Repeat: select an arbitrary person a_2 from among the 2^{2n-3} people left under consideration after a_1 has been dealt with; among the $2^{2n-3}-1$ people that a_2 may know or not know (not counting a_1), she either knows at least 2^{2n-4} of them, or doesn't know this many; in the former case, label a_2 "K" and select a subset of size 2^{2n-4} of people (other than a_1) that she knows, removing all others from consideration; in the latter case, label a_2 "D" and select a subset of size 2^{2n-4} of people (other than a_1) that she doesn't know, removing all others from consideration.

Iterate this process until we have selected $a_1, a_2, \ldots, a_{2n_2}$. Notice that when we consider a_{2n-2}, there is one person left unconsidered (if a_{2n-2} knows this person, she gets label "K"; if not, label "D"). Call this last person a_{2n-1}.

Two labels have been used to label a_1 through a_{2n-2}, so, by pigeon-hole, one of the labels must be used at least $n-1$ times [note that we could have said the same thing if we had only up to a_{2n-3}; so the "4^{n-1}" at the beginning of the problem could be replaced by "$4^{n-1}/2$"]. Say that that label is "K". Then any collection of $n-1$ of the a_i's with label "K", together with a_{2n-1}, form a collection of n people who mutually know each other (that a_i knows a_j for $i<j$ follows from the fact that a_i has label "K"). On the

other hand, if that label is “D” then any collection of $n-1$ of the a_i's with label “D”, together with a_{2n-1}, form a collection of n people who mutually don't know each other.

8. The Fibonacci numbers are defined by the recurrence $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1}+f_{n-2}$ for $n \geq 2$. Show that the Fibonacci sequence is periodic modulo any positive integer. (I.e, show that for each $k \geq 1$, the sequence whose nth term is the remainder of f_n on division by k is a periodic sequence).

   Solution: I found this on the Northwestern Putnam prep class webpage.

   Consider the sequence obtained from the Fibonacci sequence by taking the remainder of each term on division by k (so the result is a sequence, all terms in $\{0, \ldots, k-1\}$). Suppose that there are two consecutive terms in this sequence, say the mth and $(m+1)$st, taking values a, b, and two *other* consecutive terms , say the nth and $n+1$st, taking the same values a, b (with $m < n$). Then the $(m+2)$nd and $(n+2)$nd terms of the reduced sequence agree.

   [WHY? Because the $(m+2)$nd term is the remainder of F_{m+2} on division by k, which is the remainder of $F_m + F_{m+1}$ on division by k, which is the remainder of F_m on division by k PLUS the remainder of F_{m+1} on division by k, which is the remainder of a on division by k PLUS the remainder of b on division by k, which is the remainder of F_n on division by k PLUS the remainder of F_{n+1} on division by k, which is the remainder of $F_n + F_{n+1}$ on division by k, which is the remainder of F_{n+2} on division by k.]

   The same argument shows that the reduced sequence is periodic beyond the mth terms, with period (at most) $n - m$.

   So all we need to do to find periodicity is to find two consecutive terms in the sequence, that agree with two *other* consecutive terms. There are only k^2 possibilities for a pair of consecutive values in the sequence, and infinitely many consecutive values, so by PHP there has to be a coincidence of the required kind.

5 Week four (September 17) — Binomial coefficients

Binomial coefficients crop up quite a lot in Putnam problems. This handout presents some ways of thinking about them.

Introduction

The *binomial coefficient* $\binom{n}{k}$, with $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, can be defined many ways; possibly the most helpful definition from the point of view of problem-solving is the following combinatorial one:

$$\binom{n}{k} \text{ is the number of subsets of size } k \text{ of a set of size } n.$$

In particular, this definition immediately tells us that for all $n \geq 0$ we have $\binom{n}{k} = 0$ if $k > n$ or if $k < 0$, and that $\binom{n}{0} = \binom{n}{n} = 1$ (and so in particular $\binom{0}{0} = 1$).

The binomial coefficients can also be defined by a recurrence relation: for $n \geq 1$, and all $k \in \mathbb{Z}$, we have the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \qquad (\textit{Pascal's identity})$$

with initial conditions $\binom{0}{k} = 0$ if $k \neq 0$, and $\binom{0}{0} = 1$. To see that this recurrence does indeed generate the binomial coefficients, think about the combinatorial interpretation: the subsets of $\{1, \ldots, n\}$ of size k ($\binom{n}{k}$ of them) partition into those that don't include element n ($\binom{n-1}{k}$ of them) and those that do include element n ($\binom{n-1}{k-1}$ of them). The recurrence allows us to quickly compute small binomial coefficients via *Pascal's triangle*: the zeroth row of the triangle has length one, and consists just of the number 1. Below that, the first row has two 1's, one below and to the left of the 1 in the zeroth row, and one below and to the right of the 1 in the zeroth row. The second row has three entries, a 1 below and to the left of the leftmost 1 in the first row, a 1 below and to the right of the rightmost 1 in the first row, and in the center a 2. Each subsequent row contains one more entry than the previous row, starting with a 1 below and to the left of the leftmost 1 in the previous row, ending with a 1 below and to the right of the rightmost 1 in the previous row, and with all other entries being the sum of the two entries in the previous row above to the left and to the right of the entry being considered (see the picture below).

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

etc.

Pascal's triangle in numbers

The kth entry in row k (counting from 0 rather than 1 both down and across) is then $\binom{n}{k}$ (this is just a restatement of Pascal's identity) (see the picture below).

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

etc.

Pascal's triangle symbolically

Finally, there is an algebraic expression for $\binom{n}{k}$, that makes sense for all $n, k \geq 0$, using the factorial function (defined combinatorially as the number of ways of arranging n distinct objects in order, and algebraically by $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ for $n \geq 1$, with $0! = 1$):

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-(k-1))}{k!} = \frac{n!}{k!(n-k)!}.$$

To see this, note that $n(n-1)\ldots(n-(k-1))$ is fairly evidently the number of *ordered lists* of k distinct elements from $\{1, \ldots, n\}$ (often referred to in textbooks as "permutations of n

items taken k at a time" — ugh). When the ordered lists are turned into (unordered) subsets, each subset appears $k!$ times (once for each of the $k!$ ways of putting k distinct objects into an ordered list), so we need to divide the ordered count by $k!$ to get the unordered count.

When dealing with binomial coefficients, it is very helpful to bear all three definitions in mind, but in particular the first two.

Identities

The binomial coefficients satisfy a staggering number of identities. The simplest of these are easily understood using either the combinatorial or algebraic definitions; for the more involved ones, that include sums, the algebraic definition is usually next to useless, and often the easiest way to prove the identity is combinatorially, by showing that both sides of the identity count the same thing in different ways (illustration below), though it is often possible also to prove these identities by induction, using the recurrence relation. Another approach that is helpful is that of generating functions.

Here are some of the basic binomial coefficient identities:

1. (Symmetry)
$$\binom{n}{k} = \binom{n}{n-k}$$
(Proof: trivial from the algebraic definition; combinatorially, left-hand side counts selection of subsets of size k from a set of size n, by naming the selected elements; right-hand side also counts selection of subsets of size k from a set of size n, this time by naming the *un*selected elements).

2. (Lower summation)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
(Proof: close to impossible using the algebraic definition; combinatorially, very straightforward: left-hand side counts the number of subsets of a set of size n, by first deciding the size of the subset, and then choosing the subset itself; right-hand side also counts the number of subsets of a set of size n, by going through the n elements one-by-one and deciding whether they are in the subset or not).

3. (Upper summation)
$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

4. (Parallel summation)
$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n}.$$

5. (Square summation)
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

6. (Vandermonde identity, or Vandermonde convolution)

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r}.$$

The binomial theorem

This is the most important identity involving binomial coefficients: for all real x and y, and $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This can be proved by induction using Pascal's identity, but the proof is quite awkward. Here's a nice combinatorial proof. First, note that the identity is trivial if either $x = 0$ or $y = 0$, so we may assume $x, y \neq 0$. Dividing through by x^n, the identity is the same as

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

We will prove this combinatorially when z is a positive integer. The left-hand side counts the number of words of length n from alphabet $\{0, 1, 2, \ldots, z\}$, by deciding on the letters one after the other. The right-hand side also counts the number of words of length n from alphabet $\{0, 1, 2, \ldots, z\}$, as follows: first decide how many of the letters of the word are from $\{1, \ldots, z\}$ (this is the k of the summation). Next, decide the location of these k letters (this is the $\binom{n}{k}$). Finally, decide what specific letters go into those spots, one after another (this is the z^k) (note that the remaining $n-k$ letters must all be 0's).

This only shows the identity for positive integer z. But now we use the fact that both the right-hand and left-hand sides are polynomials of degree n, so if they agree at $n+1$ different values of z, they must agree at all values of z (otherwise, their difference is a not-identically-zero polynomial of degree at most n with $n+1$ distinct roots, an impossibility). And indeed, the two sides agree not just at $n+1$ different values of z, but at infinitely many (all positive integers z). So from the combinatorial argument that shows that the two sides are equal for positive integers z, we infer that they are equal for *all* real z. This argument is often called the *polynomial principle*.

There is a version of the binomial theorem also for non-positive-integral exponents: for all real α,

$$(1+z)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} z^k$$

where $\binom{\alpha}{k}$ is defined in the obvious way:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} \quad \text{(for } k \geq 1\text{; } \tbinom{\alpha}{0} = 1\text{)},$$

and the equality is valid for all real $|z| < 1$. (Check: when α is a positive integer, this reduces to the standard binomial theorem).

For example, if $\ell > 0$ is a positive integer, then

$$\binom{-\ell}{k} = \frac{(-\ell)(-\ell-1)\dots(-\ell-k+1)}{k!} = (-1)^k\binom{\ell+k-1}{k},$$

and so

$$\frac{1}{(1-z)^\ell} = \sum_{k\geq 0}\binom{\ell+k-1}{k}z^k.$$

This generalizes the familiar identity

$$\frac{1}{1-z} = 1 + z + z^2 + \dots.$$

Modulo the convergence analysis, the proof of the binomial theorem for general exponents is fair easy: the coefficient of x^k in the Taylor series Taylor series of $(1+z)^\alpha$ is

$$\frac{1}{k!}\frac{d^k}{dx^k}(1+z)^\alpha|_{z=0} = \binom{\alpha}{k}.$$

Compositions and weak compositions

A *composition* of a positive integer n into k parts is a vector $(x_1, x_2, \dots, x_k)$, with each entry a strictly positive integer, and with $\sum_{i=1}^k x_i = n$. For example, $(2,1,1,3)$ is a composition of 7, as is $(1,3,1,2)$; and, because a composition is a *vector* (ordered list), these two are considered different compositions.

A *weak composition* of a positive integer n into k parts is a vector $(x_1, x_2, \dots, x_k)$, with each entry a non-negative (possibly 0) integer, and with $\sum_{i=1}^k x_i = n$. For example, $(2,0,1,3)$ is a weak composition of 6, but not a composition.

How many weak compositions of n are there, into k parts? Put down $n+k-1$ stars in a row. Choose $k-1$ of them to turn into bars. The resulting arrangement of stars-and-bars encodes a weak composition of n into k parts — the number of stars before the first bar is x_1, the number of stars between the first and second bar is x_2, and so on, up to the number of stars after the last bar, which is x_k (notice that only $k-1$ bars are needed to determine k intervals of stars). Conversely, every weak composition of n into k parts is encoded by one such selection of $k-1$ bars from the initial list of $n+k-1$ stars. For example, the configuration $\star\star\,|\,|\,\star\,|\,\star\star\star$ encodes the weak composition $(2,0,1,3)$ of 6 into 4 parts. So, the number of weak compositions of n into k parts is a binomial coefficient, $\binom{n+k-1}{k-1}$.

How many compositions of n are there, into k parts? Each such composition $(x_1, x_2, \dots, x_k)$ gives rise to a weak composition $(x_1-1, x_2-1, \dots, x_k-1)$ of $n-k$ into k parts, and all weak composition of $n-k$ into k parts are achieved by this process. So, the number of compositions of n into k parts is the same as the number of weak compositions of $n-k$ into k parts, which is $\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$.

For example: I like plain cake, chocolate cake, blueberry cake and pumpkin cake donuts from Dunkin' Donuts. In how many different ways can I buy a dozen donuts that I like? I must buy x_1 plain, x_2 chocolate, x_3 blueberry and x_4 pumpkin, with $x_1+x_2+x_3+x_4 = 12$, and with each x_i a non-negative integer (possibly 0). So the number of different purchases I can make is the number of weak compositions of 12 into 4 parts, so $\binom{15}{4} = 1365$.

5.1 Warm-up problems

1. Give a combinatorial proof of the upper summation identity.

2. Give a combinatorial proof of the parallel summation identity.

3. Give a combinatorial proof of the square summation identity.

4. Give a combinatorial proof of the Vandermonde identity.

5. Evaluate
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}$$
for $n \geq 1$.

6. (a) Let a_n be the number of 0-1 strings of length n that do not have two consecutive 1's. Find a recurrence relation for a_n (starting with initial conditions $a_0 = 1$, $a_1 = 2$).

   (b) Let $a_{n,k}$ be the number of 0-1 strings of length n that have exactly k 1's and that do not have two consecutive 1's. Express $a_{n,k}$ as a (single) binomial coefficient.

   (c) Use the results of the previous two parts to give a combinatorial proof (showing that both sides count the same thing) of the identity
$$F_n = \sum_{k\geq 0}\binom{n-k-1}{k}$$
where F_n is the nth Fibonacci number ($F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$).

5.2 Week four problems

1. Show that the coefficient of x^k in $(1+x+x^2+x^3)^n$ is

$$\sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

2. Find a simple expression (not involving a sum) for

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}.$$

3. n points are arranged on a circle. All possible diagonals are drawn. Assuming that no three of the diagonals meet at a single point, how many intersections of diagonals are there inside the circle?

4. (a) The *kth falling power* of x is $x^{\underline{k}} = x(x-1)(x-2)\ldots(x-(k-1))$. Prove that for all real x, y, and all $n \geq 1$,

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}}.$$

   (b) The *kth rising power* of x is $x^{\overline{k}} = x(x+1)(x+2)\ldots(x+(k-1))$. Prove that for all real x, y, and all $n \geq 1$,

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{n-k}} y^{\overline{k}}.$$

5. Evaluate

$$\sum_{k=0}^{n} F_{k+1}\binom{n}{k}$$

   for $n \geq 0$, where $F_1, F_2, F_3, F_4, F_5, \ldots$ are the Fibonacci numbers $1, 1, 2, 3, 5, \ldots$.

6. Evaluate

$$\sum_{k=0}^{2n} (-1)^k k^n \binom{2n}{k}$$

   for $n \geq 1$.

7. Show that for all $k \geq 0$,

$$\int_0^{\pi/2} (2\sin x)^{2k}\, dx = \frac{\pi}{2}\binom{2k}{k}.$$

5.3 Solutions to warm-up problems

1. Give a combinatorial proof of the upper summation identity.

   Solution: RHS is number of subsets of $\{1, \ldots, n+1\}$ of size $k+1$, counted directly. LHS counts same, by first specifying largest element in subset (if largest element is $k+1$, remaining k must be chosen from $\{1, \ldots, k\}$, $\binom{k}{k}$ ways; if largest element is $k+2$, remaining k must be chosen from $\{1, \ldots, k+1\}$, $\binom{k+1}{k}$ ways; etc.).

2. Give a combinatorial proof of the parallel summation identity.

   Solution: RHS is number of subsets of $\{1, \ldots, n+m+1\}$ of size n, counted directly. LHS counts same, by first specifying the smallest element *not* in subset (if smallest missed element is 1, all n elements must be chosen from $\{2, \ldots, n+m+1\}$, $\binom{m+n}{n}$ ways, the $k = n$ term; if smallest missed element is 2, then 1 is in subset and remaining $n-1$ elements must be chosen from $\{3, \ldots, n+m+1\}$, $\binom{m+n-1}{n-1}$ ways, the $k = n-1$ term; etc., down to: if smallest missed element is $n+1$, then $\{1, \ldots, n\}$ is in subset and remaining 0 elements must be chosen from $\{n+2, \ldots, k+1\}$, $\binom{m+0}{0}$ ways, the $k = 0$ term).

3. Give a combinatorial proof of the square summation identity.

   Solution: RHS is number of subsets of $\{\pm 1, \ldots, \pm n\}$ of size n, counted directly. LHS counts same, by first specifying k, the number of positive elements chosen, then selecting k positive elements ($\binom{n}{k}$ ways), then selecting the k negative elements that are *not* chosen (so the $n-k$ that are, for n in total) ($\binom{n}{k}$ ways).

4. Give a combinatorial proof of the Vandermonde identity.

   Solution: Let $A = \{x_1, \ldots, x_m\}$ and $B = \{y_1, \ldots, y_n\}$ be disjoint sets. RHS is number of subsets of $A \cup B$ of size r, counted directly. LHS counts same, by first specifying k, the number of elements chosen from A, then selecting r elements from A ($\binom{m}{k}$ ways), then selecting the remaining $r-k$ elements from B ($\binom{n}{r-k}$ ways).

5. Evaluate
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}$$
   for $n \geq 1$.

   Solution: Applying the binomial theorem with $x = 1, y = 1$ get
$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$
   so the sum is 0.

6. (a) Let a_n be the number of 0-1 strings of length n that do not have two consecutive 1's. Find a recurrence relation for a_n (starting with initial conditions $a_0 = 1$, $a_1 = 2$).

   Solution: By considering whether the last term is a 0 or a 1, get the Fibonacci recurrence: $a_n = a_{n-1} + a_{n-2}$.

(b) Let $a_{n,k}$ be the number of 0-1 strings of length n that have exactly k 1's and that do not have two consecutive 1's. Express $a_{n,k}$ as a (single) binomial coefficient.

Solution: Add a 0 to the beginning and end of such a string. By reading off a_1, the number of 0's before the first 1, then a_2, the number of 0's between the first 1 and the second, and so on up to a_{k+1}, the number of 0's after the last 1, we get a composition $(a_1, \ldots, a_{k+1})$ of $n+2-k$ into $k+1$ parts; and each such composition can be encoded (uniquely) by such a string. So $a_{n,k}$ is the number of compositions of $n+2-k$ into $k+1$ parts, and so equals $\binom{n+1-k}{k}$.

(c) Use the results of the previous two parts to give a combinatorial proof (showing that both sides count the same thing) of the identity

$$F_n = \sum_{k \geq 0} \binom{n-k-1}{k}$$

where F_n is the nth Fibonacci number ($F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$).

Solution: From the recurrence in the first part, we get $a_n = F_{n+2}$, so F_n counts the number of 0-1 strings of length $n-2$ with no two consecutive 1's. We can count such strings by first deciding on k, the number of 1's, and by the second part, the number of such strings is $\binom{n-1-k}{k}$. Summing over k we get the result.

5.4 Week four solutions

1. Show that the coefficient of x^k in $(1+x+x^2+x^3)^n$ is

$$\sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.$$

Solution: The was on the 1992 Putnam competition, problem B2.

We have

$$\begin{aligned} (1+x+x^2+x^3)^n &= (1+x)^n(1+x^2)^n \\ &= \left(\sum_{i \geq 0} \binom{n}{i} x^i \right) \left(\sum_{i' \geq 0} \binom{n}{2i'} x^{2i'} \right). \end{aligned}$$

We get an x^k term in the product by pairing each $\binom{n}{i} x^i$ from the first sum with $\binom{n}{k-2i} x^{k-2i}$ from the second; so the coefficient of x^k in the product is

$$\sum_{i=0}^{k} \binom{n}{i} \binom{n}{k-2i},$$

as claimed.

2. Find a simple expression (not involving a sum) for

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}.$$

Solution: This was on the Putnam in 1962. It was question A5. These days, A5 is typically a much more involved question!

We claim that the (or at least *an*) answer is $n(n+1)2^{n-2}$.

From the binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad (\star)$$

Differentiating both sides with respect to x twice, get

$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^{k-2} = \sum_{k=0}^{n} (k^2-k)\binom{n}{k} x^{k-2},$$

and evaluating at $x=1$ get

$$n(n-1)2^{n-2} = (1^2-1)\binom{n}{1} + (2^2-2)\binom{n}{2} + (3^2-3)\binom{n}{3} + \cdots + (n^2-n)\binom{n}{n}. \quad (\star\star)$$

Differentiating both sides of $(\star)$ with respect to x once, get

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1},$$

and evaluating at $x=1$ get

$$n2^{n-1} = 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}. \quad (\star\star\star).$$

Adding $(\star\star)$ and $(\star\star\star)$ get

$$\begin{aligned} 1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} &= n(n-1)2^{n-2} + n2^{n-1} \\ &= n(n+1)2^{n-2}. \end{aligned}$$

3. n points are arranged on a circle. All possible diagonals are drawn. Assuming that no three of the diagonals meet at a single point, how many intersections of diagonals are there inside the circle?

Solution: This is an old classic.

We claim that the answer is $\binom{n}{4}$.

Each intersection inside the circle determines a unique collection of four of the points on the circle, by: two lines meet at each intersection, and each of the two lines has two endpoints. Conversely, each set of four points on the circle determines a unique point of intersection, by: if the four points are, in clockwise order, a, b, c, d, then the associated point of intersection is the intersection of the lines ac and bd.

It follows that there are exactly as many intersections of diagonals inside the circle, as there are sets of points on the circle; and there are $\binom{n}{4}$ such sets of points.

4. (a) The *kth falling power* of x is $x^{\underline{k}} = x(x-1)(x-2)\ldots(x-(k-1))$. Prove that for all real x, y, and all $n \geq 1$,

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}}.$$

Solution: This part and the next are standard Binomial coefficient identities. This problem was B1 on the 1962 Putnam competition.

An argument by induction is possible. But there is also a combinatorial argument: Let x and y be positive integers. The number of words in alphabet $\{1,\ldots,x\} \cup \{x_1,\ldots,x+y\}$ of length n with no two repeating letters, counted by selecting letter-by-letter, is $(x+y)^{\underline{n}}$. If instead we count by first selecting k, the number of letters from $\{x+1,\ldots,x+y\}$ used, then locate the k positions in which those letters appear, then selecting the $n-k$ letters from $\{1,\ldots,x\}$ letter-by-letter in the order that they appear in the word, and finally selecting the k letters from $\{x+1,\ldots,x+y\}$ letter-by-letter in the order that they appear in the word, we get a count of $\sum_{k=0}^{n}\binom{n}{k}x^{\underline{n-k}}y^{\underline{k}}$. So the identity is true for positive integers x, y.
The LHS and RHS are polynomials in x and y of degree n, so the difference is a polynomial in x and y of degree at most n, which we want to show is identically 0. Write the difference as $P(x,y) = p_0(x)+p_1(x)y+\ldots+p_n(x)y^n$ where each $p_i(x)$ is a polynomial in x of degree at most n. Setting $x=1$ we get a polynomial $P(1,y)$ in y of degree at most n. This is 0 for all integers $y>0$ (by our combinatorial argument), so by the polynomial principle[5] it is identically 0. So each $p_i(x)$ evaluates to 0 at $x=1$. But the same argument shows that each $p_i(x)$ evaluates to 0 at any positive integer x. So again by the polynomial principle, each $p_i(x)$ is identically 0 and so $P(x,y)$ is. This proves the identity for all real x, y.

(b) The *kth rising power* of x is $x^{\overline{k}} = x(x+1)(x+2)\ldots(x+(k-1))$. Prove that for all real x, y, and all $n \geq 1$,

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{n-k}} y^{\overline{k}}.$$

Solution: This can be derived from the result of the previous part.

[5]The *polynomial principle*: If a polynomial of degree at most n has $n+1$ or more zeroes, then it is identically 0.

Set $x' = -x$ and $y' = -y$; we have

$$(x+y)^{\overline{n}} = (-x'-y')^{\overline{n}} = (-1)^n (x'+y')^{\underline{n}}$$

and

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} x^{\overline{n-k}} y^{\overline{k}} &= \sum_{k=0}^{n} \binom{n}{k} (-x')^{\overline{n-k}} (-y')^{\overline{k}} \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x')^{\underline{n-k}} (-1)^k (y')^{\underline{k}} \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (x')^{\underline{n-k}} (y')^{\underline{k}}, \end{aligned}$$

so the identity follows from the falling power binomial theorem (the previous part).

5. Evaluate

$$\sum_{k=0}^{n} F_{k+1} \binom{n}{k}$$

for $n \geq 0$, where $F_1, F_2, F_3, F_4, F_5, \ldots$ are the Fibonacci numbers $1, 1, 2, 3, 5, \ldots$.

Solution: This is a well-known binomial coefficient identity.

We claim that the answer is F_{2n+1}.

