Basic Combinatorics (Math 40210) Sec 01, Spring 2014, Quiz 2

Solutions

February 7, 2014

1. How many different solutions are there to the equation $a_1 + a_2 + \ldots + a_k = n$, if all of the a_i have to be integers that are at least 2? [If "2" is replaced by "1", then the answer is $\binom{n-1}{k-1}$, the number of (strong) compositions of n into k parts]. Justify your answer.

   Solution: Solving $a_1 + a_2 + \ldots + a_k = n$, with all a_i being integers at least 2, is the same as solving $a'_1 + a'_2 + \ldots + a'_k = n - k$, with all a'_i being integers at least 1 (then just set $a_i = a'_i + 1$ for each i). This in turn is the same as the number of (strong) compositions of $n - k$ into k parts; there are $\binom{n-k-1}{k-1}$ such compositions, so this is the answer to the original question.

2. Write down, and justify, the recurrence relation that expresses $S(n,k)$ in terms of $S(n-1,k)$ and $S(n-1,k-1)$ (here $S(n,k)$ is the Stirling number of the second kind, the number of partitions of $[n]$ into k non-empty blocks).

   Solution: $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. The $S(n-1,k-1)$ term on the right-hand side counts those partitions of $[n]$ into k-non-empty blocks, in which element n is in a block on its own (and so the remaining $n-1$ elements must be partitioned into $k-1$ non-empty blocks). The $kS(n-1,k)$ term on the right-hand side counts those partitions of $[n]$ into k-non-empty blocks, in which element n is *not* in a block on its own: to get such a partition, first the remaining $n-1$ elements should be partitioned into k non-empty blocks, and then elements n should be added to one of these blocks (this is the fact or k). This counts all partitions of $[n]$ into k-non-empty blocks, so the sum of these terms on the right-hand side is $S(n,k)$, the total number of partitions of $[n]$ into k-non-empty blocks.