

Proofs Crash Course

Winter 2011

Today's Topics

- Why are Proofs so Hard?
- Proof by Deduction
- Proof by Contrapositive
- Proof by Contradiction
- Proof by Induction

Why are Proofs so Hard?

“If it is a miracle, any sort of evidence will answer, but if it is a fact, proof is necessary”

-Mark Twain

Why are Proofs so Hard?

- o Proofs are very different from the math problems that you're used to in High School.
- o Proofs are problems that require a whole different kind of thinking.
- o Most proofs will not give you all of the information you need to complete them.

Before we go further...

- o Understanding the purpose of proofs is fundamental to understanding how to solve them.
- o Doing proofs is like making a map
 - o The goal is to get from point A to B using paths, roads, and highways.
 - o Proofs show us how two statements logically connect to each other through theorems, definitions, and laws.

Proof by Deduction

“The two operations of our understanding, intuition and deduction, on which alone we have said we must rely in the acquisition of knowledge.”

-Rene Descartes

Proof by Deduction

- o This is the most basic proof technique.
- o By using laws, definitions, and theorems you can get from A to B by starting at A and progressively moving towards B.
- o You start by assuming the conditional (the “if” part) and showing the logical flow to the conclusion (the “then” part).

Deductive Proof Example

Suppose you know the following:

if A then B

if B then C

if C then D

Show that if A then D.

Deductive Proof Example

Remember that deductive proofs start at the beginning and proceed towards the conclusion

Proof: Assume A is true. Therefore B must be true. Since B is true, C is true. Because C is true, D is true.

Hence, D. ■

Deductive Proof Analysis

- o Notice that the path taken to get from A to D was very direct and linear.
- o We started by assuming that A was true.
- o Then we used the given “laws” to show that D was true.

Deductive Proof Example

Prove the following statement:

If Jerry is a jerk, Jerry won't get a family.

Note: Many of you likely can prove this using some form of intuition. However, in order to definitively prove something, there need to be some agreed upon guidelines.

Deductive Proof Example

If Jerry is a jerk, Jerry won't get a family.

Let's also suppose that we have some guidelines:

- If somebody doesn't date, they won't get married.
- If you don't get married, you won't get a family.
- Girls don't date jerks.

Deductive Proof Example

How would you prove that Jerry won't get a family if he's a jerk?

Given:

1. If somebody doesn't date, they won't get married.
2. If you don't get married, you won't get a family.
3. Girls don't date jerks.
4. Jerry is a jerk. (Our assumption)

Conclusion:

Jerry won't get a family.

Deductive Proof Solution

Proof:

Suppose that Jerry is a jerk. We therefore know that girls don't date him. Therefore he will never get married. Hence, he won't get a family. ■

Deductive Proof Example

Prove the following:

if $f(x)$ is even, then $f(x)$ is not one-to-one

Deductive Proof Example

if $f(x)$ is even, then $f(-x)$ is not one-to-one.

- o How is this problem different from our previous ones?
 - o What does it mean for $f(x)$ to be even?
 - o What does it mean for $f(-x)$ to be one-to-one?
- o This is a main reason why proofs are so hard; they don't give you all of the information you need to solve the problem.

Deductive Proof Example

Remember the following:

$f(x)$ is even if $f(-x) = f(x)$ for all values of x .

$f(x)$ is one-to-one if for all x , $f(x)$ is unique.

How would you solve the problem?

if $f(x)$ is even, then $f(x)$ is not one-to-one.

Deductive Proof Solution

Proof:

Suppose that $f(x)$ is even. This implies that $f(x) = f(-x)$ for all x . Therefore there exists an a and b such that $f(a) = f(b)$. Therefore $f(x)$ is not one-to-one. ■

Note: Notice how we used a lot of words instead of math symbols? They are still present, but the main way of communicating with math is through using English. Who knew?

A Word of Caution

A common pitfall students fall in while solving proofs is by assuming that the **converse** of a statement is true.

BE EXTREMELY CAREFUL WHEN DOING THIS!

If it is raining outside, then the sidewalk is wet.

If the sidewalk is wet, then it is raining outside.

When the Converse is True

- o If the door is locked, I can't open it
- o If the door is unlocked, I can open it

- o Statements whose converse is guaranteed to be true have an “if and only if” clause.

- o If and only if the door is locked, I can't open it.
- o I can't open the door if and only if it is locked.

The Biconditional

- o The clause “if and only if” means that the statement is true read both ways.
- o A if and only if B is the same as saying:
 - o If A, then B
 - o If B, then A
- o Proving statements with an “if and only if” clause require us to show they are true in both directions.

Deductive Proof Example

Prove the following:

x is even if and only if $x + 1$ is odd

Note:

An even number y can be represented by $y = 2k$ for some integer k .

Similarly, an odd number z can be represented by $z = 2j + 1$ for some integer j .

Deductive Proof Solution

Proof:

Suppose that x is even. This means that there exists an integer k such that $x = 2k$. Therefore, $x + 1 = 2k + 1$. Since k is an integer, $x + 1$ must be odd.