When $n = 0$ we get a sum of 1; when $n = 1$ we get a sum of 2; when $n = 2$ we get a sum of 5; when $n = 3$ we get a sum of 13; when $n = 4$ we get a sum of 34; this suggests strongly

$$\sum_{k=0}^{n} F_{k+1} \binom{n}{k} = F_{2n+1}.$$

One way to prove this is to iterative apply the Fibonacci recurrence to F_{2n+1}: on the zeroth iteration,

$$F_{2n+1} = \binom{0}{0} F_{2n+1}.$$

The first iteration leads to

$$F_{2n+1} = F_{2n} + F_{2n-1} = \binom{1}{0} F_{2n} + \binom{1}{0} F_{2n-1}.$$

The second leads to

$$\begin{aligned} F_{2n+1} &= F_{2n} + F_{2n-1} \\ &= (F_{2n-1} + F_{2n-2}) + (F_{2n-2} + F_{2n-3}) \\ &= \binom{2}{0} F_{2n-1} + \binom{2}{1} F_{2n-2} + \binom{2}{2} F_{2n-3}. \end{aligned}$$

This suggest that we prove to more general statement, that for each $0 \le s \le n$,

$$F_{2n+1} = \sum_{j=0}^{s} \binom{s}{j} F_{2n+1-s-j}. \qquad (\star)$$

The case $s = n$ yields

$$F_{2n+1} = \sum_{j=0}^{n} \binom{n}{j} F_{n+1-j},$$

which is the same as what we have to prove (by the symmetry relation $\binom{n}{j} = \binom{n}{n-j}$).

We can prove $(\star)$ by induction on s (for each fixed n), with the case $s = 0$ trivial. For larger s, we begin with the $s-1$ case of the induction hypothesis, then use the Fibonacci recurrence to break each Fibonacci number into the sum of two earlier ones, then use Pascals identity to gather together terms involving the same Fibonacci number. (Details omitted.)

6. Evaluate

$$\sum_{k=0}^{2n} (-1)^k k^n \binom{2n}{k}$$

for $n \ge 1$.

Solution: I don't know where I first saw this.

We claim that the sum is 0.

It's not easy to deal with this sum in isolation. But, we can generalize: define, for each $n, r \ge 1$, $a_{r,n} = \sum_{k=0}^{2n} (-1)^k k^r \binom{2n}{k}$. We claim that $a_{r,n} = 0$ (and so in particular $a_{n,n}$, our sum of interest, is 0.

We prove the claim for each n by induction on r. It will be helpful to define

$$f_n(x) = \sum_{k=0}^{2n} x^k \binom{2n}{k} = (1+x)^{2n}.$$

Differentiating,

$$f_n'(x) = \sum_{k=0}^{2n} k x^{k-1} \binom{2n}{k} = 2n(1+x)^{2n-1}.$$

Evaluating at $x = -1$ we get

$$\sum_{k=0}^{2n} (-1)^{k-1} k^1 \binom{2n}{k} = 0,$$

and multiplying by -1 gives $a_{1,n} = 0$. This is the base case of the induction.

For the induction step, assume that $a_{j,n} = 0$ for all $1 \le j < r$ (with $r \ge 2$). The rth derivative of $f_n(x)$ is

$$f_n^{(r)}(x) = \sum_{k=0}^{2n} k(k-1)\dots(k-(r-1))x^{k-r} \binom{2n}{k} = 2n(2n-1)\dots(2n-(r-1))(1+x)^{2n-r}.$$

Now $k(k-1)\dots(k-(r-1))$ is a polynomial in k, of degree r, whose leading coefficient is 1, and for which all other terms are polynomials in r; in other words,

$$k(k-1)\dots(k-(r-1)) = k^r + c_1(r)k^{r-1} + \dots + c_r(r),$$

and so

$$f_n^{(r)}(x) = \sum_{k=0}^{2n} x^{k-r}k^r\binom{2n}{k} + \sum_{j=1}^{r} c_j(r)\sum_{k=0}^{2n} x^{k-r}k^{r-j}\binom{2n}{k}.$$

Evaluating at $x=-1$, and recalling that $f_n^{(r)}(x) = 2n(2n-1)\dots(2n-(r-1))(1+x)^{2n-r}$, we get

$$\sum_{k=0}^{2n}(-1)^{k-r}k^r\binom{2n}{k} + \sum_{j=1}^{r} c_j(r)\sum_{k=0}^{2n}(-1)^{k-r}k^{r-j}\binom{2n}{k} = 0.$$

The sum corresponding to $j=r$ is

$$c_j(r)\sum_{k=0}^{2n}(-1)^{k-r}\binom{2n}{k} = (-1)^{-r}c_j(r)\sum_{k=0}^{2n}(-1)^{k}\binom{2n}{k} = (-1)^{-r}c_j(r)(1-1)^{2n} = 0.$$

The sum corresponding to each j, $1 \le j < r$ is $c_j(r)(-1)^{-r}a_{r,n}$, so is 0 by induction. So we conclude

$$\sum_{k=0}^{2n}(-1)^{k-r}k^r\binom{2n}{k} = 0,$$

which gives $a_{r,n}=0$ on multiplying by $(-1)^r$.

7. Show that for all $k \ge 0$,

$$\int_0^{\pi/2}(2\sin x)^{2k}\,dx = \frac{\pi}{2}\binom{2k}{k}.$$

Solution: This is a classic calculus exercise that appears in most calculus textbooks. I saw it on the UMass Putnam preparation website.

Using integration by parts ($u=(\sin x)^{n-1}$, $dv = \sin x\,dx$, so that $du=(n-1)(\sin x)^{n-2}\cos x\,dx$ and $v=-\cos x\,dx$) we get that for $n \ge 2$,

$$\begin{aligned}
\int(\sin x)^n\,dx &= -(\sin x)^{n-1}\cos x + (n-1)\int(\sin x)^{n-2}(\cos x)^2\,dx \\
&= -(\sin x)^{n-1}\cos x + (n-1)\int(\sin x)^{n-2}(1-(\sin x)^2)\,dx \\
&= -(\sin x)^{n-1}\cos x - (n-1)\int(\sin x)^{n}\,dx + (n-1)\int(\sin x)^{n-2}\,dx.
\end{aligned}$$

Rearranging yields the “reduction formula”

$$\int(\sin x)^n\,dx = -\frac{1}{n}(\sin x)^{n-1}\cos x + \frac{n-1}{n}\int(\sin x)^{n-2}\,dx.$$

Applying this to the definite integral, we obtain

$$\int_0^{\pi/2} (\sin x)^n \, dx = \frac{n-1}{n} \int_0^{\pi/2} (\sin x)^{n-2} \, dx. \tag{1}$$

Noting that

$$\int_0^{\pi/2} (\sin x)^0 \, dx = \frac{\pi}{2},$$

we iteratively apply (1) to obtain

$$\int_0^{\pi/2} (\sin x)^{2k} \, dx = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \ldots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2},$$

and so

$$\int_0^{\pi/2} (2\sin x)^{2k} \, dx = 2^{2k} \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \ldots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.$$

Now

$$\begin{aligned}
2^{2k} \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \ldots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} &= 2^k \frac{(2k-1)(2k-3)\ldots(3)(1)}{k!} \frac{\pi}{2} \\
&= 2^k \frac{(2k-1)(2k-3)\ldots(3)(1)k!}{k!k!} \frac{\pi}{2} \\
&= 2^k \frac{[k](2k-1)[k-1](2k-3)[k-2]\ldots[2](3)[1](1)}{k!k!} \frac{\pi}{2} \\
&= \frac{[2k](2k-1)[2k-2](2k-3)[2k-4]\ldots[4](3)[2](1)}{k!k!} \frac{\pi}{2} \\
&= \frac{(2k)!}{k!k!} \frac{\pi}{2} \\
&= \frac{\pi}{2} \binom{2k}{k},
\end{aligned}$$

as required.

6 Week five (September 24) — Calculus

Calculus is a rich source for competition problems. The Putnam problem setters try to assume minimal mathematical background, so the the topics from Calculus that come up will tend to focus on material from Calc 1 & 2. Here are the things you should for sure be familiar with:

- The definitions of limits, continuity and derivative. Some questions will ask to compute interesting limits, or make certain continuity assumptions, or give some information about the values of derivatives of a function, and of course it will be helpful to be comfortable with these concepts!

- The three basic theorems of continuity and differentiability:
  - The intermediate value theorem: if continuous $f : [a, b] \to \mathbb{R}$ is negative at a and positive at b it must be 0 at some point between a and b.
  - The extreme value theorem: a continuous function f defined on a closed interval $[a, b]$ is bounded, and there are numbers c, d such that $f(c) = \max\{f(x) : x \in [a, b]\}$ and $f(d) = \min\{f(x) : x \in [a, b]\}$ (i.e., not only is f bounded, but it *reaches* its bounds).
  - The mean value theorem: if $f : [a, b] \to \mathbb{R}$ is differentiable (and so, necessarily, continuous) then there is some $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$ (i.e., the average slope is matched *at some point* by the exact slope).

- The meaning of first and second derivates, in terms of local maxima and minima of functions.

- The idea of approximating am integral via a Riemann sum, and recognizing a sum as a Riemann sum — sometimes a complicated sum becomes very easy to understand if you realize that it is a Riemann sum for some integral.

- The fundamental theorem of calculus, which has two distinct parts:
  - if new function g is defined from old continuous function f by $g(x) = \int_a^x f(t)dt$ (some fixed a), then g is differentiable, and $g'(x) = f(x)$, and
  - if for some continuous f, the function g has the property that $g'(x) = f(x)$ (i.e., g is an antiderivative of f) then $\int_a^b f(x)dx = g(b) - g(a)$.

- Taylor's theorem, with remainder term: suppose f is infinitely differentiable at and near a. Then

  $$f(x) \approx f(a) + (x-a)f'(a) + (x-a)^2 \frac{f''(a)}{2!} + \cdots + (x-a)^n \frac{f^{(n)}(a)}{n!}.$$

  More precisely, there is some number c between a and x for which

  $$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 \frac{f''(a)}{2!} + \cdots + (x-a)^n \frac{f^{(n)}(a)}{n!} + (x-a)^{n+1} \frac{f^{(n+1)}(c)}{(n+1)!}.$$

- All the basic integrals, and all the basic integration techniques — integration by parts, u-substitutions, trigonometric substitutions, et cetera.

Paraphrasing my colleague Andrei Jorza: "you will rarely need any new calculus technique that you haven't seen before; the difficulty is to patch together all the things you know to obtain a solution. While cleverness will take you a long way in problem solving calculus, this is no place for being squeamish about algebraic manipulations."

The book *Putnam and Beyond* (available online) has a huge number of Putnam-style calculus problems. You'll also find a fair number at `https://www3.nd.edu/~ajorza/courses/2018f-m43900/handouts/lecture3.pdf` (Andrei Jorza's 43900 page from Fall 2018). Many of this week's problems come from that list.

6.1 Week five problems

1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers satisfying $x^4 + 36 \leq 13x^2$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $|f(x) - f(y)| \geq |x - y|$ for all x, y. Show that f is both injective and surjective.

3. Given $0 < \alpha < \beta$, find
$$\lim_{\lambda \to 0} \left(\int_0^1 (\beta x + \alpha(1-x))^\lambda dx \right)^{1/\lambda}.$$

4. Curves A, B, C, D are defined in the plane as follows:
$$\begin{aligned} A &= \left\{ (x,y) : x^2 - y^2 = \frac{x}{x^2+y^2} \right\} \\ B &= \left\{ (x,y) : 2xy + \frac{y}{x^2+y^2} = 3 \right\} \\ C &= \left\{ (x,y) : x^3 - 3xy^2 + 3y = 1 \right\} \\ D &= \left\{ (x,y) : 3x^2y - 3x - y^3 = 0 \right\}. \end{aligned}$$
Prove that $A \cap B = C \cap D$.

5. Compute
$$\int \frac{x + \sin x - \cos x - 1}{x + e^x + \sin x} dx.$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $g(x) = f(x) \int_0^x f(t)dt$. Show that if g is non-increasing then f must be the identically 0 function.

7. Compute
$$\int_0^{\pi/2} \frac{dx}{1 + \tan^{\sqrt{2}}(x)}.$$

8. Let A and B be points on the same branch of the hyperbola $xy = 1$. Let P be a point on the chord AB such that the triangle APB has largest area. Show that the area bounded by the hyperbola and the chord AP is the same as the area bounded by the hyperbola and the chord BP.

9. Compute
$$\lim_{n \to \infty} \left(\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right).$$

10. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that for every $x \in [0,1]$, the series
$$\sum_{n=1}^{\infty} \frac{f(x^n)}{2^n}$$
converges.

6.2 Week five solutions

1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers satisfying $x^4 + 36 \leq 13x^2$.

   Solution: This was from the 1986 Putnam competition, question A1.

   The answer is 18.

   $x^4+36 \leq 13x^2$ is equivalent to $x^4-13x^2+36 \leq 0$, which is equivalent to $(x^2-4)(x^2-9) \leq 0$, which is equivalent to $4 \leq x^2 \leq 9$, which is equivalent to

   $$-3 \leq x \leq -2 \quad \text{and} \quad 2 \leq x \leq 3.$$

   Now $f(x) = x^3-3x$ is a continuous function with critical points (points of zero derivative) at $3x^2 - 3 = 0$, or $x = \pm 1$. Neither of these are in the range of interest, so to find the maximum on the range of interest, we need only evaluate $f(x)$ at $x = -3, -2, 2$ and 3. It's an easy check that $f(3) = 18$ and this is the largest value among $f(-3), f(-2), f(2)$ and $f(3)$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $|f(x) - f(y)| \geq |x - y|$ for all x, y. Show that f is both injective and surjective.

   Solution: This is from *Putnam and beyond* by Gelca and Andreescu.

   Injectivity: Suppose that $x \neq y$. Then it must be the case that $f(x) \neq f(y)$; for if not, then $0 = |f(x) - f(y)| \geq |x - y| > 0$, a contradiction.

   Surjectivity: For any $y > 0$, we have $|f(0) - f(y)| \geq y$, so

   $$\text{either } f(y) \geq f(0) + y \text{ or } f(y) \leq f(0) - y,$$

   and similarly we have $|f(0) - f(-y)| \geq y$, so

   $$\text{either } f(-y) \geq f(0) + y \text{ or } f(-y) \leq f(0) - y.$$

   Suppose we have $f(y) \geq f(0)+y$ and $f(-y) \geq f(0)+y$. By intermediate value theorem, somewhere in $(0, y)$ f takes on the value $f(0) + y/2$, and somewhere in $(-y, 0)$ it also takes on the value $f(0)+y/2$. This contradicts injectivity. We get a similar contradiction if $f(y) \leq f(0) - y$ and $f(-y) \leq f(0) - y$.

   So either

   $$f(y) \geq f(0) + y \quad \text{and} \quad f(-y) \leq f(0) - y,$$

   or

   $$f(y) \leq f(0) - y \quad \text{and} \quad f(-y) \geq f(0) + y.$$

   In either case, by intermediate value theorem f takes on all values in the interval $[f(0)-y, f(0) + y]$ (and in particular takes them on as the argument runs between $-y$ and y). Since $f(0)+y$ can be made arbitrarily large, and $f(0)-y$ arbitrarily small, by appropriate choice of $y > 0$, we conclude that f takes on all real values.

3. Given $0 < \alpha < \beta$, find
$$\lim_{\lambda\to 0}\left(\int_0^1 (\beta x + \alpha(1-x))^\lambda dx\right)^{1/\lambda}.$$

Solution: This was problem B2 of the 1979 Putnam competition.

Solution due to John Scholes: Making the substitution $t = \beta x + \alpha(1-x)$, so $dt = (\beta - \alpha)dx$, the integral becomes
$$\frac{1}{\beta-\alpha}\int_\alpha^\beta t^\lambda\, dt = \left(\frac{1}{1+\lambda}\right)\left(\frac{\beta^{\lambda+1}-\alpha^{\lambda+1}}{\beta-\alpha}\right).$$
So we need to compute
$$\lim_{\lambda\to 0}\left(\left(\frac{1}{1+\lambda}\right)\left(\frac{\beta^{\lambda+1}-\alpha^{\lambda+1}}{\beta-\alpha}\right)\right)^{1/\lambda}$$
Now
$$\lim_{\lambda\to 0^+}\frac{1}{(1+\lambda)^{1/\lambda}} = \lim_{k\to\infty}\frac{1}{\left(1+\frac{1}{k}\right)^k} = \frac{1}{e},$$
and
$$\lim_{\lambda\to 0^-}\frac{1}{(1+\lambda)^{1/\lambda}} = \lim_{k\to-\infty}\frac{1}{\left(1+\frac{1}{k}\right)^k} = \lim_{\ell\to+\infty}\left(1-\frac{1}{\ell}\right)^\ell = \frac{1}{e},$$
so
$$\lim_{\lambda\to 0}\frac{1}{(1+\lambda)^{1/\lambda}} = \frac{1}{e}.$$
For the the other part of the limit, write
$$\left(\frac{\beta^{\lambda+1}-\alpha^{\lambda+1}}{\beta-\alpha}\right)^{1/\lambda} = e^{\frac{\log\left(\frac{\beta^{\lambda+1}-\alpha^{\lambda+1}}{\beta-\alpha}\right)}{\lambda}}.$$
The exponent is an indeterminate of the form $0/0$ at $\lambda = 0$, so we evaluate the limit of the exponent as $\lambda \to 0$ by an application of L'Hôpital's rule; it is
$$\lim_{\lambda\to 0}\left(\left(\frac{\beta-\alpha}{\beta^{\lambda+1}-\alpha^{\lambda+1}}\right)\left(\frac{(\lambda+1)\beta^\lambda-(\lambda+1)\alpha^\lambda}{\beta-\alpha}\right)\right) = \frac{1}{\beta-\alpha}.$$
So by continuity,
$$\lim_{\lambda\to 0}\left(\frac{\beta^{\lambda+1}-\alpha^{\lambda+1}}{\beta-\alpha}\right)^{1/\lambda} = e^{1/(\beta-\alpha)}.$$
It follows that
$$\lim_{\lambda\to 0}\left(\int_0^1 (\beta x + \alpha(1-x))^\lambda dx\right)^{1/\lambda} = (1/e)e^{1/(\beta-\alpha)}.$$

4. Curves A, B, C, D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\} \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\} \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\} \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

Solution: This was problem A1 on the 1987 Putnam competition.

Solution by John Scholes: Plotting the curves and actually identifying the points of intersection seems hopeless! It's necessary to manipulate the equations so that whenever the first two are simultaneously satisfied, the last two are, and vice-versa.

Write the equations as:

1 $x^2 - y^2 = \frac{x}{x^2+y^2}$

2 $2xy - 3 = -\frac{y}{x^2+y^2}$

3 $x^3 - 3xy^2 + 3y = 1$ and

4 $3yx^2 - 3x - y^3 = 0$.

Then it is easily checked that $x\mathbf{1} - y\mathbf{2}$ gives **3**, and $y\mathbf{1} + x\mathbf{2}$ gives **4**, so definitely it is the case that $A \cap B \subseteq C \cap D$.

Similarly, $(x\mathbf{3} + y\mathbf{4})/(x^2+y^2)$ gives **1** and $(-y\mathbf{3} + x\mathbf{4})/(x^2+y^2)$ gives **2**, so anything in $C \cap D$ is also in $A \cap B$, *except* possibly $(0,0)$. But it is easily checked that $(0,0)$ does not satisfy **4**, so is not in $C \cap D$.

We conclude that $A \cap B = C \cap D$.

Note (given in the official Putnam solutions): Consider the equations in the complex plane by $z^2 = 3i + 1/z$ and $z^3 = 3iz + 1$. The complex $z = 0$ is not a solution to either equation (it doesn't satisfy the second, and makes no sense for the first). So, dividing the second equation by z, we see that the two equations are defining the same three numbers $x + iy$ in the complex plane (three numbers — solutions to a cubic). But if we substitute in $z = x + iy$, expand, and equate real parts with real parts and imaginary parts with imaginary parts, we find that the solution set to the first equation is exactly $A \cap B$, and the solution set to the second equation is exactly $C \cap D$. This gives a hint as to how the problem was constructed!

5. Compute

$$\int \frac{x + \sin x - \cos x - 1}{x + e^x + \sin x} dx.$$

Solution: This was from *Putnam and Beyond* by Gelca and Andreescu.

We have

$$\begin{aligned}
\int \frac{x+\sin x-\cos x-1}{x+e^x+\sin x}dx &= \int \frac{x+e^x+\sin x-\cos x-e^x-1}{x+e^x+\sin x}dx \\
&= \int \left(1-\left(\frac{1+e^x+\cos x}{x+e^x+\sin x}\right)\right) dx \\
&= x-\int \left(\frac{1+e^x+\cos x}{x+e^x+\sin x}\right) dx \\
&= x-\int \frac{du}{u} \quad \text{where } u=x+e^x+\sin x \\
&= x-\log|u| \\
&= x-\log|x+e^x+\sin x|.
\end{aligned}$$

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Define $g(x)=f(x)\int_0^x f(t)dt$. Show that if g is non-increasing then f must be the identically 0 function.

   Solution: This was from the book *Putnam and Beyond* by Gelca and Andreescu.

   Define $h:\mathbb{R}\to\mathbb{R}$ by

$$h(x)=\frac{1}{2}\left(\int_0^x f(t)dt\right)^2.$$

   Notice that, by the fundamental theorem of calculus, h is differentiable and

$$h'(x)=g(x).$$

   Now $g(x)$ is non-increasing and $g(0)=0$, so $g(x)$ is non-negative on $(-\infty,0)$ and non-positive on $(0,\infty)$. But $g=h'$, so this implies that h is non-decreasing on $(-\infty,0)$, and non-increasing on $(0,\infty)$. And $h(0)=0$, while $h(x)\geq 0$ for all x, so it must be the case that $h(x)=0$ for *all* x. This tells us that

$$\int_0^x f(t)dt=0$$

   for all real x; and differentiating with respect to x tells us that $f(x)=0$ for all x.

7. Compute

$$\int_0^{\pi/2}\frac{dx}{1+\tan^{\sqrt{2}}(x)}.$$

   Solution: This was problem A3 of the Putnam Competition from 1980.

   Solution due to John Scholes: The integral evaluates to $\pi/4$.

   For any positive real α, set $f_\alpha(x)=1/(1+\tan^\alpha x)$. We have

$$f_\alpha(\pi/2-x)=\frac{1}{1+\tan^\alpha(\pi/2-x)}=\frac{1}{1+\cot^\alpha x}=\frac{\tan^\alpha x}{1+\tan^\alpha x}=1-f_\alpha(x).$$

So

$$\begin{aligned}
\int_0^{\pi/2} f_\alpha(x)\, dx &= \int_0^{\pi/4} f_\alpha(x)\, dx + \int_{\pi/4}^{\pi/2} f_\alpha(x)\, dx \\
&=(\star) \int_0^{\pi/4} f_\alpha(x)\, dx + \int_0^{\pi/4} f_\alpha(\pi/2 - x)\, dx \\
&= \int_0^{\pi/4} f_\alpha(x)\, dx + \int_0^{\pi/4} (1 - f_\alpha(x))\, dx \\
&= \int_0^{\pi/4} 1\, dx \\
&= \frac{\pi}{4}.
\end{aligned}$$

In particular, when $\alpha = \sqrt{2}$ the result is $\pi/2$.

Explanation of $(\star)$: we calculate

$$\int_0^{\pi/4} f_\alpha(\pi/2 - x)\, dx$$

by making the substitution $u = \pi/2 - x$. We have $du = -dx$, so $dx = -du$; at $x = 0$, $u = \pi/2$; at $x = \pi/4$, $u = \pi/4$; and the integrand $f_\alpha(\pi/2 - x)$ becomes $f_\alpha(u)$. So:

$$\int_0^{\pi/4} f_\alpha(\pi/2 - x)\, dx = -\int_{\pi/2}^{\pi/4} f_\alpha(u)\, du = \int_{\pi/4}^{\pi/2} f_\alpha(u)\, du = \int_{\pi/4}^{\pi/2} f_\alpha(x)\, dx.$$

Note that the $\sqrt{2}$ was a complete red herring(!), just introduced to make sure that the integrand does not have an elementary antiderivative.

8. Let A and B be points on the same branch of the hyperbola $xy = 1$. Let P be a point on the chord AB such that the triangle APB has largest area. Show that the area bounded by the hyperbola and the chord AP is the same as the area bounded by the hyperbola and the chord BP.

   Solution: This was problem A1 on the 2015 Putnam.

   Solution by Kiran Kedlaya: Without loss of generality, assume that A and B lie in the first quadrant with $A = (t_1, 1/t_1)$, $B = (t_2, 1/t_2)$, and $t_1 < t_2$. If $P = (t, 1/t)$ with $t_1 \le t \le t_2$, then the area of triangle APB is

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ t_1 & t & t_2 \\ 1/t_1 & 1/t & 1/t_2 \end{vmatrix} = \frac{t_2 - t_1}{2t_1t_2}(t_1 + t_2 - t - t_1t_2/t).$$

   When t_1, t_2 are fixed, this is maximized when $t + t_1t_2/t$ is minimized, which by AM-GM exactly holds when $t = \sqrt{t_1t_2}$.

The line AP is given by $y = \frac{t_1+t-x}{tt_1}$, and so the area of the region bounded by the hyperbola and AP is

$$\int_{t_1}^{t} \left(\frac{t_1+t-x}{tt_1} - \frac{1}{x} \right) dx = \frac{t}{2t_1} - \frac{t_1}{2t} - \log\left(\frac{t}{t_1}\right),$$

which at $t = \sqrt{t_1t_2}$ is equal to $\frac{t_2-t_1}{2\sqrt{t_1t_2}} - \log(\sqrt{t_2/t_1})$. Similarly, the area of the region bounded by the hyperbola and PB is $\frac{t_2}{2t} - \frac{t}{2t_2} - \log\frac{t_2}{t}$, which at $t = \sqrt{t_1t_2}$ is also $\frac{t_2-t_1}{2\sqrt{t_1t_2}} - \log(\sqrt{t_2/t_1})$, as desired.

Second solution: For any $\lambda > 0$, the map $(x, y) \mapsto (\lambda x, \lambda^{-1}y)$ preserves both areas and the hyperbola $xy = 1$. We may thus rescale the picture so that A, B are symmetric across the line $y = x$, with A above the line. As P moves from A to B, the area of APB increases until P passes through the point $(1, 1)$, then decreases. Consequently, $P = (1, 1)$ achieves the maximum area, and the desired equality is obvious by symmetry. Alternatively, since the hyperbola is convex, the maximum is uniquely achieved at the point where the tangent line is parallel to AB, and by symmetry that point is P.

9. Compute

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}} \right).$$

Solution: This was from *Putnam and Beyond* by Gelca and Andreescu.

The limit is $\pi/6$. We have

$$\sum_{k=1}^{n} \frac{1}{\sqrt{4n^2-k^2}} = \sum_{k=1}^{n} \frac{1}{n\sqrt{4-(k/n)^2}} = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{4-(k/n)^2}}.$$

This is a Riemann sum, specifically for the function $f(x) = 1/\sqrt{4-x^2}$, on the interval $[0, 1]$, with n partitions each of length $1/n$, and evaluating at the right-hand end of each interval. Since f is integrable on $[0, 1]$, and indeed

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \frac{\pi}{6},$$

we get that the limit is $\pi/6$.

10. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that for every $x \in [0, 1]$, the series

$$\sum_{n=1}^{\infty} \frac{f(x^n)}{2^n}$$

converges.

Solution: I found this on Andrei Jorza's Putnam prep class page.

Since f is a continuous function on a bounded closed interval, there is $M > 0$ such that $f|(x)| \leq M$ for all $x \in [0, 1]$. So

$$\sum_{n=1}^{\infty} \frac{|f(x^n)|}{2^n}$$

(a sum of positive terms) converges — the partial sums form an increasing sequence, bounded above by

$$\sum_{n=1}^{\infty} \frac{M}{2^n} = 2M.$$

So, for all x, $\sum_{n=1}^{\infty} \frac{f(x^n)}{2^n}$ is absolutely convergent, and so convergent.

7 Week six (October 1) — Modular arithmetic and the greatest common divisor

Modular arithmetic is something that everyone (not just mathematicians), are familiar with from a very early age, though maybe not in a formal way. For example, whenever you observe something like "it is 11 o'clock now, so in three hours time it will two o'clock", you are performing addition modulo 12, saying "$11+3=2$". In this section we will formalize modular arithmetic, and present numerous properties and applications that highlight its usefulness.

Modular arithmetic

For integers a and b, and positive integer k, say that *a is congruent to b (modulo k)*, written "$a \equiv b \pmod{k}$", if a and b leave the same remainder on division by k, or equivalently if $a-b$ is a multiple of k, or equivalently if $a-b=mk$ for some integer m. Congruence (modulo k) is an equivalence relation on the integers, that partitions $\mathbb{Z}$ into k classes, called residue classes. For example, when $k=3$ the three classes are $\{\ldots,-6,-3,0,3,6,\ldots\}$, $\{\ldots,-5,-2,1,4,7,\ldots\}$ and $\{\ldots,-4,-1,2,5,8,\ldots\}$.

Many of the standard arithmetic operations go through unchanged to modular arithmetic. For example, it is easy to establish that if

$$a \equiv b \pmod{k} \quad \text{and} \quad c \equiv d \pmod{k}$$

then each of

$$\begin{aligned} a+c &\equiv b+d \pmod{k} \\ a-c &\equiv b-d \pmod{k} \\ ac &\equiv bd \pmod{k} \end{aligned}$$

hold. Repeated application of this last relation also quickly gives that for all positive numbers n,

$$a^n \equiv b^n \pmod{k}.$$

Modular arithmetic can be a great time-saver when working with problems concerning divisibility. We give a quick and useful example.

Claim: The remainder of any integer, on division by 9, is the same as the remainder of the sum of its digits on division by 9.

Proof: Write the number in decimal form as $\sum_{i=0}^n a_i 10^i$ (with each $a_i \in \{0,\ldots,9\}$). Since $10 \equiv 1 \pmod{9}$, we immediately have $10^i \equiv 1^i \equiv 1 \pmod{9}$, and so $a_i 10^i \equiv a_i \pmod{9}$, and so $\sum_{i=0}^n a_i 10^i \equiv \sum_{i=0}^n a_i \pmod{9}$, which is exactly what we wanted to show.

Here are three more quick examples illustrating how modular arithmetic can make life easy:

Question: What are the last two digits of 3^{72}?

Answer: We are being asked: what number x between 0 and 99 is such that $3^{72} \equiv x \pmod{100}$? By repeated squaring we have

$$\begin{aligned} 3 &\equiv 3 \pmod{100} \\ 3^2 &\equiv 3^2 \equiv 9 \pmod{100} \\ 3^4 &\equiv 9^2 \equiv 81 \pmod{100} \\ 3^8 &\equiv 81^2 \equiv 6561 \equiv 61 \pmod{100} \\ 3^{16} &\equiv 61^2 \equiv 3721 \equiv 21 \pmod{100} \\ 3^{32} &\equiv 21^2 \equiv 441 \equiv 41 \pmod{100} \\ 3^{64} &\equiv 41^2 \equiv 1681 \equiv 81 \pmod{100} \end{aligned}$$

and so

$$3^{72} \equiv 3^{64}3^8 \equiv 81 \cdot 61 \equiv 4941 \equiv 41 \pmod{100},$$

so the last two digits of 3^{72} are 41.

Problem: Prove that $2^{70} + 3^{70}$ is divisible by 13.

Solution: We could use the same technique as above to discover that $2^{70} \equiv 10 \pmod{13}$ and $3^{70} \equiv 3 \pmod{13}$ so that $2^{70} + 3^{70} \equiv 10 + 3 \equiv 0 \pmod{13}$. But there is a much easier way: $2^2 \equiv -3^2 \pmod{13}$, so $2^{70} \equiv (-1)^{35}3^{70} \equiv -3^{70} \pmod{13}$, so $2^{70} + 3^{70} \equiv 0 \pmod{13}$.

Problem: Find all integers x, y satisfying $x^2 - 5y^2 = 6$.

Solution: Some experimentation shows that no small numbers x and y work. We might suspect, then, that the equation has no integer solutions. One way to verify this is to work modulo 4. If there was an x and y with $x^2 - 5y^2 = 6$, then for that x and y we would have $x^2 - 5y^2 \equiv 6 \pmod 4$.

If $x \equiv 0, 1, 2, 3 \pmod 4$ then $x^2 \equiv 0, 1, 0, 1 \pmod 4$, and if $y \equiv 0, 1, 2, 3 \pmod 4$ then $5y^2 \equiv 0, 1, 0, 1 \pmod 4$. So, modulo 4, $x^2 - 5y^2$ is equivalent to one of $-1, 0, 1$, or one of $0, 1, 3$, and so we cannot have $x^2 - 5y^2 \equiv 6 \pmod 4$.

Hence the equation indeed has no solutions.

The greatest common divisor

Closely related to modular arithmetic is the greatest common divisor function in number theory. Here is a brief introduction to some ideas around the greatest common divisor.

1. **Divisibility**: For integers a, b, $a|b$ (a divides b) if there is an integer k with $ak = b$.

   For positive integers a and b, the greatest common divisor of a, b, $\gcd(a, b)$ (sometimes just written (a, b)) is the largest positive number that is a divisor of both a and b (this exists, since 1 is a common divisor, and all common divisors are at most the minimum of a and b). This means that if $d = \gcd(a, b)$, and e is any positive number with $e|a$ and $e|b$, then $e \leq d$; but in fact it turns out that moreover $e|d$. This very useful fact follows easily from looking at the prime factorizations of a and b, see below.

   The least common multiple of a, b, $\mathrm{lcm}(a, b)$ is the smallest positive number f such that $a|f$ and and $b|f$; for any positive number g with $a|g$ and $b|g$ we have $f \leq g$; but in fact, as with gcd, it turns out that we even have $f|g$ in this case.

If $\gcd(a,b) = 1$ (so no factors in common other that 1) then a and b are said to be *coprime* or *relatively prime*.

2. **Primes**: If $p > 1$ only has 1 and p as divisors, it is said to be *prime*; otherwise it is *composite*.

   The fundamental fact about prime numbers (other that there are infinitely many of them!) is that every number $n > 1$ has a *prime factorization*:

   $$n = p_1^{a_1} \dots p_k^{a_k}$$

   with each p_i a prime, and each $a_i > 0$. Moreover, the factorization is *unique* if we assume that $p_1 < \dots < p_k$.

   The prime factorization gives one way (not the most computationally efficient way) of accessing $\gcd(a,b)$ and $\mathrm{lcm}(a,b)$. Indeed, if

   $$a = p_1^{a_1} \dots p_k^{a_k} \quad \text{and} \quad b = p_1^{b_1} \dots p_k^{b_k}$$

   (with some of the a_i and b_i possibly 0), then

   $$\gcd(a,b) = p_1^{\min(a_1,b_1)} \dots p_k^{\min(a_k,b_k)} \quad \text{and} \quad \mathrm{lcm}(a,b) = p_1^{\max(a_1,b_1)} \dots p_k^{\max(a_k,b_k)}.$$

   Using $\min(x,y) + \max(x,y) = x + y$, we get the nice identity

   $$ab = \gcd(a,b)\mathrm{lcm}(a,b).$$

   Since any common divisor of a and b must be of the form $\prod_{i=1}^{k} p_i^{\gamma_i}$ for some γ_i's satisfying $\gamma_i \leq \min(a_1,b_1)$, we quickly get the fact, alluded to earlier, that if $d = \gcd(a,b)$ and e is a common divisor of a and b, then not only do we we have $e \leq d$ but also $e|d$.

3. **Euclidean algorithm**: Euclid described a simple way to compute $\gcd(a,b)$. Assume $a > b$. Write

   $$a = kb + j$$

   where $0 \leq j < b$. If $j = 0$, then $\gcd(a,b) = b$. If $j > 0$, then it is fairly easy to check that $\gcd(a,b) = \gcd(b,j)$. Repeat the process with the smaller pair b, j, and keep repeating as long as necessary. For example, suppose I want $\gcd(63,36)$:

   $$\begin{aligned} 63 &= 1.36 + 27 \\ 36 &= 1.27 + 9 \\ 27 &= 3.9. \end{aligned}$$

   We conclude $9 = \gcd(27,9) = \gcd(36,27) = \gcd(63,36)$.

4. **Bézout's Theorem**: Given a, b, there are integers x, y such that $ax + by = \gcd(a,b)$. Moreover, the set of numbers that can be expressed in the form $ax' + by' = c$ for integers x', y' is exactly the set of multiples of $\gcd(a,b)$.

The proof comes from working the Euclidean algorithm backwards. I'll just do an example, with the pair $63, 36$. We have

$$\begin{aligned} 9 &= 36 - 1.27 \\ &= 36 - 1(63 - 1.36) \\ &= -1.63 + 2.36 \end{aligned}$$

so we can take $x = -1$ and $y = 2$.

Once we have found an x and y, the rest is easy. Suppose $c = k\gcd(a, b)$ is a multiple of $\gcd(a, b)$; then $(kx)a + (ky)b = c\gcd(a, b)$. On the other hand, if $ax' + by' = c$ then since $\gcd(a, b)|a$ and $\gcd(a, b)|$ we have $\gcd(a, b)|c$, so c is a multiple of $\gcd(a, b)$.

The most common form of Bézout's Theorem is that if $(a, b) = 1$ then every integer k can be written as a linear combination of a and b; in particular there is x, y with $ax + by = 1$.

In the language of modular arithmetic, this says that if $(a, k) = 1$, then there is a number x such that $ax \equiv 1 \pmod{k}$. We may think of x as an inverse of a (modulo k); this is a starting point for thinking about division in modular arithmetic.

5. **Useful facts/theorems concerning modular arithmetic**:

   (a) **Inverses** (repeating a previous observation): If p is a prime, and $a \not\equiv 0 \pmod{p}$, then there is a whole number b such that $ab \equiv 1 \pmod{p}$; more generally if a and k are coprime then there is a whole number b such that $ab \equiv 1 \pmod{k}$.

   (b) **Fermat's theorem**: If p is a prime, and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.

   More generally, for arbitrary m (prime or composite) define $\varphi(m)$ to be the number of numbers in the range 1 through m that are coprime with m. If $(a, m) = 1$ then $a^{\varphi(m)} \equiv 1 \pmod{m}$. (When $m = p$ this reduces to Fermat's Theorem.)

   We refer to φ as *Euler's totient function*. A quick way to calculate its value: if m has prime factorization $m = p_1^{a_1} \dots p_k^{a_k}$, then

   $$\varphi(m) = m \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

   (c) **Chinese Remainder Theorem**: Suppose $n_1, n_2, \dots, n_k$ are pairwise relatively prime. If $a_1, a_2, \dots, a_k$ are any integers, there is a number x that simultaneously satisfies $x \equiv a_k \pmod{n_k}$. Moreover, modulo $n_1 n_2 \dots n_k$, this solution is unique.

7.1 Week six problems

1. Prove that the product of three consecutive integers is divisible by 504 if the middle one is a perfect cube.

2. Find all integers n such that $(2^n + n)|(8^n + n)$.

3. Compute the sum of the digits of the sum of the digits of the sum of the digits of the number 4444^{4444}.

4. Prove that the expression
$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$
is an integer for all pairs of integers $n \geq m \geq 1$.

5. Let $a \geq b \geq 0$ be integers and let p be a prime number. Show that $\binom{pa}{pb}$ and $\binom{a}{b}$ are congruent modulo p.

6. Several positive integers are written on a chalk board. One can choose two of them, erase them, and replace them with their greatest common divisor and least common multiple. Prove that eventually the numbers on the board do not change.

7. Is it possible to place 2019 integers on a circle such that for every pair of adjacent numbers the ratio of the larger one to the smaller one is a prime?

8. How many primes numbers have the following (decimal) form: digits alternating between 1's and 0's, beginning and ending with 1?

9. Define a sequence recursively by: $u_1 = 1$, $u_2 = 2$, $u_3 = 24$ and
$$u_n = \frac{6u_{n-1}^2 u_{n-3} - 8u_{n-1}u_{n-2}^2}{u_{n-2}u_{n-3}}.$$

   (a) Solve for u_n, and show that u_n is always an integer.

   (b) Show moreover that u_n is always a multiple of n.

7.2 Week six solutions

1. Prove that the product of three consecutive integers is divisible by 504 if the middle one is a perfect cube.

   Solution: I found this problem on the Putnam prep page at Kenyon College.

   Let the middle integer be m^3 where m is an integer. Then the product of the three integers is

   $$(m^3-1)m^3(m^3+1) = (m-1)(m^2+m+1)m^3(m+1)(m^2-m+1).$$

   The prime factorization of 504 is $504 = 2^3 \times 3^2 \times 7$.

   We first show that $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$ by looking at m modulo 7.

   - If $m \equiv 0$ (modulo 7) then clearly $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 1$ (modulo 7) then $m-1 \equiv 0$ (modulo 7) and $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 2$ (modulo 7) then $m^2+m+1 \equiv 0$ (modulo 7) and $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 3$ (modulo 7) then $m^2-m+1 \equiv 0$ (modulo 7) and $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 4$ (modulo 7) then $m^2+m+1 \equiv 0$ (modulo 7) and $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 5$ (modulo 7) then $m^2-m+1 \equiv 0$ (modulo 7) and $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 6$ (modulo 7) then $m+1 \equiv 0$ (modulo 7) and $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.

   Next we show that $3^2|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$ by looking at m modulo 3.

   - If $m \equiv 0$ (modulo 3) then $3^2|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$ because of the m^3 factor.
   - If $m \equiv 1$ (modulo 3) then $m-1 \equiv 0$ (modulo 3) and $m^2+m+1 \equiv 0$ (modulo 3), so $3^2|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
   - If $m \equiv 2$ (modulo 3) then $m+1 \equiv 0$ (modulo 3) and $m^2-m+1 \equiv 0$ (modulo 3), so $7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.

   Finally we show that $2^3|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$ by looking at m modulo 2, and modulo 4.

   - If $m \equiv 0$ (modulo 2) then $2^3|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$ because of the m^3 factor.

- If $m \equiv 1$ (modulo 2) and $m \equiv 1$ (modulo 4) then $m-1 \equiv 0$ (modulo 4) and $m+1 \equiv 0$ (modulo 2), so $2^3|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.
- If $m \equiv 1$ (modulo 2) and $m \equiv 3$ (modulo 4) then $m-1 \equiv 0$ (modulo 2) and $m+1 \equiv 0$ (modulo 4), so $2^3|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$.

We conclude that $2^3 3^2 7|(m-1)(m^2+m+1)m^3(m+1)(m^2-m+1)$, as required.

2. Find all integers n such that $(2^n+n)|(8^n+n)$.

   Solution: This was on a Putnam prep page from Northwestern.

   The solutions are $n = 0, 1, 2, 4$ and 6.

   We may assume $n \geq 0$: note that for $n \leq -1$, neither $2^n + n$ nor $8^n + n$ are integers; and also 2^n+n and 8^n+n are both positive, with $2^n+n > 8^n+n$, so even with a broad interpretation of "divides", no integer below 0 will work.

   We have $8^n + n = 2^n 4^n + n = (2^n+n)4^n - n4^n + n$, so if 2^n+n divides 8^n+n, then

$$(2^n+n)|(n4^n-n).$$

   We have $n4^n - n = n2^n 2^n - n = (2^n+n)n2^n - n^2 2^n - n$, so if 2^n+n divides $n4^n - n$, then

$$(2^n+n)|(n^2 2^n+n).$$

   We have $n^2 2^n + n = (2^n+n)n^2 - n^3 + n$, so if 2^n+n divides $n^2 2^n + n$, then

$$(2^n+n)|(n^3-n).$$

   It is an easy (but slightly tedious, so omitted) induction that for $n \geq 10$, $2^n+n > n^3-n$, so we conclude that if 2^n+n divides 8^n+n, then $n \leq 9$.

   It is another tedious but easy check that $n = 0, 1, 2, 4$ and 6 all lead to integers, but not any other $n \leq 9$.

3. Compute the sum of the digits of the sum of the digits of the sum of the digits of the number 4444^{4444}.

   Solution: International Mathematical Olympiad 1975, problem 4. Appropriately enough, if you had gotten this question fully correct at the IMO, you would have scored 7 points!

   The answer is 7.

   We start with

$$4444^{4444} < 10000^{10000} = 10^{40000}.$$

   Among all numbers below 10^{40000}, none has a larger sum of digits than $10^{40000}-1$ (a string of 40000 9's). So the sum of the digits of 4444^{4444} is at most $9 \times 40000 < 1000000$. Among all numbers below 1000000, none has a larger sum of digits than 999999. So the sum of the digits of the sum of the digits of 4444^{4444} is at most 54. Among all numbers at most 54, none has a larger sum of digits than 49. So the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} is at most 13.

Now we use a useful fact: the remainder of a number, on division by 9, is the same as the remainder of the sum of the digits on division by 9. . This fact implies that the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} leaves the same remainder on division by 9 as 4444^{4444} itself does.

To calculate the remainder of 4444^{4444} on division by 9, we can use a repeated multiplication trick. It's easy that

$$4444 \equiv 7 \pmod 9$$

and so

$$\begin{aligned}
4444^2 &\equiv 49 \equiv 4 \pmod 9\\
4444^4 &\equiv 16 \equiv 7 \pmod 9\\
4444^8 &\equiv 49 \equiv 4 \pmod 9\\
4444^{16} &\equiv 16 \equiv 7 \pmod 9\\
4444^{32} &\equiv 49 \equiv 4 \pmod 9\\
4444^{64} &\equiv 16 \equiv 7 \pmod 9\\
4444^{128} &\equiv 49 \equiv 4 \pmod 9\\
4444^{256} &\equiv 16 \equiv 7 \pmod 9\\
4444^{512} &\equiv 49 \equiv 4 \pmod 9\\
4444^{1024} &\equiv 16 \equiv 7 \pmod 9\\
4444^{2048} &\equiv 49 \equiv 4 \pmod 9\\
4444^{4096} &\equiv 16 \equiv 7 \pmod 9).
\end{aligned}$$

It follows that

$$4444^{4444} = 4444^{4096}4444^{256}4444^{64}4444^{16}4444^{8}4444^{4} \equiv 7.7.7.7.4.7 \equiv 7 \pmod 9).$$

So 4444^{4444} leaves a remainder of 7 on division by 9, and also the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} leaves a remainder of 7 on division by 9; but we've calculated that this last is at most 13. The only number at most 13 that leaves a remainder of 7 on division by 9 is 7 it self; so the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} must be 7.

4. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Solution: This was from the 2000 Putnam competition, problem B2.

We know that $\gcd(m,n) = am + bn$ for some integers a, b; but then

$$\frac{\gcd(m,n)}{n}\binom{n}{m} = a\left(\frac{m}{n}\binom{n}{m}\right) + b\left(\frac{n}{n}\binom{n}{m}\right).$$

Since

$$\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$$

(the "committee-chair" identity, or easy algebra), we get

$$\frac{\gcd(m,n)}{n}\binom{n}{m} = a\binom{n-1}{m-1} + b\binom{n}{m},$$

and so (since a, b, $\binom{n-1}{m-1}$ and $\binom{n}{m}$ are all integers) we get the desired result.

5. Let $a \geq b \geq 0$ be integers and let p be a prime number. Show that $\binom{pa}{pb}$ and $\binom{a}{b}$ are congruent modulo p.

Solution: This was from the 1977 Putnam competition, Problem A5.

Solution from John Scholes. Denote by $f(n)$ the highest power of p dividing n (so, e.g., $f(2^3 5^8 p^7) = p^7$, if $p \neq 2, 5$). The multiples of p in $(pa)!$ are pa, $p(a-1)$, ..., $2p$, and p. Hence $f((pa)!) = p^a f(a!)$. Similarly, $f((pb)!) = p^b f(b!)$ and $f((p(a-b))!) = p^{a-b} f((a-b)!)$. Hence

$$f\left(\binom{a}{b}\right) = f\left(\binom{pa}{pb}\right).$$

This says that $\binom{pa}{pb} - \binom{a}{b}$ can be expressed as xp^y where x and y are non-negative integers, and x is not divisible by p. If $y > 0$, this gives the result.

I'm not sure what happens for this line of attack if $y = 0$.

Here is the solution as posted in the American Mathematical Monthly shortly after the 1977 competition:

A-5. (66, 24, 9, 7, 1, 1, 1, 0, 5, 1, 22, 49)

It is well known that $\binom{p}{i} \equiv 0 \bmod p)$ for $i = 1, 2, \cdots, p-1$ or equivalently that in $Z_p[x]$ one has $(1+x)^p = 1 + x^p$, where Z_p is the field of the integers modulo p. Thus in $Z_p[x]$,

$$\sum_{k=0}^{pa} \binom{pa}{k} x^k = (1+x)^{pa} = \left[(1+x)^p\right]^a = [1+x^p]^a = \sum_{j=0}^{a} \binom{a}{j} x^{jp}.$$

Since coefficients of like powers must be congruent modulo p in the equality

$$\sum_{k=0}^{pa} \binom{pa}{k} x^k = \sum_{j=0}^{a} \binom{a}{j} x^{jp}$$

in $Z_p[x]$, one sees that

$$\binom{pa}{pb} \equiv \binom{a}{b} (\bmod p)$$

for $b = 0, 1, \ldots, a$.

6. Several positive integers are written on a chalk board. One can choose two of them, erase them, and replace them with their greatest common divisor and least common multiple. Prove that eventually the numbers on the board do not change.

Soution: I found this problem on a Stanford Putnam prep class page.

Here's a very quick, slick solution shown to me by Do Trong Thanh: If you pick two numbers a, b with $a|b$ or $b|a$, then since $\gcd(a, b) = \min\{a, b\}$ and $\mathrm{lcm}(a, b) = \max\{a, b\}$ in this case, the numbers do not change. In general $\gcd(a, b)|\mathrm{lcm}(a, b)$, so if it is not the case that $a|b$ or $b|a$, then after the swap it *is* the case for that particular pair. Initially there are only finitely many pairs (a, b) with $a \not| b$ and $b \not| a$; either eventually we replace all these pairs with pairs of which one divides to other (in which case we are done), or we eventually commit to avoiding all remaining such pairs (in which case we are done).

Here's my laborious solution: When we take a pair of numbers (a, b), and replace them with $(\gcd(a, b), \mathrm{lcm}(a, b))$, we preserve something, namely the product of the pair of numbers (that $ab = \gcd(a, b)\mathrm{lcm}(a, b)$ is easily seen from the prime factorization of a and b: if

$$a = \prod_{i=1}^{n} p_i^{a_i}, \quad b = \prod_{i=1}^{n} p_i^{b_i}$$

(with maybe some of the a_i, b_i zero) then

$$ab = \prod_{i=1}^{n} p_i^{a_i+b_i},$$

$$\gcd(a, b) = \prod_{i=1}^{n} p_i^{\min\{a_i,b_i\}}, \quad \mathrm{lcm}(a, b) = \prod_{i=1}^{n} p_i^{\max\{a_i,b_i\}},$$

so

$$\gcd(a, b)\mathrm{lcm}(a, b) = \prod_{i=1}^{n} p_i^{\min\{a_i,b_i\}+\max\{a_i,b_i\}}.$$

Thus $ab = \gcd(a, b)\mathrm{lcm}(a, b)$ follows from $x + y = \min\{x, y\} + \max\{x, y\}$, valid for any positive integers x, y.)

For any fixed positive number, there are only finitely many ways to write it as the product of a fixed number of positive numbers (if the target of the product is N, and we are using d numbers, then each of the d numbers must be a divisor of N, so the number of ways of writing N as a product of d terms is at most $a(N)^d$, where $a(N)$ is the number of divisors of N). This shows that there are only finitely many possibilities for the numbers written on the board.

Consider the sum of the numbers. How does this change with the swap operation? It depends on how $a + b$ compares to $\gcd(a, b) + \mathrm{lcm}(a, b)$. Experimentation suggests that $\gcd(a, b) + \mathrm{lcm}(a, b) \geq a + b$, with equality iff the pair (a, b) coincides (in some order) with the pair $(\mathrm{lcm}(a, b), \gcd(a, b))$. To prove this, first consider $a = b$, for which the result is trivial. For all other cases, assume without loss of generality that $a > b$. We have

$$\mathrm{lcm}(a, b) \geq a > b \geq \gcd(a, b).$$

If any one of $a = \mathrm{lcm}(a, b)$, $b = \gcd(a, b)$ holds then by the conservation of product the other must too, and the result we are trying to prove is true. So now we may assume

$$\mathrm{lcm}(a, b) > a > b > \gcd(a, b),$$

and what we want to show is that this implies $\gcd(a,b) + \operatorname{lcm}(a,b) > a + b$. Let $n = ab = \gcd(a,b)\operatorname{lcm}(a,b)$, so

$$\gcd(a,b) + \operatorname{lcm}(a,b) = \gcd(a,b) + \frac{n}{\gcd(a,b)} \quad \text{and} \quad a + b = b + \frac{n}{b}.$$

A little calculus shows that the function $f(x) = x + n/x$ is *decreasing* on the interval $(0, \sqrt{n}]$. Since $\gcd(a,b) < b < \sqrt{n}$, this shows that

$$\gcd(a,b) + \frac{n}{\gcd(a,b)} > b + \frac{n}{b}$$

which is exactly what we want to show.

So, suppose we have the bunch of numbers in front of us, and we perform the swap operation infinitely often. All swaps preserve the product. Some swaps also preserve the sum; these swaps are exactly the swaps that don't change the set of numbers. All other swaps increase the sum. We can only increase the sum finitely many times (there are only finitely many different configurations of numbers). Therefore there must be some point (curiously, not boundable as a function of the original numbers!) after which we make no more sum-increasing swaps; from that point on, the numbers remain unchanged.

7. Is it possible to place 2019 integers on a circle such that for every pair of adjacent numbers the ratio of the larger one to the smaller one is a prime?

   Solution: I found this on Andrei Jorza's webpage, from his 2018 Putnam prep class.

   No. We argue by contradiction.

   Suppose it were possible. Consider two consecutive numbers on the circle, a and b. Either $b = pa$ for some prime p, in which case label the arc of the circle between a and b "p" (and call the arc an UP arc), or $a = pb$, in which case label the arc "$1/p$" (and call it a DOWN arc).

   Starting from a particular (arbitrarily chosen) number, A, say, on the circle, the number one step away clockwise from A is A multiplied by the label on the arc of the circle between A and that number one step away clockwise. In general, the number k steps away from A (clockwise) is A multiplied by all the labels encountered along those k arcs. So the number 2019 steps away from A (clockwise) is A multiplied by all the labels on the arcs (since 2019 steps takes us all the way around the circle).

   But this last number is A itself. So we have an equation:

$$A = A \times \frac{\text{product of bunch of primes — the primes on the UP arcs}}{product of bunch of primes --- the primes on the DOWN arcs}$$

   or

$$\text{product of primes on UP arcs} = \text{product of primes on DOWN arcs.}$$

   But there are 2019 arcs, and odd number, so one side of the above equation has an odd number of primes in it, and the other side has an even number, contradicting the fundamental theorem of arithmetic.

   Notice that all we used here was that 2019 is odd.

8. How many primes numbers have the following (decimal) form: digits alternating between 1's and 0's, beginning and ending with 1?

   Solution: This was from an NYU Putnam prep class webpage.

   The only prime of this form is 101.

   The number $x_n = 1010\ldots101$, with n 0's, can be written as

   $$1 + 100 + 1000 + 1000000 + \ldots + 1000\ldots000 = 1 + (100) + (100)^2 + \ldots + (100)^n,$$

   in other words, $x_n = P_n(100)$ where $P_n(x)$ is the polynomial $1 + x + x^2 + \ldots + x^n$.

   We want to know for which n the polynomial $P_n(x)$ is prime for $x = 100$.

   For $n = 0$, it is not ($P_0(100) = 1$), and for $n = 1$ it is ($P_1(100) = 101$). So we assume $n \geq 2$.

   Since $(x - 1)(1 + x + x^2 + \ldots + x^n) = (x^{n+1} - 1)$, we have

   $$99P_n(100) = 100^{n+1} - 1 = 10^{2(n+1)} - 1 = \left(10^{n+1}\right)^2 - 1 = \left(10^{n+1} - 1\right)\left(10^{n+1} + 1\right).$$

   What happens if $P_n(100)$ is prime? It must divide one of $10^{n+1} - 1$, $10^{n+1} + 1$. But, for $n \geq 2$,

   $$P_n(100) = 1 + (100) + (100)^2 + \ldots + (100)^n > 1 + 10^{2n} > 1 + 10^{n+1},$$

   so $P_n(100)$ is too big to divide either $10^{n+1} + 1$ or $10^{n+1} - 1$. Hence for $n \geq 2$, $P_n(100)$ can't be prime.

   The conclusion is that the only prime of the given form is 101.

9. Define a sequence recursively by: $u_1 = 1$, $u_2 = 2$, $u_3 = 24$ and

   $$u_n = \frac{6u_{n-1}^2 u_{n-3} - 8u_{n-1}u_{n-2}^2}{u_{n-2}u_{n-3}}.$$

   (a) Solve for u_n, and show that u_n is always an integer.

   (b) Show moreover that u_n is always a multiple of n.

   Solution: This was from the 1999 Putnam competition, problem A6.

   Here is a screenshot of a solution by Prof. W. Kahan, UC Berkeley:

Problem A6: The sequence $\{a_n\}_{n\ge1}$ is defined by $a_1 := 1$, $a_2 := 2$, $a_3 := 24$, and $a_n := \dfrac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$ for $n \ge 4$. Show that every a_n is an integer multiple of n .

Solution A6: Substitute $r_n := a_n/a_{n-1}$ into the given recurrence to obtain $r_2 = 2$, $r_3 = 12$ and $r_n = 6r_{n-1} - 8r_{n-3}$. This is a linear recurrence whose characteristic polynomial $r^2 - 6r + 8$ factors into $(r-2)(r-4)$; from this soon follows that $r_n = 4^{n-1} - 2^{n-1} = (2^{n-1} - 1)2^{n-1}$ and then $a_n = \prod_{2\le k\le n} r^k = \prod_{2\le k\le n} (2^{k-1} - 1)2^{k-1} = 2^{(n-1)n/2}\cdot\prod_{2\le k\le n} (2^{k-1} - 1)$. Next we shall prove that n divides this a_n by showing that every prime power that divides n divides a_n too.

If n is divisible by 2^m for some $m > 0$ then $m \le n-1$ (because $2^n \ge n+1$), and therefore a_n is divisible by 2^m too. If n is divisible by p^m for some odd prime p and integer $m > 0$, then again $m \le n-1$ (because $p^n > 2^n \ge n+1$), and each of the m integers $k = p, p^2, \ldots, p^m$ appears among the consecutive integers $k = 2, 3, \ldots, n$. Now we appeal to Fermat's "little" theorem to the effect that $L^{p-1} - 1$ is divisible by prime p whenever integer L is not divisible by p . Apply this for $L = 2^{(k-1)/(p-1)}$ whenever k is a power of p to infer that then $2^{k-1} - 1$ is divisible by p . Therefore, $a_n = 2^{(n-1)n/2}\cdot\prod_{2\le k\le n} (2^{k-1} - 1)$ has $n-1$ odd factors $(2^{k-1} - 1)$ of which at least m are divisible by p , so p^m divides a_n too. Since every prime power that divides n divides a_n too, n must divide a_n . End of proof.

8 Week seven (October 8) — Graphs

A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*, each of which is a two-element subset of V. Think of the vertices as points put down on a piece of paper, and of the edges as arcs joining pairs of points. There is no inherent geometry to a graph — all that matters is which pairs of points are joined, not the exact position of the points, or the nature of the arcs joining them.

Thinking about the data of a problem as a graph can sometimes be helpful. Although some Putnam problems in the past have been non-trivial results from graph theory in disguise, there is no real need to know much graph theory, so in this discussion I'll just mention some basic ideas that might be useful. A little more background on graph theory can be found, for example, at `http://www.math.ucsd.edu/~jverstra/putnam-week6.pdf`.

Problem: n people go to a party, and each one counts up the number of other people she knows at the party. Show that there are an even number of people who come up with an answer that is an odd number. (Assuming that "knowing" is a two-way relation; I know you if and only if you know me.)

Solution: Model the problem as a graph. V is the set of n party goers, and E consists of all pairs of people who know each other. For person i, denote by d_i the number of edges that involve i (d_i is the *degree* of vertex i). We have

$$\sum_{i=1}^{n} d_i = 2|E|$$

since as we run over all vertices and count degrees, each edge gets counted exactly twice (once for each vertex in that edge). So the sum of degrees is even. But if there were an odd number of vertices with odd degree, the sum would be odd; so there are indeed an even number of people who know an odd number of people.

The useful fact that is true about all graphs that lies at the heart of the solution is this: in any graph $G = (V, E)$,

$$\sum_{i=1}^{n} d_i = 2|E|.$$

Problem: Show that two of the people at the party have the same number of friends.

Solution: The possible values for d_i are 0 through $n-1$, n of them, so the pigeon-hole principle doesn't immediately apply. But: it's not possible for there to be one vertex with degree 0, and another with degree $n-1$. So the possible values of d_i are either 1 through $n-1$, $n-1$ of them, or 0 through $n-2$, $n-1$ of them, and in either case the pigeon-hole principle gives that there are two people with the same number of friends.

The useful fact that is true about all graphs that lies at the heart of the solution is this: in any graph $G = (V, E)$, there must be two vertices with the same degree.

A *walk* in a graph from vertex u to vertex v is list of (not necessarily distinct) edges, with u in the first, v in the last, and every pair of consecutive edges sharing a vertex in common — graphically, a walk is a way to trace a path from u to v, always using complete arcs of the drawing, and never taking pencil of paper.

Problem: How many different walks are there from u to v, that use k edges?

Solution: Form the *adjacency matrix* of the graph: rows and columns indexed by vertices, entry (a, b) is 1 if $\{a, b\}$ is an edge, and 0 otherwise. Then form the matrix A^k. The (u, v) entry of this matrix is exactly the number of different walks from u to v, that use k edges. The proof uses induction of k, and the definition of matrix multiplication. The key point is that the number of walks from u to v that use k edges is the sum, over all neighbours w of u (i.e., vertices w such that $\{u, w\}$ is an edge), of the number of walks from w to v that use $k - 1$ edges. I'll skip the details.

The relation "there is a walk between" is an equivalence relation on vertices, so any graph can be partitioned into equivalence classes, with each class having the property that between any two vertices in the class, there is a walk, but there is no walk between any two vertices in different classes. These classes are called *components* of the graph. If a graph has just one component, meaning that between any two vertices in the graph, there is a walk, it is said to be connected.

Problem: Given a graph G, under what circumstances is it possible to take a walk that uses every edge of the graph exactly once, and ends up at the same vertex that it started at?

Solution: Such a walk is called an *Euler circuit*, after the man who first studied them (google "Bridges of Konigsberg"). Such a circuit is a tracing of the graphical representation of the graph, with each arc traced out exactly once, the pencil never leaving the paper, ending where it started. Two fairly obvious necessary conditions for the existence of an Euler circuit are:

- the graph is connected, and
- every degree is even (because each time an Euler circuit visits a vertex, it eats up two edges — one going in and one coming out).

Euler proved that these necessary conditions are sufficient: a connected graph has an Euler circuit if and only if all vertex degrees are even. The details are given in any basic graph theory textbook.

What if we don't require the tracing to end at the same vertex it began?

Problem: Given a graph G, and two distinct vertices u and v, under what circumstances is it possible to take a walk from u to v that uses every edge of the graph exactly once?

Solution: Such a walk is called an *Euler trail.* Euler proved that a connected graph has an Euler trail from u to v if and only if all vertex degrees are even except the degrees of u and v, which must be odd. It follows easily from his result on Euler circuits: just add an edge from u to v, apply the Euler trails theorem, and delete the added edge.

Problem: Given a graph G with n vertices, under what circumstances is it possible to list the vertices in some order $v_1, \ldots, v_n$, in such a way that each of $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$ are all edges?

Solution: Such a list is called a *Hamiltonian cycle*, after the man who first studied them (google "icosian game"). Unlike with Eulerian trials, there is no simple set of necessary-and-sufficient conditions known to allow one to determine whether such a thing exists in a given graph. There is one useful sufficient condition, due to Dirac, that has an elementary but involved proof that can be found in any graph theory textbook.

Dirac's theorem: A graph G with n vertices has a Hamiltonian cycle if all vertices have degree at least $n/2$.

A connected graph with the fewest possible number of edges is called a *tree*. It turns out that all trees on n vertices have the same number of edges, namely $n-1$. One way to see this is to imagine building up the tree from a set of n totally disconnected vertices, edge-by-edge. At each step, you should add an edge that bridges two components, since adding an edge within a component does not help; in the end such an edge can be removed without hurting connectivity. Since two components get merged each time an edge is added, exactly $n-1$ are needed to get to a single component.

A characterization of trees is that they are connected, but have no cycles — a *cycle* in a graph is a list of distinct vertices $u_1, u_2, \ldots, u_k$ such that each of $\{u_1, u_2\}$, $\{u_2, u_3\}$, $\ldots$, $\{u_{k-1}, u_k\}$, $\{u_k, u_1\}$ are all edges.

A *planar* graph is a graph that can be drawn in the plane with no two arcs meeting except at a vertex (if they have one in common). A planar drawing of a graph partitions the plane into connected regions, called *faces*. Euler discovered a remarkable formula that relates the number of vertices, edges and faces in a planar graph:

Euler's formula: Let G be a planar graph with V vertices, E edges and F faces. Then

$$V - E + F = 2.$$

Proof sketch: By induction on F. If $F = 1$ then the graph has only one face, so it must be a tree. A tree on V vertices has $V - 1$ edges, and so fits the formula.

Now suppose $F > 1$. Then the graph contains a cycle. If we remove an edge e of that cycle then F drops by one, V stays the same, and E drops by 1. Now by induction, $V - (E - 1) + (F - 1) = 2$ and this gives $V - E + F = 2$.

Problem: Show that five points can't be connected up with arcs in the plane in such a way that no two arcs meet each other except at a vertex (if they have one in common).

Solution: Suppose such a connection was possible. The resulting planar graph would have 5 vertices and $\binom{5}{2} = 10$ edges, so by Euler's formula would have 7 faces. The sum, over the faces, of the number of edges bounding the faces, is then at least 21, since each faces has at least three bounding edges. But this sum is at most twice the number of edges, since each each edge can be on the boundary of at most two faces; so it is at most 20, a contradiction.

A *bipartite* graph is a graph whose vertex set can be partitioned into two classes, X and Y, such that the graph only has edges that go from X to Y (and so none that are entirely within X or entirely within Y). It's fairly easy to see that any odd-length cycle is *not* bipartite, so any graph that has an odd-length cycle sitting inside it is also not bipartite. This turns out to be a characterization of bipartite graphs; the proof can be found in any textbook on graph theory.

Theorem: A graph is bipartite if an only if it has no odd cycle.

A *matching* in a graph is a set of edges, no two of which share a vertex. A *perfect matching* is a matching that involves all the edges. A famous result, whose proof can be found in any graph theory textbook, is Hall's marriage theorem. A consequence of it says that if there are n women and n men, each women likes exactly d men, and each man is liked by exactly d women, then it is possible to pair the men and women off into n pairs, such that each women is paired with a man she likes. Here's the statement in graph-theory language:

Theorem: Let G be a bipartite graph that is regular (all vertices have the same degree). G has a perfect matching.

8.1 Week seven problems

1. In a town there are three new houses, and each needs to be connected by a line to the gas, water, and electricity factories. The lines are only allowed to run along the ground. Is there a way to make all nine connections without any of the lines crossing each other?

2. n teams play each other in a round-robin tournament (so each team plays each of the other teams exactly once). There are no ties. Show that there exists an ordering of the teams, $(a_1, a_2, a_3, \ldots, a_n)$, such that team a_1 beats team a_2, team a_2 beats team a_3, $\ldots$, team a_{n-1} beats team a_n.

3. An airline operates flights out of $2n$ airports. In all, the airline operates $n^2 + 1$ different routes (all there-and-back: South Bend to Chicago is considered the same route as Chicago to South Bend). Prove that there are three airports a, b, c such that it is possible to fly directly from a to b, then b to c, then c back to a.

4. An airline operates flights out of n airports. In all, the airline operates at least $3n/2$ different routes (all there-and-back, as in the last question). Prove that it is possible the find a collection of $2k$ airports $a_1, a_2, \ldots, a_{2k-1}, a_{2k}$ (for some $k \geq 2$), such that it is possible to fly directly from a_1 to a_2, then a_2 to a_3, wt cetera, then a_{2k-1} to a_{2k}, and then close the loop by flying from a_{2k} back to a_1.

5. The complete graph K_n on vertex set $\{1, \ldots, n\}$ is the graph in which all $\binom{n}{2}$ possible edges are present. Suppose that the edges of K_n are colored with two colors, say Red and Blue (meaning, each edge is either assigned the color Red or the color Blue, but not both). Prove that is possible to partition $\{1, \ldots, n\}$ as $A \cup B$, such that there is a Red path that covers all the vertices in A and a Blue path that covers all the vertices in B. (By this is meant: the elements of A can be ordered as $(a_1, a_2, \ldots, a_\ell)$ in such a way that each of the edges $a_1a_2, a_2a_3, \ldots, a_{\ell-1}a_\ell$ are all colored Red, and similarly for B.)

6. Is there a way to list the 2^n subsets of $\{1, \ldots, n\}$ (with each subset appearing on the list once and only once) in such a way that the first element of the list is the empty set, and every element on the list is obtained from the previous element either by adding an element or deleting an element?

7. Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence $g_1, g_2, g_3, \ldots, g_{2n}$ such that every element of G occurs exactly twice in the sequence, and, for each $i = 1, 2, \ldots, 2n$, g_{i+1} equals g_ia or g_ib. (Interpret g_{2n+1} as g_1.)

8. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an n by n grid so that the k-th row, from left to right, is $(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n$. (So you are writing the numbers in order, starting at the top left and moving left to right along each row, then continuing at the left of the next row down, and so on.)

   Color the squares of the grid so that half of the squares in each row are red (and the other half are black), and half of the squares in each column are red (and the other half are black). (A checkerboard coloring is one possibility, but there are many others).

   Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

8.2 Week seven solutions

1. In a town there are three new houses, and each needs to be connected by a line to the gas, water, and electricity factories. The lines are only allowed to run along the ground. Is there a way to make all nine connections without any of the lines crossing each other?

   Solution: This is a standard result in graph theory.

   The question is asking whether the graph on six vertices, $1, 2, \ldots, 6$, with an edge from i to j if and only if $1 \leq i \leq 3$ and $4 \leq j \leq 6$, can be drawn in the plane without crossing edges. It has 6 vertices and 9 edges, so if it could, any representation would have 5 faces (Euler's formula). Each face is bounded by at least 4 edges (note that the graph we are working with clearly has no triangles), so summing "#(bounding edges)" over all faces, get at least 20. But this sum counts each edge at most twice, so we get at most 18, a contradiction that reveals that there is no such planar representation.

2. n teams play each other in a round-robin tournament (so each team plays each of the other teams exactly once). There are no ties. Show that there exists an ordering of the teams, $(a_1, a_2, a_3, \ldots, a_n)$, such that team a_1 beats team a_2, team a_2 beats team a_3, …, team a_{n-1} beats team a_n.

   Solution: This was on the 1958 Putnam competition, but is also a standard result in graph theory.

   Proof by induction on n, with $n = 1$, and indeed $n = 2$, trivial. So assume $n \geq 3$. Fix an arbitrary vertex x. By induction there exists an ordering of the remaining teams, $(a_1, a_2, a_3, \ldots, a_{n-1})$, such that team a_1 beats team a_2, team a_2 beats team a_3, …, team a_{n-2} beats team a_{n-1}.

   If x beats a_1, then the ordering $(x, a_1, \ldots, a_{n-1})$ works. If x looses to everyone, then the ordering $(a_1, \ldots, a_{n-1}, x)$ works. If neither of these things happen, then there must be an i such that x looses to a_i, but beats a_{i+1}, and then the ordering $(a_1, \ldots, a_i, x, a_{i+1}, \ldots, a_{n-1})$ works.

3. An airline operates flights out of $2n$ airports. In all, the airline operates $n^2 + 1$ different routes (all there-and-back: South Bend to Chicago is considered the same route as Chicago to South Bend). Prove that there are three airports a, b, c such that it is possible to fly directly from a to b, then b to c, then c back to a.

   Solution: This is Mantel's theorem, one of the first results proved in the vast area of extremal graph theory.

   Suppose there were no such three cities a, b, c. For each city x, denote by $d(x)$ the number of cities with direct connection to x. If there is a connection between cities x and y, then there cannot be a third city y directly connected to both (or we would have a triangle), so

   $$(d(x) - 1) + (d(y) - 1) \leq 2n - 2, \qquad d(x) + d(y) \leq 2n.$$

Now in the sum of $d(x) + d(y)$ over all pairs of connected cites, for each x we have that $d(x)$ appears exactly $d(x)$ times (once for each city directly connected to x), so

$$\sum (d(x) + d(y)) = \sum_x d^2(x) \leq (n^2 + 1)2n.$$

Now use the Cauchy-Schwartz-Bunyakovski inequality to bound:

$$2n \left(\sum_x d^2(x) \right) \geq \left(\sum_x d(x) \right)^2 = (2(n^2 + 1))^2$$

(note that in $\sum_x d(x)$, each route gets counted exactly twice, once for each endpoint). We conclude that

$$4(n^2 + 1)^2 \leq 4n^2(n^2 + 1),$$

which fails to hold for any $n \geq 0$.

Here's an alternate solution taken from `https://mks.mff.cuni.cz/kalva/putnam/psoln/psol5612.html`: Model the problem as one about graphs: we are given a graph on $2n$ vertices with $n^2 + 1$ edges, and we want to find a triangle.

Induction. For $n = 2$, the result is obviously true, because there is only one graph with 4 points and 5 edges and it certainly contains a triangle. Suppose the result is true for some $n \geq 2$. Consider a graph G with $2n + 2$ vertices and $n^2 + 2n + 2$ edges. Take any two vertices x and y joined by an edge. We consider two cases. If there are fewer than $2n + 1$ other edges joined to either x or y (or both), then if we remove x and y we get a graph with $2n$ vertices and at least $n^2 + 1$ edges, which must contain a triangle (by induction), so G does also. If there are at least $2n + 1$ other edges joined to either x or y (or both) then by pigeon-hole principle there is at least one vertex joined to both, and that gives a triangle.

4. An airline operates flights out of n airports. In all, the airline operates at least $3n/2$ different routes (all there-and-back, as in the last question). Prove that it is possible the find a collection of $2k$ airports $a_1, a_2, \ldots, a_{2k-1}, a_{2k}$ (for some $k \geq 2$), such that it is possible to fly directly from a_1 to a_2, then a_2 to a_3, wt cetera, then a_{2k-1} to a_{2k}, and then close the loop by flying from a_{2k} back to a_1.

   Solution: This is a standard result in graph theory; I was reminded of it by looking at a UCSD Putnam prep class webpage

   In the language of graph theory, we want to show that a graph on n vertices with at least $3n/2$ edges must have an even-length cycle.

   Let G be a graph on n vertices with at least $3n/2$ edges must but with no even-length cycle. We will argue a contradiction.

   We may assume that G is connected. For if it has components, $C_1, \ldots, C_m$ say, with $n_1, \ldots, n_m$ vertices, then at least one of the components, C_i say, must have at least $3n_i/2$ edges (if C_j has strictly less than $3n_j/2$ edges, for each j, then the graph would have strictly fewer than $3n/2$ edges, since $\sum_{j=1}^{n} n_j = n$). We can then run the argument we are about to describe on C_j.

Since G is connected, it has a spanning tree (a tree that touches all the vertices), that uses $n-1$ edges. Each of the remaining at least $3n/2-(n-1)$ edges must belong to at least one cycle (the remaining edges of the cycle being from the spanning tree).

Now, note that no edge in the graph can belong to more than one cycle. For suppose an edge e is in two cycles, both, by assumption, odd. Unioning together these two cycles and removing e we get a circuit (vertices and edges allowed to repeat) that is even. But then the graph has an even cycle (contradiction), since a minimal length even circuit is a cycle.

So, consider each of the at least $3n/2-(n-1)$ edges that are not on the spanning tree. Each is involved in exactly one cycle, which must use at least two edges from the spanning tree (cycles have length at least 3). But no edge from the spanning tree can be used more than once (else it would be an edge of the graph involved in more than one cycle). So we can draw on the spanning tree to create cycles (that the non-spanning tree edges are involved in) at most $(n-1)/2$ times. So

$$3n/2-(n-1) \leq (n-1)/2,$$

or $n < n-1$, a contradiction.

5. The complete graph K_n on vertex set $\{1,\ldots,n\}$ is the graph in which all $\binom{n}{2}$ possible edges are present. Suppose that the edges of K_n are colored with two colors, say Red and Blue (meaning, each edge is either assigned the color Red or the color Blue, but not both). Prove that is possible to partition $\{1,\ldots,n\}$ as $A \cup B$, such that there is a Red path that covers all the vertices in A and a Blue path that covers all the vertices in B. (By this is meant: the elements of A can be ordered as $(a_1, a_2, \ldots, a_\ell)$ in such a way that each of the edges $a_1a_2, a_2a_3, \ldots, a_{\ell-1}a_\ell$ are all colored Red, and similarly for B.)

Solution: I learned of this problem in a recent paper of András Gyárfás , at `http://arxiv.org/pdf/1509.05539.pdf`.

Let A and B be disjoint subsets of $\{1,\ldots,n\}$, with a Red path covering A and a Blue path covering B, and with $A \cup B$ as large as possible subject to this condition. If $A \cup B = \{1,\ldots,n\}$ then we are done. If not, then we may assume that both A and B are not empty, since if one of them, B say, was empty, then we could replace B by $\{x\}$ where x is any vertex not in A, and the result would be a valid pair (A, B) with the size of the union one larger, a contradiction of maximality (note that a Blue path covers a single vertex).

Let A be covered by the Red path given by the ordering $(a_1, a_2, \ldots, a_\ell)$, and let B be covered by the Blue path given by the ordering $(b_1, b_2, \ldots, b_k)$. Let x be any vertex not in $A \cup B$. If either the edge $a_\ell x$ is Red or the edge $b_k x$ is Blue, then we can either add x to A or add x to B and get a valid pair that covers more vertices, a contradiction. So we may assume that $a_\ell x$ is Blue and $b_k x$ is Red.

Now look at edge $a_\ell b_k$. If this is Red, then we can replace A by $\{a_1, \ldots, a_\ell, b_k, x\}$ and replace B by $\{b_1, \ldots, b_{k-1}\}$ and get a valid pair that covers more vertices, a contradiction.

If $a_\ell b_k$ is Blue, then we can replace A by $\{a_1, \ldots, a_{\ell-1}\}$ and replace B by $\{b_1, \ldots, b_k, a_\ell, x\}$ and again get a valid pair that covers more vertices, a contradiction.

We conclude that $A \cup B = \{1, \ldots, n\}$.

6. Is there a way to list the 2^n subsets of $\{1, \ldots, n\}$ (with each subset appearing on the list once and only once) in such a way that the first element of the list is the empty set, and every element on the list is obtained from the previous element either by adding an element or deleting an element?

   Solution: I learned of this problem from Imre Leader.

   We'll prove by induction on n that it is possible, and that moreover it is possible to do so in such a way that the last element listed is a singleton (so that the list can be considered a cycle). In fact, the question is asking for a Hamiltonian path (a walk that visits every vertex once) in the graph whose vertex set in the power set of $\{1, \ldots, n\}$, with two vertices adjacent if they have symmetric difference of size exactly 1; this graph is called the n-dimensional hypercube (when $n = 2$ it is just a square, when $n = 3$ it is the usual 3-cube). We'll prove by induction on $n \geq 2$ that the n-dimensional hypercube has a Hamiltonian cycle, from which we can clearly construct a Hamiltonian path of the required kind by deleting an edge out of $\emptyset$.

   The case $n = 2$ is trivial: $\emptyset, \{1\}, \{1, 2\}, \{2\}, \emptyset$ works.

   For $n \geq 2$, start with a Hamiltonian cycle C, $\emptyset, \{1\}, \ldots, \{2\}, \emptyset$, of the $(n-1)$-dimensional hypercube (we known there's one by induction), and then also consider the sequence C', $\{n\}, \{1, n\}, \ldots, \{2, n\}, \{n\}$, obtained by unioning every term of C with $\{n\}$. C' has the property that it is a cycle list of the elements of the n-dimensional hypercube that are not listed in C, and also has the property that adjacent elements have symmetric difference of size exactly 1. A Hamiltonian cycle of the n dimensional hypercube is now obtained by starting with all the elements of C except the final $\emptyset$, then going to the second-from-last element of C', and then listing the remaining element of C' (except for the final $\emptyset$) in reverse order.

7. Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence $g_1, g_2, g_3, \ldots, g_{2n}$ such that every element of G occurs exactly twice in the sequence, and, for each $i = 1, 2, \ldots, 2n$, g_{i+1} equals $g_i a$ or $g_i b$. (Interpret g_{2n+1} as g_1.)

   Solution: This was from the 1990 Putnam competition. It is (an instance of) a very basic result in graph theory, probably the first ever result proved, namely the necessary and sufficient conditions for the existence of an Eulerian circuit in a directed graph.

   See `https://mks.mff.cuni.cz/kalva/putnam/psoln/psol9010.html` for a solution.

8. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an n by n grid so that the k-th row, from left to right, is $(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n$. (So you are writing the numbers in order, starting at the top left and moving left to right along each row, then continuing at the left of the next row down, and so on.)

Color the squares of the grid so that half of the squares in each row are red (and the other half are black), and half of the squares in each column are red (and the other half are black). (A checkerboard coloring is one possibility, but there are many others).

Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Solution: This was problem B1 on the 2001 Putnam competition.

Here is a solution due to Kiran Kedlaya:

Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n + g(s) + 1$ denote the number written in square s, where $0 \leq f(s), g(s) \leq n - 1$. Then it is clear that the value of $f(s)$ depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} f(s) = \sum_{s \in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} g(s) = \sum_{s \in B} g(s).$$

It follows that

$$\sum_{s \in R} f(s)n + g(s) + 1 = \sum_{s \in B} f(s)n + g(s) + 1,$$

as desired.

Note: Richard Stanley points out a theorem of Ryser (see Ryser, *Combinatorial Mathematics*, Theorem 3.1) that can also be applied. Namely, if A and B are $0 - 1$ matrices with the same row and column sums, then there is a sequence of operations on 2×2 matrices of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or vice versa, which transforms A into B. If we identify 0 and 1 with red and black, then the given coloring and the checkerboard coloring both satisfy the sum condition. Since the desired result is clearly true for the checkerboard coloring, and performing the matrix operations does not affect this, the desired result follows in general.

9 Week eight (October 15) — Inequalities

Many Putnam problem involve showing that a particular inequality between two expressions holds always, or holds under certain circumstances. There are a huge variety of general inequalities between sets of numbers satisfying certain conditions, that are quite reasonable for you to quote as "well-known". I've listed some of them here, mostly without proofs, with stars next to the most important ones. If you are interested in knowing more about inequalities, consider looking at the book *Inequalities* by Hardy, Littlewood and Pólya (QA 303 .H223i at the math library).

Squares are positive $\star\star$

Surprisingly many inequalities reduce to the obvious fact that $x^2 \geq 0$ for all real x, with equality iff $x = 0$. I'll highlight one example in what follows.

The triangle inequality

For real or complex x and y, $|x + y| \leq |x| + |y|$ (called the triangle inequality because it says that the distance travelled along the line in going from x to $-y$ — $|x+y|$ — does not decrease if we demand that we go through the intermediate point 0).

Arithmetic mean — Geometric mean — Harmonic mean inequality $\star\star$

For positive $a_1, \ldots, a_n$

$$\frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_1}} \leq \sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n}$$

with equalities in both inequalities iff all a_i are equal. The three expressions above are the *harmonic mean*, the *geometric mean* and the *arithmetic mean* of the a_i.

For $n = 2$, here's a proof of the second inequality: $\sqrt{a_1 a_2} \leq (a_1+a_2)/2$ iff $4a_1a_2 \leq (a_1+a_2)^2$ iff $a_1^2 - 2a_1a_2 + a_2^2 \geq 0$ iff $(a_1 - a_2)^2 \geq 0$, which is true by the "squares are positive" inequality; there's equality all along iff $a_1 = a_2$.

For $n = 2$ the first inequality is equivalent to $\sqrt{a_1 a_2} \leq (a_1 + a_2)/2$.

Power means inequality

For a non-zero real r and positive $a_1, \ldots, a_n$ define

$$M^r(a_1, \ldots, a_n) = \left(\frac{a_1^r + \ldots + a_n^r}{n}\right)^{1/r},$$

and set $M^0(a_1, \ldots, a_n) = \sqrt[n]{a_1 \ldots a_n}$. For real numbers $r < s$,

$$M^r(a_1, \ldots, a_n) \leq M^s(a_1, \ldots, a_n)$$

with equality iff all a_i are equal.

Notice that $M^{-1}(a_1, \ldots, a_n)$ is the harmonic mean of the a_i's, and $M^1(a_1, \ldots, a_n)$ is their geometric mean, so this inequality generalizes the Arithmetic mean — Geometric mean — Harmonic mean inequality.

There is a weighted power means inequality: let $w_1, \ldots, w_n$ be positive reals that add to 1, and define

$$M_w^r(a_1, \ldots, a_n) = (w_1 a_1^r + \ldots + w_n a_n^r)^{1/r}$$

for non-zero real r, with $M_w^0(a_1, \ldots, a_n) = a_1^{w_1} \ldots a_n^{w_n}$. For real numbers $r < s$,

$$M_w^r(a_1, \ldots, a_n) \leq M_w^s(a_1, \ldots, a_n).$$

(This reduces to the power means inequality when all $w_i = 1/n$.)

Cauchy-Schwarz-Bunyakovsky inequality ⋆⋆

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. We have

$$(x_1 y_1 + \ldots + x_n y_n)^2 \leq \left(x_1^2 + \ldots + x_n^2\right)\left(y_1^2 + \ldots + y_n^2\right).$$

Equality holds if one of the sequences $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ is identically zero. If both are not identically zero, then there is equality iff there is some real number t_0 such that $x_i = t_0 y_i$ for each i.

Here's a quick proof: If either sequence is identically 0, both sides are zero. So assume that neither is identically 0. For any real t we have

$$\sum_{i=1}^{n} (x_i - t y_i)^2 \geq 0.$$

But also,

$$\sum_{i=1}^{n} (x_i - t y_i)^2 = \sum_{i=1}^{n} x_i^2 - 2t \sum_{i=1}^{n} x_i y_i + t^2 \sum_{i=1}^{n} y_i^2,$$

so for all real t, so

$$\sum_{i=1}^{n} x_i^2 - 2t \sum_{i=1}^{n} x_i y_i + t^2 \sum_{i=1}^{n} y_i^2 \geq 0.$$

This means that viewed as a polynomial in t, the expression above must have either complex roots or a repeated real root, i.e., that

$$\left(2 \sum_{i=1}^{n} x_i y_i\right)^2 \leq 4 \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right),$$

which is exactly the inequality we wanted. (Notice the key point — squares are positive!). If the inequality is an equality, then the polynomial has a repeated root, which means there is some real t_0 at which the polynomial evaluates to 0. But the polynomial at this point is equal to $\sum_{i=1}^{n} (x_i - t_0 y_i)^2$, and the only way this can happen is if each $x_i - t_0 y_i$ is 0, as claimed.

This is really a very general inequality: if you are familiar with inner products from linear algebra, the Cauchy-Schwarz-Bunyakovsky inequality really says that if $\mathbf{x}$, $\mathbf{y}$ are vectors in an inner product space over the reals then

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle.$$

Equivalently

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}||\ ||\mathbf{y}||.$$

There is equality iff $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Hölder's inequality

Fix $p > 1$ and define q by $1/p + 1/q = 1$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. We have

$$|x_1 y_1 + \ldots + x_n y_n| \leq \left(|x_1|^p + \ldots + |x_n|^p\right)^{1/p} \left(|y_1|^q + \ldots + |y_n|^q\right)^{1/q}.$$

Notice that Hölder becomes Cauchy-Schwarz-Bunyakovsky in the case $p = 2$.

Chebyshev's sum inequality

If $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ are sequences of reals, then

$$\frac{a_1 b_1 + \ldots + a_n b_n}{n} \geq \left(\frac{a_1 + \ldots + a_n}{n}\right)\left(\frac{b_1 + \ldots + b_n}{n}\right).$$

The same holds if $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$; if either $a_1 \geq \ldots \geq a_n$ and $b_1 \leq \ldots \leq b_n$ or $a_1 \leq \ldots \leq a_n$ and $b_1 \geq \ldots \geq b_n$, then

$$\frac{a_1 b_1 + \ldots + a_n b_n}{n} \leq \left(\frac{a_1 + \ldots + a_n}{n}\right)\left(\frac{b_1 + \ldots + b_n}{n}\right).$$

The rearrangement inequality

If $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$ are sequences of reals, and $a_{\pi(1)}, \ldots, a_{\pi(n)}$is a permutation (rearrangement) of $a_1 \leq \ldots \leq a_n$, then

$$a_n b_1 + \ldots + a_1 b_n \leq a_{\pi(1)} b_1 + \ldots + a_{\pi(n)} b_n \leq a_1 b_1 + \ldots + a_n b_n.$$

If $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$, then there is equality in the first inequality iff π is the reverse permutation $\pi(i) = n + 1 - i$, and there is equality in the second inequality iff π is the identity permutation $\pi(i) = i$.

Jensen's inequality $\star\star$

A real function $f(x)$ is *convex* on the interval $[c, d]$ if for all $c \leq a < b \leq d$, the line segment joining $(a, f(a))$ to $(b, f(b))$ lies entirely above the graph $y = f(x)$ on the interval (a, b), or equivalently, if for all $0 \leq t \leq 1$ we have

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

If $f(x)$ is convex on the interval $[c,d]$, and $c \leq a_1 \leq \ldots \leq a_n \leq d$, then

$$f\left(\frac{a_1 + \ldots + a_n}{n}\right) \leq \frac{f(a_1) + \ldots + f(a_n)}{n}$$

(note that when $n = 2$, this is just the definition of convexity).

We say that $f(x)$ is *concave* on $[c,d]$ if for all $c \leq a < b \leq d$, and for all $0 \leq t \leq 1$, we have

$$f((1-t)a + tb) \geq (1-t)f(a) + tf(b).$$

If $f(x)$ is concave on the interval $[c,d]$, and $c \leq a_1 \leq \ldots \leq a_n \leq d$, then

$$f\left(\frac{a_1 + \ldots + a_n}{n}\right) \geq \frac{f(a_1) + \ldots + f(a_n)}{n}.$$

As an example, consider the convex function $f(x) = x^2$; for this function Jensen says that

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n},$$

which is equivalent to the powers means inequality $M^1(a_1, \ldots, a_n) \leq M^2(a_1, \ldots, a_n)$; and when $f(x) = -\ln x$ we get

$$\sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n},$$

the AM-GM inequality.

Four miscellaneous comments

1. Maximization/minimization problems are often problems about inequalities in disguise. For example, to find the minimum of $f(a,b)$ as (a,b) ranges over a set R, it is enough to first guess that the minimum is m, then find an $(a,b) \in R$ with $f(a,b) = m$, and then use inequalities to show that $f(a,b) \geq m$ for all $(a,b) \in R$.

2. If an expression is presented as a sum of n squares, it is sometimes helpful to think of it as the (square of the) distance between two points in n dimensional space, and then think of the problem geometrically.

3. Sometimes a little calculus is all that is needed. For example, here is a very useful inequality:
$$1 + x \leq e^x \quad \text{for all } x \in \mathbb{R}.$$
To prove this for $x \geq 0$ note that both sides are equal at $x = 0$, and the derivative of $1 + x$, which is 1, is smaller than the derivative of e^x, which is e^x, for all $x \geq 0$; so the two sides start together but always the right-hand side is growing faster than the left-hand side, so the right-hand side is always bigger. A similar argument proves the inequality for $x \leq 0$: $1 + x$, with derivative 1, falls faster as we move along the x-axis negatively away from 0, than does e^x, which has derivative positive but strictly less than 1 for $x < 0$. (To formalize this second half of the argument, consider $f(y) = 1 - y$ and $g(y) = e^{-y}$, defined for $y \geq 0$. We have $f(0) = g(0)$, and $f'(y) = -1 \leq -e^{-y} = g'(y)$ for $y \geq 0$, so $f(y) \leq g(y)$ for $y \geq 0$. It follows that for $x \leq 0$, $1 - x \leq e^{-x}$.)

4. If $f(x)$ is a positive, increasing function on $(0, \infty)$, then by considering Riemann sums we have

$$\int_0^n f(x)\ dx \leq \sum_{k=1}^n f(k) \leq \int_1^{n+1} f(x)\ dx$$

(assuming the left-hand integral converges). For example, consider $f(x) = x^k$ for $k > 0$. We have

$$\int_0^n x^k\ dx = \frac{n^{k+1}}{k+1}$$

and

$$\int_1^{n+1} x^k\ dx = \frac{(n+1)^{k+1}}{k+1} - \frac{1}{k+1}.$$

It easy to check that

$$\left(\frac{(n+1)^{k+1}}{k+1} - \frac{1}{k+1}\right) / \left(\frac{n^{k+1}}{k+1}\right) \to 1 \quad \text{as } n \to \infty,$$

so we have a quick proof that for each fixed $k > 0$ (not necessarily an integer)

$$\lim_{n\to\infty} \frac{1^k + \ldots + n^k}{n^{k+1}} = \frac{1}{k+1};$$

in other words, the sum of the first n perfect kth powers grows like $n^{k+1}/(k+1)$.

Some warm-up problems

You should find that these are all fairly easy to prove by direct applications of an appropriate inequality from the list above.

1. $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$.

2. $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$ for positive a, b, c.

3. Minimize $x_1 + \ldots + x_n$ subject to $x_i \geq 0$ and $x_1 \ldots x_n = 1$.

4. Minimize

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$

subject to $x, y, z \geq 0$ and $xyz = 1$.

5. If triangle has side lengths a, b, c and opposite angles (measured in radians) A, B, C, then

$$\frac{aA + bB + cC}{a+b+c} \geq \frac{\pi}{3}.$$

6. Identify which is bigger:

$$1999!^{(2000)} \quad \text{or} \quad 2000!^{(1999)}.$$

(Here $n!^{(k)}$ indicates iterating the factorial function k times, so for example $4!^{(2)} = 24!$.)

7. Identify which is bigger:

$$1999^{1999} \quad \text{or} \quad 2000^{1998}.$$

8. Minimize

$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$$

on the interval $0 < x < \pi/2$.

9.1 Week eight problems

1. Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of
$$\frac{A(R)+A(S)}{A(T)},$$
where T ranges over all triangles and R, S over all rectangles as above.

2. Show that for non-negative reals $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,
$$(a_1 \ldots a_n)^{1/n} + (b_1 \ldots b_n)^{1/n} \leq ((a_1+b_1)\ldots(a_n+b_n))^{1/n}.$$

3. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

4. For positive integers m, n, show
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

5. Minimize
$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v^2}\right)^2$$
in the range $0 \leq u \leq \sqrt{2}$, $v \geq 0$.

6. Given that $\{x_1, \ldots, x_n\} = \{1, \ldots, n\}$ (i.e., the numbers $x_1, \ldots, x_n$ are 1 through n in some order), find (with proof!) the maximum value of
$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

7. Show that for every positive integer n,
$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} \leq 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

8. Suppose that $f(x)$ is a polynomial with all real coefficients, satisfying $f(x)+f'(x) > 0$ for all x. Show that $f(x) > 0$ for all x.

9. Show that in a triangle with side lengths a, b, c and area A one has
$$a^2+b^2+c^2 \geq 4\sqrt{3}A.$$

Solutions to the warm-up problems

All of these problems were all taken from a Northwestern Putnam prep problem set.

1. $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$.

   Solution: Use the geometric mean — arithmetic mean inequality, with $(a_1, \ldots, a_n) = (1, \ldots, n)$.

2. $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$ for positive a, b, c.

   Solution: Use the power means inequality, with $(a_1, a_2, a_3) = (a, b, c)$ and $r = 1/2, s = 1$.

3. Minimize $x_1 + \ldots + x_n$ subject to $x_i \geq 0$ and $x_1 \ldots x_n = 1$.

   Solution: Guess: the minimum is n, achieved when all $x_1 = 1$. Then use geometric mean - arithmetic mean inequality to show

   $$\left(\frac{x_1 + \ldots + x_n}{n}\right) \geq \sqrt[n]{x_1 \ldots x_n} = 1$$

   for positive x_i satisfying $x_1 \ldots x_n = 1$.

4. Minimize

   $$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$

   subject to $x, y, z \geq 0$ and $xyz = 1$.

   Solution: Apply Cauchy-Schwartz with the vectors $\left(\sqrt{y+z}, \sqrt{z+x}, \sqrt{x+y}\right)$ and

   $$\left(\frac{x}{\sqrt{y+z}}, \frac{y}{\sqrt{z+x}}, \frac{z}{\sqrt{x+y}}\right)$$

   to get

   $$(x+y+z)^2 \leq \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) 2\,(x+y+z)\,,$$

   leading to

   $$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2}.$$

   By the AM-GM inequality,

   $$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1,$$

   so

   $$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

   This lower bound can be achieved by taking $x = y = z = 1$, so the minimum is $3/2$.

5. If triangle has side lengths a, b, c and opposite angles (measured in radians) A, B, C, then

$$\frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3}.$$

Solution: Assume, without loss of generality, that $a \leq b \leq c$. Then also $A \leq B \leq C$, so by Chebychev,

$$\frac{aA + bB + cC}{3} \geq \left(\frac{a+b+c}{3}\right)\left(\frac{A+B+C}{3}\right) = \left(\frac{a+b+c}{3}\right)\frac{\pi}{3},$$

from which the result follows.

6. Identify which is bigger:

$$1999!^{(2000)} \quad \text{or} \quad 2000!^{(1999)}.$$

(Here $n!^{(k)}$ indicates iterating the factorial function k times, so for example $4!^{(2)} = 24!$.)

Solution: For $n \geq 1$, $n!$ is increasing in n ($1 \leq n < m$ implies $n! < m!$). So, starting from the easy

$$1999! > 2000,$$

apply the factorial function 1999 more times to get

$$1999!^{(2000)} > 2000!^{(1999)}.$$

7. Identify which is bigger:

$$1999^{1999} \quad \text{or} \quad 2000^{1998}.$$

Solution: Consider $f(x) = (1999 - x)\ln(1999 + x)$. We have $e^{f(0)} = 1999^{1999}$ and $e^{f(1)} = 2000^{1998}$, so we want to see what f does on the interval $[0, 1]$: increase or decrease? The derivative is

$$f'(x) = -\ln(1999 + x) + \frac{1999 - x}{1999 + x},$$

which is negative on $[0, 1]$ (since, for example,

$$\frac{1999 - x}{1999 + x} \leq 1 = \ln e < \ln(1999 + x)$$

on that interval). So

$$2000^{1998} < 1999^{1999}.$$

8. Minimize

$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$$

on the interval $0 < x < \pi/2$.

Solution: We can use the rearrangement inequality on the pairs $\left(\sin^3 x, \cos^3 x\right)$ (which satisfies $\sin^3 x \le \cos^3 x$ on $[0, \pi/4]$, and $\sin^3 x \ge \cos^3 x$ on $[\pi/4, \pi/2]$), and $(1/\cos x, 1/\sin x)$ (which also satisfies $1/\cos x \le 1/\sin x$ on $[0, \pi/4]$, and $1/\cos x \ge 1/\sin x$ on $[\pi/4, \pi/2]$), to get

$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x} \ge \frac{\sin^3 x}{\sin x} + \frac{\cos^3 x}{\cos x} = \sin^2 x + \cos^2 x = 1$$

on the whole interval. Since 1 can be achieved (at $x = \pi/4$) the minimum is 1.

9.2 Week eight solutions

1. Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of

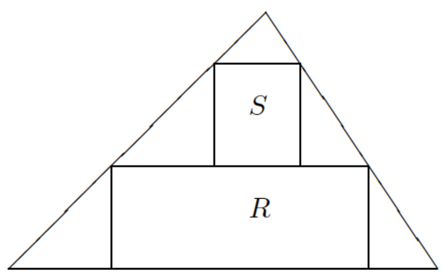
$$\frac{A(R)+A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above.

Solution: This problem was on the 1985 Putnam Competition, Problem A2.

Here's a pictorial version of the problem:

Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:



Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

We claim that the answer is $2/3$.

Assume, without loss of generality, that the horizontal base of T has length 1. Let the base of R have length x, and the base of S have base y, where $0 < y < x < 1$.

We have

$$\frac{A(S)}{A(T)} = 2y(x-y)$$

and

$$\frac{A(R)}{A(T)} = 2x(1-x),$$

so the quantity we want to maximize is

$$2y(x-y) + 2x(1-x),$$

subject to the constraint $0 < y < x < 1$.

For each fixed x, as y varies over $0 < y < x$ the quantity $2y(x-y) + 2x(1-x)$ achieves its maximum at $y = x/2$, where it takes value $x^2/22x(1-x) = (4x - 3x^2)/2$. This achieves its maximum over $0 < x < 1$ at $x = 2/3$, where it takes the value $2/3$.

2. Show that for non-negative reals $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$(a_1 \ldots a_n)^{1/n} + (b_1 \ldots b_n)^{1/n} \le ((a_1 + b_1) \ldots (a_n + b_n))^{1/n}.$$

Solution: This was from the 2003 Putnam competition, problem A2.

If any a_i is 0, the result is trivial, so we may assume all $a_i > 0$. Dividing through by $(a_1 \ldots a_n)^{1/n}$, the inequality becomes

$$1 + (c_1 \ldots c_n)^{1/n} \le ((1 + c_1) \ldots (1 + c_n))^{1/n}$$

for $c_i \ge 0$. Raising both sides to the power n, this is the same as

$$\sum_{k=0}^{n} \binom{n}{k} (c_1 \ldots c_n)^{k/n} \le \sum_{k=0}^{n} e_k$$

where e_k is the sum of the products of the c_i's, taken k at a time. So it is enough to show that for each k,

$$\binom{n}{k} (c_1 \ldots c_n)^{k/n} \le \sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \prod_{i \in A} c_i.$$

We apply the AM-GM inequality to the numbers $\prod_{i \in A} c_i$ as A ranges over all subsets of size k of $\{1, \ldots, n\}$. Note that each a_i appears exactly $\binom{n-1}{k-1}$ times in all these numbers. So we we get

$$(c_1 \ldots c_n)^{\binom{n-1}{k-1}/\binom{n}{k}} \le \frac{\sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \prod_{i \in A} c_i}{\binom{n}{k}}.$$

Since $\binom{n-1}{k-1}/\binom{n}{k} = k/n$, this is the same as

$$(c_1 \ldots c_n)^{k/n} \le \frac{\sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \prod_{i \in A} c_i}{\binom{n}{k}},$$

which is exactly what we wanted to show.

Much quicker solution, shown to me by Jonathan Sheperd: If there is any i for which $a_i + b_i = 0$, then the inequality trivially holds. If not, divide both sides by the right-hand side to get the equivalent inequality

$$\left(\prod_{i=1}^{n} \left(\frac{a_i}{a_i + b_i} \right) \right)^{\frac{1}{n}} \left(\prod_{i=1}^{n} \left(\frac{b_i}{a_i + b_i} \right) \right)^{\frac{1}{n}} \le 1.$$

Applying the arithmetic mean – geometric mean inequality to both terms on the left-hand side, we find that the left-hand side is at most

$$\frac{1}{n}\left(\sum_{i=1}^{n}\frac{a_i}{a_i+b_i}\right)+\frac{1}{n}\left(\sum_{i=1}^{n}\frac{b_i}{a_i+b_i}\right)$$

which is the same as

$$\frac{1}{n}\left(\sum_{i=1}^{n}\frac{a_i+b_i}{a_i+b_i}\right),$$

which is indeed at most (in fact exactly) 1.

3. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

   Solution: This was from the 1988 Putnam competition, Problem B2.

   Here is a solution written by John Scholes:

   The claimed inequality ($y(y-1) \leq x^2$) is true. We consider 3 cases.

   Case 1: Suppose first that $x \geq 0$, and that $y(y+1) = (x+1)^2$. Since $(x+1/2)(x+3/2) = (x+1)^2 - 1/4$, we have $y > x+1/2$. Hence $y(y-1) = y(y+1) - 2y < (x+1)^2 - 2(x+1/2) = x^2$.

   Case 2: If $x \geq 0$ and $y(y+1) < (x+1)^2$, then take $y' > y$ with $y'(y'+1) = (x+1)^2$. Clearly $y - 1 < y' - 1$ and since y is positive, $y(y-1) < y(y'-1) < y'(y'-1)$, which (by the analysis of Case 1) is $< x^2$.

   So it just remains to consider the case $x < 0$. But in this case $-|x| - 1 < x + 1 < |x| + 1$, so $(x+1)^2 < (|x|+1)^2$ and $x^2 = |x|^2$, and the result follows from the result for y, $|x|$.

4. For positive integers m, n, show

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

   Solution: This was from the 2004 Putnam competition, Problem B2.

   Rearranging, this is the same as

$$\binom{m+n}{m}\left(\frac{m}{m+n}\right)^m\left(\frac{n}{m+n}\right)^n < 1.$$

   This suggests looking at the binomial expansion

$$\left(\frac{m}{m+n}+\frac{n}{m+n}\right)^{m+n}.$$

   The whole binomial expansion sums to 1; one term of the expansion is

$$\binom{m+n}{m}\left(\frac{m}{m+n}\right)^m\left(\frac{n}{m+n}\right)^n.$$

   Since all terms are strictly positive, we get the required inequality.

5. Minimize

$$(u-v)^2+\left(\sqrt{2-u^2}-\frac{9}{v^2}\right)^2$$

in the range $0 \le u \le \sqrt{2}$, $v \ge 0$.

Solution: This was on the Putnam competition 1984 problem B2.

The expression to be minimized is the (square of the) distance between a point of the form $(u, \sqrt{2-u^2})$ on $0 < u < \sqrt{2}$, and a point of the form $(v, 9/v)$ on $v > 0$; in other words, we are looking for the (square of the) distance between the circle $x^2 + y^2 = 2$ in the first quadrant and the hyperbola $xy = 9$ in the same quadrant. By symmetry, it strongly seems that the two closed points are $(3, 3)$ on the hyperbola and $(1, 1)$ on the circle (squared distance 8). To prove that this is the minimum, note that the tangent lines to the two curves at those two points are parallel, that the distance between them at these points is the perpendicular distance between the two tangent lines, and that the hyperbola (in the first quadrant) lies completely above its tangent line, while the circle (in the first quadrant) lies completely below its tangent line; so the distance between any other two points is at least the distance between the two tangent lines.

6. Given that $\{x_1, \ldots, x_n\} = \{1, \ldots, n\}$ (i.e., the numbers $x_1, \ldots, x_n$ are 1 through n in some order), find (with proof!) the maximum value of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Solution: This was from the 1996 Putnam Competition, problem B3

Here is a solution written by Kiran Kedlaya:

View $x_1, \ldots, x_n$ as an arrangement of the numbers $1, 2, \ldots, n$ on a circle. We prove that the optimal arrangement is

$$\ldots, n-4, n-2, n, n-1, n-3, \ldots$$

To show this, note that if a, b is a pair of adjacent numbers and c, d is another pair (read in the same order around the circle) with $a < d$ and $b > c$, then the segment from b to c can be reversed, increasing the sum by

$$ac + bd - ab - cd = (d-a)(b-c) > 0.$$

Now relabel the numbers so they appear in order as follows:

$$\ldots, a_{n-4}, a_{n-2}, a_n = n, a_{n-1}, a_{n-3}, \ldots$$

where without loss of generality we assume $a_{n-1} > a_{n-2}$. By considering the pairs a_{n-2}, a_n and a_{n-1}, a_{n-3} and using the trivial fact $a_n > a_{n-1}$, we deduce $a_{n-2} > a_{n-3}$. We then compare the pairs a_{n-4}, a_{n-2} and a_{n-1}, a_{n-3}, and using that $a_{n-1} > a_{n-2}$, we deduce $a_{n-3} > a_{n-4}$. Continuing in this fashion, we prove that $a_n > a_{n-1} > \cdots > a_1$

and so $a_k = k$ for $k = 1, 2, \ldots, n$, i.e. that the optimal arrangement is as claimed. In particular, the maximum value of the sum is

$$\begin{aligned}
1 \cdot 2 + (n-1) \cdot n + 1 \cdot 3 + 2 \cdot 4 + \cdots + (n-2) \cdot n \\
&= 2 + n^2 - n + (1^2 - 1) + \cdots + [(n-1)^2 - 1] \\
&= n^2 - n + 2 - (n-1) + \frac{(n-1)n(2n-1)}{6} \\
&= \frac{2n^3 + 3n^2 - 11n + 18}{6}.
\end{aligned}$$

Alternate solution: We prove by induction that the value given above is an upper bound; it is clearly a lower bound because of the arrangement given above. Assume this is the case for $n-1$. The optimal arrangement for n is obtained from some arrangement for $n-1$ by inserting n between some pair x, y of adjacent terms. This operation increases the sum by $nx + ny - xy = n^2 - (n-x)(n-y)$, which is an increasing function of both x and y. In particular, this difference is maximal when x and y equal $n-1$ and $n-2$. Fortunately, this yields precisely the difference between the claimed upper bound for n and the assumed upper bound for $n-1$, completing the induction.

7. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} \leq 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Solution: This was from the 1996 Putnam Competition, problem B2.

We estimate the integral of $\ln x$, which is convex and hence easy to estimate. Take the integral from 1 to $2n-1$. This is less than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. But the antiderivative of $\ln x$ is $x \ln x - x$, so the integral evaluates to $(2n-1)\ln(2n-1) - 2n + 2$. Hence $(2n-1)\ln(2n-1) - (2n-1) < (2n-1)\ln(2n-1) - 2n + 2 < 2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. Exponentiating gives the right-hand inequality.

Similarly, the integral from e to $2n+1$ is greater than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$, and an explicit evaluation of the antiderivative here leads to the right-hand side of the inequality. The choice of lower bound e for the integral here is just the right thing to make the computations work out nicely.

8. Suppose that $f(x)$ is a polynomial with all real coefficients, satisfying $f(x) + f'(x) > 0$ for all x. Show that $f(x) > 0$ for all x.

Solution: **Source**: I got this problem from a Northwestern Putnam preparation class.

$f(x)$ and $f(x) + f'(x)$ have the same leading coefficient, so the same limiting behavior as x goes to $\pm\infty$, namely they both tend to $+\infty$ (since $f(x) + f'(x) > 0$ always, the limits cannot be $-\infty$).

$f(x)$ cannot have a repeated root: at a repeated root, the derivative is also 0, so $f(x) + f'(x) = 0$ at this point. So all of $f(x)$'s real roots (if it has any) are simple. Since $f(x)$ goes to $+\infty$ as x approaches both $\pm\infty$, it must thus have an even number of real zeroes.

Suppose it has any. Let x_1 and x_2 be the first two. Between x_1 and x_2, at some point the derivative is 0 (Rolle's theorem); at that point $f(x)+f'(x)$ must be negative (since $f(x)$ negative here). This contradiction shows that $f(x)$ has no real roots, so can't change sign, so must be always positive.

Remark: The example of $f(x) = -e^{-2x}$ shows that the hypothesis that $f(x)$ is a polynomial is crucial here.

9. Show that in a triangle with side lengths a, b, c and area A one has

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A.$$

Solution: I saw this on Andrei Jorza's 2018 ND Putnam Prep class; it was also on International Mathematical Olympiad problem from 1961.

The inequality is *Weitzenböck's inequality*. See

`https://en.wikipedia.org/wiki/Weitzenb%C3%B6ck%27s_inequality`

for many solutions!

10 Week nine (October 29) — Writing solutions

This week's handout is concerned with the art of writing — presenting — solutions. I've made a few notes myself, but I also strongly recommend that you also look at both of these essays:

- (short, with very practical advice and a few examples): `http://web.evanchen.cc/handouts/english/english.pdf`
- (long, with lots of examples of both good and bad writing): `https://artofproblemsolving.com/news/articles/how-to-write-a-solution`

In what follows, I'm going to repeat some of the advice I've given on a previous handout. I encourage you to look over this carefully, and bear it in mind as you go over this week's problems and (more importantly) as you take the Putnam competition. I have shamelessly appropriate much of this from Ravi Vakil's Stanford Putnam Preparation website (`http://math.stanford.edu/~vakil/putnam05/05putnam7.pdf`), and from Ioana Dumitriu's UWashington's "The Art of Problem Solving" website (`http://www.math.washington.edu/~putnam/index.html`). Both of these websites are filled with what I think is great advice.

This weeks problem set is the 2019 Virginia Tech Regional Math Contest, which took place last Saturday (October 26). For next Tuesday, fully write up a solution to at least one of the problems, *after* you have read the handout and (at the very least) the 4-page essay by Evan Chen (`http://web.evanchen.cc/handouts/english/english.pdf`).

Some general problem-solving tips

Remember that problem solving is a full-contact sport: throw everything you know at the problem you are tackling! Sometimes, the solution can come from an unexpected quarter. Here are some slogans to keep in mind when solving problems:

- Try small cases!
- Plug in small numbers!
- Do examples!
- Look for patterns!
- Draw pictures!
- Write lots!
- Talk it out!
- Choose good notation!
- Look for symmetry!
- Break into cases!
- Work backwards!

- Argue by contradiction!
- Consider extreme cases!
- Modify the problem!
- Make a generalization!
- Don't give up after five minutes!
- Don't be afraid of a little algebra!
- Take a break!
- Sleep on it!
- Ask questions!

And above all:

- Enjoy!

Some specific mathematical tips

Here are some very simple things to remember, that can be very helpful, but that people tend to forget to do.

1. Try a few small cases out. Try a lot of cases out. Remember that the three hours of the Putnam competition is a long time — you have time to spare! If a question asks what happens when you have n things, or 2015 things, try it out with 1, 2, 3, 4 things, and try to form a conjecture. This is especially valuable for questions about sequences defined recursively.
2. Don't be afraid to use lots and lots of paper.
3. Don't be afraid of diving into some algebra. (Again, three hours is a long time ....) You shouldn't waste that much time, thanks to the 15-minute rule.
4. If a question asks to determine whether something is true or false, and the direction you initially guess doesn't seem to be going anywhere, then try guessing the opposite possibility.
5. Be willing to try (seemingly) stupid things.
6. Look for symmetries. Try to connect the problem to one you've seen before. Ask yourself "how would [person X] approach this problem"? (It's quite reasonable here for [Person X] to be [Chuck Norris]!)
7. Putnam problems always have slick solutions. That leads to a helpful meta-approach: "The only way this problem could have a nice solution is if this particular approach worked out, unlikely as it seems, so I'll try it out, and see what happens."
8. Show no fear. If you think a problem is probably too hard for you, but you have an idea, try it anyway. (Three hours is a long time.)

Some specific writing tips

You're working hard on a problem, focussing all your energy, applying all the good tips above, and all of a sudden a bulb lights above your head: you have figured out how to solve one of the problems! Awesome! But here's the hard, unforgiving truth: that warm tingle you're feeling is no guarantee

- that you have actually solved the problem (are you sure you have all the cases covered? Are you sure that all the little details work out? ...)

  and

- that you will get any or all credit for it.

Solving the problem is only half the battle. Now you have launch into the other half, convincing the grader that you have actually done so.

Life is tough, and Putnam graders may be even tougher. But here's a list of things which, when done properly, will yield a nice write-up which will appease any reasonable grader.

1. All that scrap paper, filled with your musings on the problem to date? Put it aside; you **must** write a clean, coherent solution on a fresh piece of paper.

2. Before you start writing, organize your thoughts. Make a list of all steps to the solution. Figure out what intermediary results you will need to prove. For example, if the problem involves induction, always start with the base case, and continue with proving that "true for n implies true for $n+1$". Make sure that the steps follow from each other logically, with no gaps.

3. After tracing a "road to proof" either in your head or (preferably) on scratch paper, start writing up the solution on a fresh piece of paper. The best way to start this is by writing a quick outline of what you propose to do. Sometimes, the grader will just look at this outline, say "Yes, she knows what she is doing on this problem!", and give the credit.

4. Lead with a **clear** statement of your final solution.

5. Complete each step of the "road" before you continue to the next one.

6. When making statements like "it follows trivially" or "it is easy to see", listen for quiet, nagging doubts. If you yourself aren't 100% convinced, how will you convince someone else? Even if it seems to follow trivially, check again. Small exceptions may not be obvious. The strategy "I am sure it's true, even if I don't see it; if I state that it's obvious, maybe the grader will believe I know how to prove it" has occasionally led its user to a score of 0 out of 10.

7. Organize your solution on the page; avoid writing in corners or perpendicular to the normal orientation. Avoid, if possible, post-factum insertion (if you discover you've missed something, rather than making a mess of the paper by trying to write it over, start anew. You have the time!)

8. Before writing each phrase, formulate it completely in your mind. Make sure it expresses an idea. Starting to write one thing, then changing course in mid-sentence and saying another thing is a sure way to create confusion.

9. Be as clear as possible. Avoid, if possible, long-winded phrases. Use as many words as you need – just do it clearly. Also, avoid acronyms.

10. If necessary, state intermediary results as “claims” or “lemmas” which you can prove right after stating them. If you cannot prove one of these results, but can prove the problem’s statement from it, state that you will assume it, then show the path from it to the solution. You may get partial credit for it.

11. Rather than using vague statements like “and so on” or “repeating this process”, formulate and prove by induction.

12. When you’re done writing up the solution, go back and re-read it. Put yourself in the grader’s shoes: can someone else read your write-up and understand the solution? Must one look for things in the corners? Are there “miraculous” moments?, etc..

10.1 Week nine problems

Here are the problems from this year's Virginia Tech Regional Math Contest:

41st Annual Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 26, 2019

Fill out the individual registration form

1. For each positive integer n, define $f(n)$ to be the sum of the digits of 2771^n (so $f(1) = 17$). Find the minimum value of $f(n)$ (where $n \geq 1$). Justify your answer.

2. Let X be the point on the side AB of the triangle ABC such that $BX/XA = 9$. Let M be the midpoint of BX and let Y be the point on BC such that $\angle BMY = 90°$. Suppose AC has length 20 and that the area of the triangle XYC is 9/100 of the area of the triangle ABC. Find the length of BC.

3. Let n be a nonnegative integer and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial with real coefficients a_i. Suppose that

$$\frac{a_n}{(n+1)(n+2)} + \frac{a_{n-1}}{n(n+1)} + \cdots + \frac{a_1}{6} + \frac{a_0}{2} = 0.$$

   Prove that $f(x)$ has a real zero.

4. Compute $\displaystyle\int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}\,dx$ (the answer is a rational number).

5. Find the general solution of the differential equation

$$x^4 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} + (1-2x)y = 0$$

   valid for $0 < x < \infty$.

6. Let S be a subset of $\mathbb{R}$ with the property that for every $s \in S$, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap S = \{s\}$. Prove there exists a function $f\colon S \to \mathbb{N}$, the positive integers, such that for all $s,t \in S$, if $s \neq t$ then $f(s) \neq f(t)$.

7. Let S denote the positive integers that have no 0 in their decimal expansion. Determine whether $\sum_{n\in S} n^{-99/100}$ is convergent.

11 Week ten (November 5) — Polynomials

This week's problem are all about polynomials, which come up in virtually every Putnam competition.

Things to know about polynomials

- **Fundamental Theorem of Algebra**: Every polynomial $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n$, with real or complex coefficients, has a root in the complex numbers, that is, there is $c \in \mathbb{C}$ such that $p(c) = 0$.

- **Factorization**: In fact, every polynomial $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n_2} + \ldots + a_{n-1}x + a_n$, with real or complex coefficients, has exactly n roots, in the sense that there is a vector $(c_1, \ldots, c_n)$ (perhaps with some repetitions) such that

  $$p(x) = (x - c_1)(x - c_2)\ldots(x - c_n).$$

  If a c appears in this vector exactly k times, it is called a *root* or *zero* of *multiplicity* k. The next bullet point gives a very useful consequence of this.

- **Two different polynomials of the same degree can't agree too often**: If $p(x)$ and $q(x)$ (over $\mathbb{R}$ or $\mathbb{C}$) both have degree at most n, and there are $n+1$ distinct numbers $x_1, \ldots, x_{n+1}$ such that $p(x_i) = q(x_i)$ for $i = 1, \ldots, n+1$, then $p(x)$ and $q(x)$ are equal for all x. [Because then $p(x) - q(x)$ is a polynomial of degree at most n with at least $n+1$ roots, so must be identically zero].

- **Complex conjugates**: If the coefficients of $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n_2} + \ldots + a_{n-1}x + a_n$ are all **real**, then the complex roots occur in complex-conjugate pairs: if $s + it$ (with s, t real, and $i = \sqrt{-1}$) is a root, then $s - it$ is also a root.

- **Coefficients in terms of roots**: If $(c_1, \ldots, c_n)$ is the vector of roots of a polynomial $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n_2} + \ldots + a_{n-1}x + a_n$ (over $\mathbb{R}$ or $\mathbb{C}$), then each of the coefficients can be expressed simply in terms of the roots: a_1 is the negative of the sum of the c_i's; a_2 is the sum of the products of the c_i's, taken two at a time, a_3 is the negative of the sum of the products of the c_i's, taken three at a time, etc. Concisely:

  $$a_k = (-1)^k \sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \ \prod_{i \in A} c_i.$$

- **Elementary symmetric polynomials**: The kth *elementary symmetric polynomial* in variables $x_1, \ldots, x_n$ is

  $$\sigma_k = \sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \ \prod_{i \in A} x_i$$

  (these polynomials have already appeared in the last bullet point). A polynomial $p(x_1, \ldots, x_n)$ in n variables is *symmetric* if for every permutation π of $\{1, \ldots, n\}$, we have

  $$p(x_1, \ldots, x_n) \equiv p(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

(For example, $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is symmetric, but $x_1^2 + x_2^2 + x_3^2 + x_1x_4$ is not.) Every symmetric polynomial in variables $x_1, \ldots, x_n$ can be expressed as a linear combination of the σ_k's.

- **Some special values tell things about the coefficients**: (Rather obvious, but worth keeping in mind) If $p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n_2} + \ldots + a_{n-1}x + a_n$, then

$$\begin{aligned} p(0) &= a_n \\ p(1) &= a_0 + a_1 + a_2 + \ldots + a_n \\ p(-1) &= a_n - a_{n-1} + a_{n-2} - a_{n-3} + \ldots + (-1)^n a_0. \end{aligned}$$

- **Intermediate value theorem**: If $p(x)$ is a polynomial with real coefficients (or in fact any continuous real function) such that for some $a < b$, $p(a)$ and $p(b)$ have different signs, then there is some c, $a < c < b$, with $p(c) = 0$.

- **Lagrange interpolation**: Suppose that $p(x)$ is a real polynomial of degree n, whose graph passes through the points (x_0, y_0), (x_1, y_1), $\ldots$, (x_n, y_n). Then we can write

$$p(x) = \sum_{i=0}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

- **The Rational Roots theorem**: Suppose that $p(x)$ is a polynomial of degree n with **integer** coefficients, and that x is a *rational* root a/b with a and b having no common factors. Then the leading coefficient of $p(x)$ (the coefficient of x^n) is a multiple of b, and the constant term is a multiple of a. An immediate corollary of this is that if $p(x)$ is a *monic* polynomial (integer coefficients, leading coefficient 1), then any rational root must in fact be an integer; conversely, if a real number x is a root of a monic polynomial but is not an integer, it must be irrational (for example, $\sqrt{2}$ is a root of monic $x^2 - 2$, but is clearly not an integer, so it must be irrational)!

- **Gauss' lemma**: Here is a weak form of Gauss' lemma, but one that is very useful: if c is an integer root of a monic polynomial $p(x)$ (integer coefficients, leading coefficient 1), then $p(x)$ factors as $(x - c)q(x)$, where $q(x)$ is also a monic polynomial (the surprise being not that $q(x)$ has leading coefficient 1, but that it has all integer entries).

- **One more fact about integer polynomials**: Let $p(x)$ be a (not necessarily monic) polynomial of degree n with **integer** coefficients. For any integers a, b,

$$(a - b)|(p(a) - p(b)).$$

  (So also,

$$(p(a) - p(b))|(p(p(a)) - p(p(b))),$$

  etc.)

11.1 Week ten problems

1. For which real values of p and q are the roots of the polynomial $x^3 - px^2 + 11x - q$ three consecutive integers? Give the roots in these cases.

2. Let $p(x)$ be a polynomial with integer coefficients, for which $p(0)$ and $p(1)$ are odd. Can $p(x)$ have any integer zeroes?

3. (a) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x+1) = p(x) + 1$ for all x.

   (b) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x^2+1) = (p(x))^2 + 1$ for all x.

4. Does there exist a non-zero polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$ for all x?

5. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. Suppose that there exist four distinct integers a, b, c, d with $p(a) = p(b) = p(c) = p(d) = 5$. Prove that there is no integer k with $p(k) = 8$.

6. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

   has exactly n distinct real roots?

11.2 Week ten solutions

1. For which real values of p and q are the roots of the polynomial $x^3 - px^2 + 11x - q$ three consecutive integers? Give the roots in these cases.

   Solution: From a Harvey Mudd Putnam prep class.

   A polynomial with roots being three consecutive integers is of the form

   $$(x-(a-1))(x-a)(x-(a+1)) = x^3 - 3ax^2 + (3a^2-1)x - (a^3-a)$$

   for some integer a. So, matching coefficients, we must have $3a^2 - 1 = 11$, or $a = \pm 2$. When $a = 2$ we get roots $1, 2, 3$ and $p = 6$, $q = 6$; when $a = -2$ we get roots $-3, -2, -1$ and $p = -6$, $q = -6$.

2. Let $p(x)$ be a polynomial with integer coefficients, for which $p(0)$ and $p(1)$ are odd. Can $p(x)$ have any integer zeroes?

   Solution: From a Northwestern Putnam prep class.

   No. If k is an even integer we have $p(k) \equiv p(0) \equiv 1 \pmod 2$ (Why? Suppose $a \equiv b \pmod m$. Then $a^\ell \equiv b^\ell \pmod m$ for any ℓ, so $ca^\ell \equiv cb^\ell \pmod m$ for any c, so (summing), $p(a) \equiv p(b) \pmod m$ for any polynomial p). By the same token, if k is odd then $p(k) \equiv p(1) \equiv 1 \pmod 2$. So we never have $p(k) \equiv 0 \pmod 2$, and never have $p(k) = 0$.

3. (a) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x+1) = p(x) + 1$ for all x.

   (b) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x^2+1) = (p(x))^2 + 1$ for all x.

   Solution: Both parts are modified from the Putnam competition, 1971 problem A2.

   For part (a), the only such polynomial is the identity polynomial.

   By induction, $p(x) = x$ for all positive integers x, so $p(x) - x$ is a polynomial with infinitely many zeros, so must be identically 0. We conclude that $p(x) = x$ is the only possible polynomial satisfying the given conditions.

   For part (b), again, the only such polynomial is the identity polynomial.

   We have $p(0) = 0$, $p(1) = p(0)^2 + 1 = 1$, $p(2) = p(1)^2 + 1 = 2$, $p(5) = p(2)^2 + 1 = 5$, $p(26) = p(5)^2 + 1 = 26$ and in general, by induction, if the sequence (a_n) is defined recursively by $a_0 = 0$ and $a_{n+1} = a_n^2 + 1$, then $p(a_n) = a_n$. Since the sequence (a_n) is strictly increasing, we find that there are infinitely many distinct values x for which $p(x) = x$; as in the last part, this tells us that $p(x) = x$ is the only possible polynomial satisfying the given conditions.

4. Does there exist a non-zero polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$ for all x?

   Solution: From a Northwestern Putnam prep class.

   No. For positive integer n we have

   $$f(n) = \frac{n}{n+1}f(n-1) = \frac{n-1}{n+1}f(n-2) = \ldots = 0f(-1) = 0.$$

   Hence $f(x)$ has infinitely many zeros, and must be identically zero; $f(x) \equiv 0$.

5. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. Suppose that there exist four distinct integers a, b, c, d with $p(a) = p(b) = p(c) = p(d) = 5$. Prove that there is no integer k with $p(k) = 8$.

   Solution: From a Northwestern Putnam prep class.

   Set $q(x) = p(x) - 5$. We have $q(a) = q(b) = q(c) = q(d) = 0$ and so $q(x) = r(x)(x-a)(x-b)(x-c)(x-d)$, where $r(x)$ is some rational polynomial; but in fact (by Gauss' Lemma), $r(x)$ is a polynomial over integers.

   Aside: Why is $r(k)$ above a polynomial over integers? Suppose $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ (call this expression 1), with all a_i integers, factors as $(x-c)(x^{n-1} + r_{n-2}x^{n-2} + \ldots + r_1 x + r_0)$ (call this expression 2), where c is an integer. Then necessarily the r_i are rational numbers; but in fact, we can show that they are all integers. This is obvious when $c = 0$, so assume $c \neq 0$. Expanding out the factorization and equating coefficients, we get

   $$\begin{aligned} a_{n-1} &= r_{n-2} - c \\ a_{n-2} &= r_{n-3} - cr_{n-2} \\ a_{n-3} &= r_{n-4} - cr_{n-3} \\ \ldots & \\ a_2 &= r_1 - cr_2 \\ a_1 &= r_0 - cr_1 \\ a_0 &= -cr_0. \end{aligned}$$

   Now evaluating both expression 1 and expression 2 at $x = c$, we get

   $$c^n + a_{n-1}c^{n-1} + \ldots + a_2 c^2 + a_1 c + a_0 = 0.$$

   plugging in $a_0 = -cr_0$ yields

   $$c\left(c^{n-1} + a_{n-1}c^{n-2} + \ldots + a_2 c + a_1 - r_0\right) = 0.$$

   Utilizing $c \neq 0$, we conclude that

   $$c^{n-1} + a_{n-1}c^{n-2} + \ldots + a_2 c + a_1 = r_0$$

and so, since the left-hand side is clearly an integer, so is the right-hand side, r_0. Now plugging $a_1 = r_0 - cr_1$ into this last inequality, and dividing by c, we get

$$c^{n-2} + a_{n-1}c^{n-3} + \ldots + a_2 = r_1$$

so r_1 is also an integer. Continuing in this manner we get, for general k,

$$c^{n-(k+1)} + a_{n-1}c^{n-(k+2)} + \ldots + a_{k+1} = r_k$$

for $k \leq n-2$ (this could be formally proved by induction), which allows us to conclude that all of the r_i's are integers.

Back to solution: Now suppose there is an integer k with $p(k) = 8$. Then $q(k) = 3$, so $r(k)(k-a)(k-b)(k-c)(k-d) = 3$. Since $r(k)$, $(k-a)$, $(k-b)$, $(k-c)$ and $(k-d)$ are all integers, and 3 is prime, one of the five must be ± 3 and the remaining four must be ± 1. It follows that at least three of $(k-a)$, $(k-b)$, $(k-c)$ and $(k-d)$ must be ± 1, and so at least two of them must take the same value; this contradicts the fact that a, b, c and d are distinct.

6. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

has exactly n distinct real roots?

Solution: Putnam competition, 1990 problem B5.

We can explicitly construct such a sequence. Start with $a_0 = 1$ and $a_1 = -1$ (so case $n = 1$ works fine). We'll construct the a_i's inductively, always alternating in sign. Suppose we have $a_0, a_1, \ldots, a_{n-1}$. The polynomial $p_{n-1}(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1}$ has real distinct roots $x_1 < \ldots < x_{n-1}$. Choose $y_1, \ldots, y_n$ so that

$$y_1 < x_1 < y_2 < x_2 < \ldots < y_{n-1} < x_{n-1} < y_n.$$

The sequence $p_{n-1}(y_1), p_{n-1}(y_2), \ldots, p_{n-1}(y_n)$ alternates in sign (think about the graph of $y = p_{n-1}(x)$). As long as we choose a_n sufficiently close to 0, the sequence $p_n(y_1), p_n(y_2), \ldots, p_n(y_n)$ alternates in sign (this is by continuity). So, choose such an a_n. Now choose a y_{n+1} sufficiently large that $p_n(y_{n+1})$ has the opposite sign to $p_n(y_n)$ (this is where alternating the signs of the a_i's comes in — such a y_{n+1} exists exactly because a_n and a_{n-1} have opposite signs). We get that the sequence $p_n(y_1), p_n(y_2), \ldots, p_n(y_{n+1})$ alternates in sign. Hence $p_n(x)$ has n distinct real roots: one between y_1 and y_2, one between y_2 and y_3, etc., up to one between y_n and y_{n+1}. This accounts for all its roots, and we are done.

12 Week eleven (November 12) — Probability

Discrete probability may be thought about along the following lines: an experiment is performed, with a set S, the *sample space*, of possible observable outcomes (e.g., roll a dice and note the uppermost number when the dice lands; then S would be $\{1, 2, 3, 4, 5, 6\}$). S may be finite or countable for our purposes. An *event* is a subset A of S; the event *occurs* if the observed outcome is one of the elements of A (e.g., if $A = \{2, 4, 6\}$, which we might describe as the event that an even number is rolled, then we would say that A occurred if we rolled a 4, and that it did not occur if we rolled a 5). The compound event $A \cup B$ is the event that at least one of A, B occur; $A \cap B$ is the event that both A and B occur, and A^c $(= S \setminus A)$ is the event that A did not occur.

A *probability function* is a function P that assigns to each event a real number, which is intended to measure how likely A is to occur, or, in what proportion of a very large numbers of independent repetitions of the experiment does A occur. P should satisfy the following three rules:

1. $P(A) \geq 0$ always (events occur with non-negative probability);
2. $P(S) = 1$ (something always happens); and
3. if A and B are disjoint events (no outcomes in common) then $P(A \cup B) = P(A) + P(B)$; more generally, if $A_1, A_2, \ldots$ is a countable collection of mutually disjoint events, then
$$P(U_i A_i) = \sum_i P(A_i).$$

Three consequences of the rules are the following relations that one would expect:

1. If $A \subseteq B$ then $P(A) \leq P(B)$;
2. $P(\emptyset) = 0$; and
3. $P(A^c) = 1 - P(A)$.

Usually one constructs the probability function in the following way: intuitition/experiment/some underlying theory suggests that a particular $s \in S$ will occur a proportion p_s of the time, when the experiment is repeated many times; a reality check here is that p_s should be non-negative, and that $\sum_{s \in S} p_s = 1$. One then sets
$$P(A) = \sum_{s \in A} p_s;$$
it is readily checked that this function satisfies all the axioms.

In the particular case when S is finite and intuition/experiment/some underlying theory suggests that all outcomes $s \in S$ are equally likely to occur, we get the classical "definition" of the probability of an event:
$$P(A) = \frac{|A|}{|S|},$$

and calculating probabilities comes down to counting.

Example: I toss a coin 100 times. How likely is it that I get exactly 50 heads?

Solution: All 2^{100} lists of outcomes of 100 tosses are equally likely, so each one should occur with probability $1/2^{100}$. The number of outcomes in which there are exactly 50 heads is $\binom{100}{50}$, so the required probability is

$$\frac{\binom{100}{50}}{2^{100}}.$$

A *random variable* X is a function that assigns to each outcome of an experiment a (usually real) numerical value. For example, if I toss a coin 100 times, I may not be interested in the particular list of heads and tails I get, just in the total number of heads, so I could define X to be the function that takes in a string of 100 heads and tails, and returns as the numerical value the number of heads in the string. The *density function* of the random variable X is the function $p_X(x) = P(X = x)$, where "$P(X = x)$" is shorthand for the event "the set of all outcomes for which X evaluates to x". For tossing a coin 00 times and counting the number of heads, the density function is

$$p_X(x) = \begin{cases} \binom{100}{x}2^{-100} & \text{if } x = 0, 1, 2, \ldots, 100 \\ 0 & \text{otherwise.} \end{cases}$$

More generally, we have the following:

Binomial distribution: I toss a coin n times, and each time it comes up heads with some probability p. Let X be the number of heads that comes up. The random variable X is called the *binomial distribution with parameters n and p*, and has density function

$$p_X(x) = \begin{cases} \binom{n}{x}p^x(1-p)^{n-x} & \text{if } x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Note that by the binomial theorem.

$$\sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k} = (p + (1-p))^n = 1.$$

The *expected value* of a probability distribution/random variable is a measure of the average value of a long sequence of readings from that distribution; it is calculated as a weighted average:

$$E(X) = \sum_x x p_X(x)$$

with reading x being given weight $p_X(x)$. For example, if X is the binomial distribution with parameters n and p, then $E(X)$ is

$$\begin{aligned} \sum_{k=0}^{n} k\binom{n}{k} p^k(1-p)^{n-k} &= np\sum_{k=0}^{n} \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} \\ &= np(p + (1-p))^{n-1} \\ &= np, \end{aligned}$$

as we would expect.

It is worth knowing that expectation is a linear function:

Linearity of expectation: If a probability distribution/random variable X can be written as the sum $X_1 + \ldots + X_n$ of n (usually simpler) probability distributions/random variables, then

$$E(X) = E(X_1) + \ldots + E(X_n)$$

Example: n boxes have labels 1 through n. n cards with numbers 1 through n written on them (one number per card) are distributed among the n boxes (one card per box). On average how many boxes get the card whose number is the same as the label on the box?

Solution: Let X_i be the random variable that takes the value 1 if card i goes into box i, and 0 otherwise; $p_{X_i}(1) = 1/n$, $p_{X_i}(0) = 1 - (1/n)$ and $p_{X_i}(x) = 0$ for all other x's, so $E(X_i) = 1/n$. Let X be the random variable that counts the number of boxes that get the right card; since $X = X_1 + \ldots X_n$ we have

$$E(X) = E(X_1) + \ldots + E(X_n) = n(1/n) = 1$$

(independent of n!) [This is the famous *problem of derrangements*.]

One of the rules of probability is that for disjoint events A, B, we have $P(A \cup B) = P(A) + P(B)$. If A and B have overlap, this formula overcounts by including outcomes in $A \cap B$ twice, so should be corrected to

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

For three events A, B, C, a Venn diagram readily shows that

$$P(A \cup \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

There is a natural generalization:

Inclusion-exclusion (also called the sieve formula):

$$\begin{aligned} P(\cup_{i=1}^{n} A_i) \;\; = \;\; & \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \\ & + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots \\ & +(-1)^{\ell-1} \sum_{i_1<i_2<\ldots<i_\ell} P(A_{i_1} \cap A_{i_2} \cap A_{i_\ell}) + \ldots \\ & +(-1)^{n-1} P(A_1 \cap A_2 \cap A_n). \end{aligned}$$

Inclusion-exclusion is often helpful because calculating probabilities of intersections is easier than calculation probabilities of unions.

Example: In the problem of derrangements discussed above, what is the exact probability that no box gets the correct card?

Solution: Let A_i be the event that box i gets the right card. We have $P(A_i) = 1/n$, and more generally for $i_1 < i_2 < \ldots < i_\ell$ we have

$$P(A_{i_1} \cap A_{i_2} \cap A_{i_\ell}) = \frac{(n-\ell)!}{n!}$$

(there are $n!$ distributions of cards, and to make sure that boxes i_1 through i_ℓ get the right card, we are forced to place these ℓ cards each in a predesignated box; but the remaining $n-\ell$ cards can be completely freely distributed among the remaining boxes). By inclusion-exclusion,

$$P(\cup_{i=1}^n A_i) = \sum_{\ell=1}^n (-1)^{\ell-1}\binom{n}{\ell}\frac{(n-\ell)!}{n!},$$

the binomial term coming from selecting $i_1 < i_2 < \ldots < i_\ell$. We want the probability of *none* of the boxes getting the right card, which is the complement of $\cup_{i=1}^n A_i$:

$$\begin{aligned} P((\cup_{i=1}^n A_i)^c) &= 1 - \sum_{\ell=1}^n (-1)^{\ell-1}\binom{n}{\ell}\frac{(n-\ell)!}{n!} \\ &= \sum_{\ell=0}^n (-1)^{\ell}\binom{n}{\ell}\frac{(n-\ell)!}{n!} \\ &= \sum_{\ell=0}^n \frac{(-1)^\ell}{\ell!}. \end{aligned}$$

Note that this is the sum of the first $n+1$ terms in the power series of e^x around 0, evaluated at $x = -1$, so as n gets larger the probability of there being no box with the right card approaches $1/e$.

There is a counting version of inclusion-exclusion, that is very useful to know:

Inclusion-exclusion (counting version):

$$\begin{aligned} |\cup_{i=1}^n A_i| &= \sum_{i=1}^n |A_i| - \sum_{i<j} |A_i \cap A_j| \\ &\quad + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots \\ &\quad + (-1)^{\ell-1} \sum_{i_1<i_2<\ldots<i_\ell} |A_{i_1} \cap A_{i_2} \cap A_{i_\ell}| + \ldots \\ &\quad + (-1)^{n-1}|A_1 \cap A_2 \cap A_n|. \end{aligned}$$

Example: How many numbers are there, between 1 and n, that are relatively prime to n (have no factors in common)?

Solution: Let n have prime factorization $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$. Let A_i be the set of numbers between 1 and n that are multiples of p_i. We have $|A_i| = n/p_i$, and more generally for $i_1 < i_2 < \ldots < i_\ell$ we have

$$|A_{i_1} \cap A_{i_2} \cap A_{i_\ell}| = \frac{n}{p_{i_1} \ldots p_{i_\ell}}.$$

We want to know $|(A_{i_1} \cap A_{i_2} \cap A_{i_\ell})^c|$ (complement taken inside of $\{1, \ldots, n\}$), because this is exactly the set of numbers below n that share no factors in common with n. By inclusion-exclusion,

$$\begin{aligned} |\left(\cup_{i=1}^n A_i\right)^c| &= n\left(1 - \sum_{i=1}^k \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \ldots + \frac{(-1)^n}{p_1 p_2 \ldots p_k}\right) \\ &= n\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_k}\right). \end{aligned}$$

The function

$$\varphi(n) = n\prod_{i=1}^k \left(1 - \frac{1}{p_i}\right)$$

counting the number of numbers between 1 and n that are relatively prime to n, is called the *Euler totient function.*

The only mention I'll make of probability with uncountable underlying sample spaces is this: if R is a region in the plane, then a natural model for "selecting a point from R, all points equally likely", is to say that for each subset R' of R, the probability that the selected point will be in R' is Area(R')/area(R), that is, proportional to the area of R'. This idea naturally extends to more general spaces.

Example: I place a small coin at a random location on a 3 foot by 5 foot table. How likely is it that the coin is within one foot of some edge of the table?

Solution: There's a 1 foot by 3 foot region at the center of the table, consisting of exactly those points that are not within one foot of some edge of the table; assuming that the coin is equally likely to be placed at any location, the probability of landing in this region is $(1 \times 3)/(3 \times 5) = .2$, so the probability of landing withing one foot of some edge is $1 - .2 = .8$.

12.1 Week eleven problems

1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

2. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b}+c)/d$ where a, b, c and d are integers.

3. A bag contains 2019 red balls and 2019 black balls. We remove two balls at a time repeatedly and (i) discard both if they are the same color and (ii) discard the black ball and return the red ball to the bag if their colors differ. What is the probability that this process will terminate with exactly one red ball in the bag?

4. You have coins $C_1, C_2, \ldots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

5. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

6. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

12.2 Week eleven solutions

1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

   Solution: This was Putnam Competition 2002, Problem B1.

   Some doodling with small examples suggests the following: if Shanille throws a total of n free throws, $n \geq 3$, then for each k in the range $[1, n-1]$ the probability that she makes exactly k shots in $1/(n-1)$ (independent of k).

   We can prove this by induction on n, with $n = 3$ very easy. For $n > 3$, we start with the extreme case $k = 1$. The probability that she makes exactly one shot in total is the probability that she misses each of shots 3 through n, which is

   $$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n-3}{n-2} \cdot \frac{n-2}{n-1} = \frac{1}{n-1}.$$

   For $k > 1$, there are two (mutually exclusive) ways that she can make k shots in total:

   (a) Make $k-1$ of the first $n-1$, and make the last; the probability of this happening is, by induction,
   $$\frac{1}{n-2} \cdot \frac{k-1}{n-1},$$
   or

   (b) make k of the first $n-1$, and miss the last; the probability of this happening is, by induction,
   $$\frac{1}{n-2} \cdot \frac{n-1-k}{n-1}.$$

   Thus the net probability of making k shots is

   $$\frac{1}{n-2} \cdot \frac{k-1}{n-1} + \frac{1}{n-2} \cdot \frac{n-1-k}{n-1} = \frac{1}{n-1},$$

   and we are done by induction.

   The answer to the given question is $1/99$ ($n = 100$).

2. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$ where a, b, c and d are integers.

   Solution: This was on the Putnam Competition 1989, Problem B1. Note that this is an example of a Putnam question with a typo: the correct (and officially sanctioned) answer is as given below, but notice that $c = -5$, which is not a positive integer; whereas the question (as it officially appeared) demanded that a, b, c, d all be *positive* integers.

Place the dartboard on the x-y plane, with vertices at $(0,0)$, $(0,2)$, $(2,2)$ and $(2,0)$, so center at $(1,1)$. We want to compute the are of the set of points inside this square which are closer to $(1,1)$ than any of $x=0,2$, $y=0,2$. We'll just consider the triangle T bounded by vertices $(0,0)$, $(1,0)$, $(1,1)$; by symmetry, this is one eight of the desired area.

For a point (x,y) in T, the distance to $(1,1)$ is $\sqrt{(x-1)^2+(y-1)^2}$, and the distance to the nearest of $x=0,2$, $y=0,2$ is just y. So the curve $\sqrt{(x-1)^2+(y-1)^2}=y$, or

$$y=\frac{x^2-2x+2}{2}$$

cuts T into two regions, one (containing $(1,1)$) being the points that are closer to $(1,1)$ than the nearest of $x=0,2$, $y=0,2$. This curve hits the line $x=y$ at $(2-\sqrt{2},2-\sqrt{2})$. So the desired area inside T is the total area of T (which is $1/2$) minus the area bounded by $x=y$ from $(0,0)$ to $(2-\sqrt{2},2-\sqrt{2})$, then $y=(x^2-2x+2)/2$ to $(1,1/2)$, then $x=1$ to $(1,0)$, then the x-axis back to $(0,0)$. This area is the area of the triangle bounded by $(0,0)$, $(2-\sqrt{2},2-\sqrt{2})$, and $(2-\sqrt{2},0)$ (which is $(2-\sqrt{2})^2/2$), plus

$$\int_{2-\sqrt{2}}^{1}\frac{x^2-2x+2}{2}\,dx=\frac{1}{3}(4\sqrt{2}-5);$$

grand total $(1/3)(4-2\sqrt{2})$. It follows that the desired area inside T is

$$\frac{1}{2}-\frac{1}{3}(4-2\sqrt{2})=\frac{1}{6}(4\sqrt{2}-5),$$

and so the total desired area is eight times this, or $(4/3)(4\sqrt{2}-5)$.

Since the total area of the square is 4, the desired probability is thus

$$\frac{1}{3}(4\sqrt{2}-5)\approx .218951.$$

3. A bag contains 2019 red balls and 2019 black balls. We remove two balls at a time repeatedly and (i) discard both if they are the same color and (ii) discard the black ball and return the red ball to the bag if their colors differ. What is the probability that this process will terminate with exactly one red ball in the bag?

Solution: I heard this problem from David Cook.

It helps to generalize to r red balls and b black balls, since as the process goes along the number of balls of the two colors will not be equal. A little experimentation suggests the following: if the process is started with and odd number $r\geq 1$ of red balls, and $b\geq 0$ balls, then it *always* ends with one red ball. We prove this by induction on $r+b$. Formally: for each $n\geq 1$, $P(n)$ is the proposition "if the process is started with and odd number $r\geq 1$ of red balls, and $b\geq 0$ balls, with $r+b=n$, then it *always* ends with one red ball", and we prove $P(n)$ by induction on n.

Base case $n=1$ is trivial, as is base case $n=2$. For base case $n=3$, we either start with three red balls, in which case after one step we are down to one red, or we start with one red and two blues. In this case, one third of the time we first pick the two blacks, and we are down to one red, while two thirds of the time we first pick a black and a red, and we are down to one red and one black, leaving us with one red after step two.

Now consider $n \geq 4$, and start with r red balls and b black balls, r odd and $r + b = n$. If on the first step we pick two reds, then we are left with $r - 2$ red balls and b black balls. Note that $r - 2$ is odd and $(r - 2) + b = n - 2$, so by induction in this case we always end with one red. If on the first step we pick two blacks, then we are left with r red balls and $b - 2$ black balls. Note that r is odd and $r + (b - 2) = n - 2$, so by induction in this case we always end with one red. Finally, if on the first step we pick a black and a red, then we are left with r red balls and $b - 1$ black balls. Note that r is odd and $r + (b - 1) = n - 1$, so by induction in this case we always end with one red. This completes the induction.

Since 2019 is odd, the probability of ending with one red, starting with 2019 red balls and 2019 black balls is 1.

4. You have coins $C_1, C_2, \ldots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

   Solution: This was Putnam Competition 2001, A2.

   Let p_n be the required probability. We have $p_1 = 1/3$. For $n \geq 2$ we can express p_n in terms of p_{n-1} as follows: we get an odd number of heads either by getting an odd number of heads among the first $n - 1$ (probability p_{n-1}) and a tail on the nth coin (probability $2n/(2n+1)$), or by getting an even number of heads among the first $n - 1$ (probability $1 - p_{n-1}$) and a head on the nth coin (probability $1/(2n+1)$). This leads to the recurrence

$$p_n = \frac{2np_{n-1}}{2n+1} + \frac{1 - p_{n-1}}{2n+1} = \frac{1 + (2n-1)p_{n-1}}{2n+1}$$

   valid for $n \geq 2$. We claim that the solution to this recurrence is $p_n = n/(2n+1)$. We prove this claim by induction on n. This base case $n = 1$ is clear. For $n \geq 2$ we have, using the inductive hypothesis in the second equality,

$$\begin{aligned} p_n &= \frac{1 + (2n-1)p_{n-1}}{2n+1} \\ &= \frac{1 + (2n-1)\frac{n-1}{2(n-1)+1}}{2n+1} \\ &= \frac{n}{2n+1}, \end{aligned}$$

   completing the induction.

5. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

Solution: This was Putnam competition 1993, problem B3.

The closest integer to x/y is 0 if $x < 2y$. It is $2n$ (for $n > 0$) if $2x/(4n+1) < y < 2x/(4n-1)$. (We can ignore $y/x = 2/(2m+1)$ since it has probability zero.)

Hence the required probability is $p = 1/4 + (1/3 - 1/5) + (1/7 - 1/9) + \dots$. But now recall that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$, so $p = 5/4 - \pi/4$.

6. Let k be a positive integer. Suppose that the integers $1, 2, 3, \dots, 3k+1$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

   Solution: This was Putnam competition, problem A3.

   The official solution, published in the *American Mathematical Monthly*, is very nicely presented, so I reproduce it here verbatim:

   "The number of ways to write down $1, 2, 3, \dots, 3k+1$ in random order is $(3k+1)!$, so we want to count the number of ways in which none of the "partial sums" is divisible by 3. First, consider the integers modulo 3 : $1, 2, 0, 1, 2, 0, \dots, 1, 2, 0, 1$. To write these with none of the partial sums divisible by 3, we must start with a 1 or a 2. After that, we can include or omit 0's at will without affecting whether any of the partial sums are divisible by 3, so suppose [initially] we omit all 0's. The remaining sequence of 1's and 2's must then be of the form

   $$1, 1, 2, 1, 2, 1, 2, \dots$$

   or

   $$2, 2, 1, 2, 1, 2, 1, \dots$$

   (once you start, the rest of the sequence is forced by the condition that no partial sum is divisible by 3). However, a sequence of the form $2, 2, 1, 2, 1, 2, 1, \dots$ has one more 2 than 1, and we need to have one more 1 than 2. So the only possibility for our sequence modulo 3, once the 0's are omitted, is $1, 1, 2, 1, 2, 1, 2, \dots$. There are $2k+1$ numbers in this sequence, and the k 0's can be returned to the sequence arbitrarily except at the beginning. So the number of ways to form the complete sequence modulo 3 equals the number of ways to distribute the k identical 0's over $2k+1$ boxes (the "slots" after the 1's and 2's), which by a standard "stars and bars" argument is $\binom{3k}{k}$. Once this is done, there are $k!$ ways to replace the k 0's in the sequence modulo 3 by the actual integers $3, 6, \dots, 3k$. Also, there are $k!$ ways to "reconstitute" the 2's and $(k+1)!$ ways for the 1's. So the answer is

   $$\frac{\binom{3k}{k} k! k! (k+1)!}{(3k+1)!}."$$

13 Week twelve (November 19) — Games

These problem are all about games played between two players. Usually when these problems appear in the Putnam competition, you are asked to determine which player wins when both players play as well as possible. Once you have decided which player wins (maybe based on analyzing small examples), you need to prove this in general. Often this entails demonstrating a *winning strategy*: for each possible move by the losing player, you can try to identify a single appropriate response for the winning player, such that if the winning player always uses these responses as the game goes on, then she will indeed win. It's important to remember that you must produce a response for the winning player for *every* possible move of the losing player — not just a select few.

13.1 Week twelve problems

1. A chocolate bar is made up of a rectangular m by n grid of small squares. Two players take turns breaking up the bar. On a given turn, a player picks a rectangular piece of chocolate and breaks it into two smaller rectangular pieces, by snapping along one whole line of subdivisions between its squares. The player who makes the last break wins. If both players play optimally, who wins?

2. Two players alternately draw diagonals between vertices of a regular polygon. They may connect two vertices if they are non-adjacent (i.e. not a side) and if the diagonal formed does not cross any of the previous diagonals formed. The last player to draw a diagonal wins.

   Who wins if the polygon has 2019 vertices?

3. Two players play a game in which the first player places a king on an empty 8 by 8 chessboard, and then, starting with the second player, they alternate moving the king (in accordance with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy?

4. I shuffle a regular deck of cards (26 red, 26 black), and begin to turn them face-up, one after another. At some point during this process, you say "STOP!". You can say stop as early as before I've even turned over the first card, or as late as when there is only one card left to be turned over; the only rule is that at some point you must say it. Once you've said stop, I turn over the next card. If it is red, you win the game, and if it is black, you lose.

   If you play the strategy "say stop before even a single card has been turned over", you have a 50% chance of winning the game. Is there a more clever strategy that gives you a better than 50% chance of winning the game?

5. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, then the cost of jumping from a to b is $b^3 - ab^2$.

   For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

6. Alan and Barbara play a game in which they take turns filling entries of an initially empty 1024 by 1024 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

7. Two players, A and B, take turns naming positive integers, with A playing first. No player may name an integer that can be expressed as a linear combination, with positive integer coefficients, of previously named integers. The player who names "1" loses. Show that no matter how A and B play, the game will always end.

8. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

13.2 Week twelve solutions

1. A chocolate bar is made up of a rectangular m by n grid of small squares. Two players take turns breaking up the bar. On a given turn, a player picks a rectangular piece of chocolate and breaks it into two smaller rectangular pieces, by snapping along one whole line of subdivisions between its squares. The player who makes the last break wins. If both players play optimally, who wins?

   Solution: I learned this (infuriating!) problem from Peter Winkler.

   There is one piece of chocolate to start, and mn pieces at the end. Each turn by a player increases the number of squares by 1. Hence a game lasts $mn - 1$ turns, completely independently of the strategies of the two players! The winner is determined by the parity of m and n (*no* strategy involved!): if m and n are both odd, $mn - 1$ is even and player 2 wins. Otherwise $mn - 1$ is odd and player 1 wins.

2. Two players alternately draw diagonals between vertices of a regular polygon. They may connect two vertices if they are non-adjacent (i.e. not a side) and if the diagonal formed does not cross any of the previous diagonals formed. The last player to draw a diagonal wins.

   Who wins if the polygon has 2019 vertices?

   Solution: UTexas Putnam prep problem.

   It's easy to prove (by induction) that if the game is played on an n-sided polygon ($n \geq 4$) then it will have exactly $n - 3$ moves. So on a 2015-sided polygon, there will by 2012 moves, and player 2 must move last (and win). Again, no strategy is involved!

3. Two players play a game in which the first player places a king on an empty 8 by 8 chessboard, and then, starting with the second player, they alternate moving the king (in accordance with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy?

   Solution: UTexas Putnam prep problem.

   Player 2 has a winning strategy. She can imagine the board as being covered with non-overlapping 2-by-1 dominos (there are many ways to cover an 8 by 8 board with dominos). Wherever player 1 puts the king, player 2 moves it to the *other* square in the corresponding domino. She then repeats this strategy until the game is over.

4. I shuffle a regular deck of cards (26 red, 26 black), and begin to turn them face-up, one after another. At some point during this process, you say "STOP!". You can say stop as early as before I've even turned over the first card, or as late as when there is only one card left to be turned over; the only rule is that at some point you must say it. Once you've said stop, I turn over the next card. If it is red, you win the game, and if it is black, you lose.

If you play the strategy "say stop before even a single card has been turned over", you have a 50% chance of winning the game. Is there a more clever strategy that gives you a better than 50% chance of winning the game?

Solution: Asked me by David Wilson, in an interview for a job at Microsoft Research.

Here's the quick-and-dirty solution: The game fairly easily seen to be equivalent to the following: exactly as before, except now when you say "STOP", I turn over the *bottom* card in the pile of cards that remains. In this formulation, it is clear that there cannot be a strategy that gives you better than a 50% chance of winning.

Here's a more prosaic solution. Suppose that instead of being played with a balanced deck, it is played with a deck that has a red cards and b black cards. We claim that if there are a red cards and b black cards, then there is no strategy better than the naive one of saying stop before a single card has been turned over; note that with this strategy you win with probability $a/(a+b)$. We prove this by induction on $a+b$. If $a+b=1$, then (whether $a=1$ or $a=0$) the result is trivial. Suppose $a+b \geq 2$. To get a strategy that potentially improves on the proposed best strategy, you must at least wait for the first card to be turned over. Two things can happen:

- The first card turned over is red; this happens with probability $a/(a+b)$. Once this happens, you are playing a new version of the game, with $a-1$ red cards and b black cards, and by induction your best winning strategy has you winning with probability $(a-1)/(a-1+b)$.
- The first card turned over is black; this happens with probability $b/(a+b)$. Once this happens, you are playing a new version of the game, with a red cards and $b-1$ black cards, and by induction your best winning strategy has you winning with probability $a/(a+b-1)$.

Your probability of winning the original game is therefore at most

$$\frac{a}{a+b} \times \frac{a-1}{a+b-1} + \frac{b}{a+b} \times \frac{a}{a+b-1} = \frac{a}{a+b}.$$

5. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, then the cost of jumping from a to b is $b^3 - ab^2$.

   For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

   Solution: This was problem B2 of the 1999 Putnam Competition.

   All c such that $1/3 \leq c \leq 1$. For a detailed solution, see e.g. `http://math.hawaii.edu/home/pdf/putnam/Putnam_2009.pdf`.

6. Alan and Barbara play a game in which they take turns filling entries of an initially empty 1024 by 1024 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled.

Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Solution: Putnam competition 2008, problem A2.

Barbara has a winning strategy. For example, Whenever Alan plays x in row i, Barbara can play $-x$ in some other place in row i (since there are an even number of places in row i, Alan will never place the last entry in a row if Barbara plays this strategy). So Barbara can ensure that all row-sums of the final matrix are 0, so that the column vector of all 1's is in the kernel of the final matrix, so it has determinant zero.

7. Two players, A and B, take turns naming positive integers, with A playing first. No player may name an integer that can be expressed as a linear combination, with positive integer coefficients, of previously named integers. The player who names "1" loses. Show that no matter how A and B play, the game will always end.

   Solution: This is the game of *Sylver coinage*, invented by John H. Conway; see `http://en.wikipedia.org/wiki/Sylver_coinage`. It is named after J. J. Sylvester, who proved that if a and b are relatively prime positive integers, then the largest positive integer that cannot be expressed as a positive linear combination of a and b is $(a-1)(b-1)-1$.

   Suppose the first k moves consist of naming $x_1, \ldots, x_k$. Let g_k be the greatest common divisor of the x_i's. Consider the set of numbers expressible as a linear combination of the x_i's over positive integers. Each x in this set is an integer multiple of g_k (g_k divides the right-hand side of $x = \sum_i a_i x_i$, so it divides the left-hand side). We claim that there is some m such that all multiples of g_k greater than mg_k are in this set.

   If we can prove this claim, we are done. The sequence $(g_1, g_2, g_3, \ldots)$ is non-increasing. It stays constant in going from g_i to g_{i+1} exactly when x_{i+1} is a multiple of g_i, and drops exactly when x_{i+1} is not a multiple of g_i. By our claim, once the sequence has reached a certain g, it can only stay there for a finite length of time. So eventually that sequence becomes constantly 1. But once the sequence reaches 1, there are only finitely many numbers that can be legitimately played, and so eventually 1 must be played.

   Here's what we'll prove, which is equivalent to the claim: if $x_1, \ldots, x_k$ are relatively prime positive integers (greatest common divisor equals 1) then there exists an m such that all numbers greater than m can be expressed as a positive linear combination of the x_i's. We prove this by induction on k. When $k = 1$, $x_k = 1$ and the result is trivial. For $k > 1$, consider $x_1, \ldots, x_{k-1}$. These may not be relatively prime; say their greatest common divisor is d. By induction, there's an m' such that all positive integer multiples of d greater than $m'd$ can be expressed as a positive linear combination of the $x_1, \ldots, x_{k-1}$. Now d and x_k must be relatively prime (otherwise the x_i's would not be relatively prime), which means that there must be some positive integer e (which way may assume is between 1 and $x_k - 1$) with $ed \equiv 1$ (modulo x_k). If we add any multiple of x_k to e to get e', we still get $e'd \equiv 1$ (modulo x_k). Pick a multiple large enough that $e' > m'$. By induction, $e'd$ can be expressed as a positive integer combination of $x_1, \ldots, x_{k-1}$. So too can $2e'd, 3e'd, \ldots, x_k e'd$. These x_k numbers cover all the residue classes modulo x_k. Let m be one less than the largest of these numbers. For $\ell > m$, we can express ℓ as a positive linear combination of $x_1, \ldots, x_k$ as follows: first, determine the residue class of ℓ modulo x_k, say it's p. Then add the appropriate positive integer multiple of x_k to $pe'd$ (which can can be expressed as a positive integer combination of $x_1, \ldots, x_{k-1}$).

8. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

   Solution: Putnam 2006, A2.

   Here is, verbatim, the solution published in the *American Mathematical Monthly.*

   Suppose there are only finitely many n such that Bob will win if Alice starts with n stones, say all such $n < N$. Take $K > N$ so that $K - m + 1$ is composite for $m = 0, \ldots, N$. Starting with $n = K$, Alice must remove $p - 1$ stones, for p a prime number, leaving $m = K - p + 1$ stones. But $m > N$ since $K - m + 1 = p$ is prime and $K - m + 1$ is composite for $m < N$. By assumption, Alice can win starting from a heap of m stones. But it is Bob's turn to move, and so he could use the same strategy Alice would have used to win. This applies for any first move Alice could have made from a heap of K stones. Hence Bob has a winning strategy for a number $K > N$ of stones, contrary to hypothesis. Instead there must be infinitely many n for which Bob has a winning strategy.

   Implicit in this solution is the following useful fact: in a finite, two-person game with no draws allowed, one of the players must have a winning strategy.

14 Week thirteen (November 26) — a grab-bag

This handout is a "mock Putnam" exam. Look over the problems, pick out some that you feel good about, and tackle them! You'll do best if you engage your conscious brain fully on a single problem, rather than hopping back-and-forth between problems every few minutes (but it's also a good idea to read all the problems before tackling one, to allow your subconscious brain to mull over the whole set). The ideal way to approach this set would be to give yourself three quiet, uninterrupted hours, and see how you do.

14.1 Week 13 problems

1. A sequence $a_0, a_1, a_2, \ldots$ of real numbers is defined recursively by
$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \qquad (n = 0, 1, 2, \ldots).$$
Find a general formula for a_n.

2. Consider a 7 by 7 checkerboard with the squares at the four corners removed (so that the remaining board has 45 squares). Is it possible to cover this board with 1 by 3 tiles so that no two tiles overlap? Explain!

3. Let f be a function on $[0, 2\pi]$ with continuous first and second derivatives and such that $f''(x) > 0$ for $0 < x < 2\pi$. Show that the integral
$$\int_0^{2\pi} f(x) \cos x \; dx$$
is positive.

4. Given a nonnegative integer b, call a nonnegative integer $a \leq b$ a *subordinate* of b if each decimal digit of a is at most equal to the decimal digit of b in the same position (counted from the right). For example, 1329 and 316 are subordinates of 1729, but 1338 is not since the second-last digit of 1338 is greater than the corresponding digit in 1729. Let $f(b)$ denote the number of subordinates of b. For example, $f(13) = 8$, since 13 has exactly 8 subordinates: 13, 12, 11, 10, 3, 2, 1, 0. Find a simple formula for the sum
$$S(n) = \sum_{0 \leq b < 10^n} f(b).$$

5. Let $a_1, a_2, \ldots, a_{65}$ be positive integers, none of which has a prime factor greater than 13. Prove that, for some i, j with $i \neq j$, the product $a_i a_j$ is a perfect square.

6. Let n be an even positive integer, and let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Given two permutations $\sigma_1, \sigma_2 \in S_n$, define their distance $d(\sigma_1, \sigma_2)$ by
$$d(\sigma_1, \sigma_2) = \sum_{k=1}^{n} |\sigma_1(k) - \sigma_2(k)|.$$
Determine, with proof, the maximal distance between two permutations in S_n, i.e., determine the exact value of $\max_{\sigma_1, \sigma_2 \in S_n} d(\sigma_1, \sigma_2)$.

15 Week fourteen (December 3) — Preparing for the Putnam Competition

The Putnam Competition will take place on

Saturday December 7,

in

Hayes-Healy 231,

with these being the times of the two papers:

10am-1pm (paper A), 3pm-6pm (paper B).

Lunch will be provided between the two papers, in the Math department lounge.

Paper, pencils, erasers and sharpeners will be provided. Feel free to bring brain fuel. Notes, calculators, book, laptops etc. are **not** allowed.

In what follows, I'm going to repeat some of the advice I've given on some previous handouts. I encourage you to look over this carefully, and bear it in mind as you go over this week's problems and (more importantly) as you take the Putnam competition. I have shamelessly appropriate much of this from Ravi Vakil's Stanford Putnam Preparation website (`http://math.stanford.edu/~vakil/putnam05/05putnam7.pdf`), and from Ioana Dumitriu's UWashington's "The Art of Problem Solving" website (`http://www.math.washington.edu/~putnam/index.html`). Both of these websites are filled with what I think is great advice.

Some general problem-solving tips

Remember that problem solving is a full-contact sport: throw everything you know at the problem you are tackling! Sometimes, the solution can come from an unexpected quarter. Here are some slogans to keep in mind when solving problems:

- Try small cases!
- Plug in small numbers!
- Do examples!
- Look for patterns!
- Draw pictures!
- Write lots!
- Talk it out!
- Choose good notation!
- Look for symmetry!

- Break into cases!
- Work backwards!
- Argue by contradiction!
- Consider extreme cases!
- Modify the problem!
- Make a generalization!
- Don't give up after five minutes!
- Don't be afraid of a little algebra!
- Take a break!
- Sleep on it!
- Ask questions!

And above all:

- Enjoy!

General Putnam tips

Getting ready

1. Try, if possible, to get a good night's sleep beforehand.
2. Wear comfortable clothes.
3. There's no need to bring paper, pencils, erasers or pencil sharpeners (unless you have some "lucky" ones); I will provide all of these necessary materials.
4. Bring snacks and drinks, to keep your energy up during each three-hour session. I'll provide pizza, fruit and soda for lunch, so if you are happy with that, there's no need to bring any lunch supplies.
5. During the competition, you can't use outside notes, computers, calculators, etc..

During the competition

1. Spend some time at the start looking over the questions. The earlier questions in each half tend to be easier, but this isn't always the case — you may be lucky/inspired and spot a way in to A4, before A1. Remember that you have three hours for each paper; that's a lot of time, so take some time to begin calmly.
2. When (rather than if) your mind gets tired, take a break; go outside, get a snack, use the bathroom, clear your head.

3. The "fifteen minute rule" can be helpful: if you find that you've been thinking about a problem aimlessly without having any serious new ideas for 15 minutes, then jump to a different problem, or take a break.

4. Don't become discouraged. Don't think of this as a test or exam. Think of it instead as a stimulating challenge. Often a problem will break open a couple of hours into the session. And the problem with your name on it might be in the afternoon session. Remember that the national median score for the entire competition is usually 0 or 1; getting somewhere on one question is significant (it puts you on the top half of the curve); getting a full question is a big deal.

5. If you solve a problem, write it up *very* well (rather than starting a new problem); grading on the Putnam competition is very severe. See the later section on writing for more on this.

Some specific mathematical tips

Here are some very simple things to remember, that can be very helpful, but that people tend to forget to do.

1. Try a few small cases out. Try a lot of cases out. Remember that the three hours of the Putnam competition is a long time — you have time to spare! If a question asks what happens when you have n things, or 2015 things, try it out with 1, 2, 3, 4 things, and try to form a conjecture. This is especially valuable for questions about sequences defined recursively.

2. Don't be afraid to use lots and lots of paper.

3. Don't be afraid of diving into some algebra. (Again, three hours is a long time . . ..) You shouldn't waste that much time, thanks to the 15-minute rule.

4. If a question asks to determine whether something is true or false, and the direction you initially guess doesn't seem to be going anywhere, then try guessing the opposite possibility.

5. Be willing to try (seemingly) stupid things.

6. Look for symmetries. Try to connect the problem to one you've seen before. Ask yourself "how would [person X] approach this problem"? (It's quite reasonable here for [Person X] to be [Chuck Norris]!)

7. Putnam problems always have slick solutions. That leads to a helpful meta-approach: "The only way this problem could have a nice solution is if this particular approach worked out, unlikely as it seems, so I'll try it out, and see what happens."

8. Show no fear. If you think a problem is probably too hard for you, but you have an idea, try it anyway. (Three hours is a long time.)

Some specific writing tips

As in a previous week, I highly recommend reading the essays:

- (short, with very practical advice and a few examples): `http://web.evanchen.cc/handouts/english/english.pdf`
- (long, with lots of examples of both good and bad writing): `https://artofproblemsolving.com/news/articles/how-to-write-a-solution`

You're working hard on a problem, focussing all your energy, applying all the good tips above, and all of a sudden a bulb lights above your head: you have figured out how to solve one of the problems! Awesome! But here's the hard, unforgiving truth: that warm tingle you're feeling is no guarantee

- that you have actually solved the problem (are you sure you have all the cases covered? Are you sure that all the little details work out? ...)

  and

- that you will get any or all credit for it.

Solving the problem is only half the battle. Now you have launch into the other half, convincing the grader that you have actually done so.

Life is tough, and Putnam graders may be even tougher. But here's a list of things which, when done properly, will yield a nice write-up which will appease any reasonable grader.

1. All that scrap paper, filled with your musings on the problem to date? Put it aside; you **must** write a clean, coherent solution on a fresh piece of paper.
2. Before you start writing, organize your thoughts. Make a list of all steps to the solution. Figure out what intermediary results you will need to prove. For example, if the problem involves induction, always start with the base case, and continue with proving that "true for n implies true for $n+1$". Make sure that the steps follow from each other logically, with no gaps.
3. After tracing a "road to proof" either in your head or (preferably) on scratch paper, start writing up the solution on a fresh piece of paper. The best way to start this is by writing a quick outline of what you propose to do. Sometimes, the grader will just look at this outline, say "Yes, she knows what she is doing on this problem!", and give the credit.
4. Lead with a **clear** statement of your final solution.
5. Complete each step of the "road" before you continue to the next one.
6. When making statements like "it follows trivially" or "it is easy to see", listen for quiet, nagging doubts. If you yourself aren't 100% convinced, how will you convince someone else? Even if it seems to follow trivially, check again. Small exceptions may not be obvious. The strategy "I am sure it's true, even if I don't see it; if I state that it's obvious, maybe the grader will believe I know how to prove it" has occasionally led its user to a score of 0 out of 10.

7. Organize your solution on the page; avoid writing in corners or perpendicular to the normal orientation. Avoid, if possible, post-factum insertion (if you discover you've missed something, rather than making a mess of the paper by trying to write it over, start anew. You have the time!)

8. Before writing each phrase, formulate it completely in your mind. Make sure it expresses an idea. Starting to write one thing, then changing course in mid-sentence and saying another thing is a sure way to create confusion.

9. Be as clear as possible. Avoid, if possible, long-winded phrases. Use as many words as you need – just do it clearly. Also, avoid acronyms.

10. If necessary, state intermediary results as "claims" or "lemmas" which you can prove right after stating them. If you cannot prove one of these results, but can prove the problem's statement from it, state that you will assume it, then show the path from it to the solution. You may get partial credit for it.

11. Rather than using vague statements like "and so on" or "repeating this process", formulate and prove by induction.

12. When you're done writing up the solution, go back and re-read it. Put yourself in the grader's shoes: can someone else read your write-up and understand the solution? Must one look for things in the corners? Are there "miraculous" moments?, et cetera.