Now suppose that $x + 1$ is odd. This means that there exists an integer j such that $x + 1 = 2j + 1$, or in other words, $x = 2j$. Since j is an integer, x must therefore be even. ■

Deductive Proof Challenge

Prove the following:

$$x^2 - 2x + 2 \geq 0 \text{ for all } x$$

Proof by Contrapositive

If the dog is dead, he smells.

If he doesn't smell, he's not dead.

Proof by Contrapositive

- o Remember that the **Converse** is not always true.
- o The **Contrapositive** is similar to the converse, but is always true.
 - o If A then B \equiv If not B then not A
 - o If it is raining, the sidewalk is wet \equiv If the sidewalk is **not** wet then it is **not** raining.

Proof by Contrapositive

- o This proof technique makes a lot of proofs so much easier.
- o Sometimes the direct route is just too difficult to deduce.
- o Example: If $3x + 7$ is odd, then x is even.

Contrapositive Example

Prove the following:

If $3x + 7$ is even, then x is odd.

Contrapositive Solution

Proof:

Suppose that x is even. This implies that $x = 2k$ for some integer k . Thus, $3x + 7 = 3(2k) + 7 = 6k + 7 = 6k + 6 + 1 = 2(3k + 3) + 1$. And since $3k + 3$ is an integer, then $3x + 7$ is even. ■

Note: this may seem odd, assuming the opposite of the conclusion and proving the opposite of the condition, but this is perfectly legitimate.

Contrapositive Example

Prove the following:

If x^2 is even, then x is even.

Contrapositive Solution

Proof:

Suppose that x is odd. This implies that $x = 2k + 1$ for some integer k . Thus,

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer, then x^2 is odd. ■

Contrapositive Challenge

Prove the following:

If $3n^5$ is an odd integer, then n is an odd integer.

Proof by Contradiction

“Contradictions do not exist. Whenever you think you are facing a contradiction, check your premises. You will find that one of them is wrong.”

-Ayn Rand

Proof by Contradiction

- If you could prove that a statement could **never** be false, then it **must** be true.
- For example, Let's assume that somebody in this room robbed the bookstore about 2 minutes ago.
- Can we prove that they didn't commit the crime?

Proof by Contradiction

- Well, you were all here during the last two minutes. Therefore you couldn't have committed the crime. Therefore you are innocent.
- That is the basic structure of a proof by contradiction:
 - Assume the conclusion is false
 - Find a contradiction

Contradictive Proof Example

Prove the following:

No odd integer can be expressed as the sum of three even integers.

Note: there is no “if” or “then” clause, and the statement sounds negative. This is usually a hint that proof by contradiction is the method of choice.

Contradictive Proof Solution

Proof:

Assume to the contrary that an odd integer x can be expressed as the sum of three even integers, a , b , and c . This implies that there exist integers i , j , and k such that $a = 2i$, $b = 2j$, $c = 2k$. Thus:

$$x = a + b + c = 2i + 2j + 2k = 2(i + j + k)$$

Since $i + j + k$ is an integer, then x is even, a contradiction. ■

Contradictive Proof Example

Prove the following:

There is no largest positive integer.

Contradictive Proof Solution

Proof:

Assume to the contrary that there exists a largest positive integer, notated by k . However, $k+1$ is an integer and $k+1 > k$. Therefore $k+1$ is a larger integer, a contradiction. ■

Contradictive Proof Challenge

Prove the following:

There do not exist integers x and y such that
 $9 = 4x + 2y$.

Proof by Induction

“The only hope [of science] ... is in genuine induction”

-Sir Francis Bacon

Proof by Induction

- o This proof technique is used when you want to show that something works for several cases.
- o These cases are typically denoted by “ n ,” representing positive integers.

Proof by Induction

- Proof by Induction can prove statements such as:

$$1 + 2 + \dots + n = n(n + 1)/2, \text{ for all } n$$

Proof by Induction

- o There is a very systematic way to prove this:
 1. Prove that it works for a base case ($n = 1$)
 2. Assume it works for $n = k$
 3. Show that it works for $n = k + 1$
- o Think of this as a row of dominoes.
 1. Knock over the first domino
 2. Assume that a random one will get knocked over
 3. Show that the random one will hit the next one.
- o Thus all of the dominoes get knocked down.

Inductive Proof Example

Prove the following:

$$1 + 2 + \dots + n = n(n + 1)/2, \text{ for all } n$$

Inductive Proof Solution

Proof:

Let $n = 1$. Since $1 = 1(2)/2 = 1(1 + 1)/2$, then the statement holds for a base case.

Now assume that it holds for $n = k$, or in other words:

$$1 + 2 + \dots + k = k(k + 1)/2$$

Hence,

$$\begin{aligned} 1 + 2 + \dots + k + (k + 1) &= k(k + 1)/2 + (k + 1) \\ &= (k + 1)(k/2 + 1) = (k + 1)(k + 2)/2 = (k + 1)((k + 1) + 1)/2 \end{aligned}$$

Therefore it holds for $n = k + 1$, and the statement is true by induction. ■

Inductive Proof Example

Prove the following:

$2^n > n$ for all nonnegative integers

Inductive Proof Solution

Proof:

Let $n = 0$. Thus $2^0 = 1 > 0$, and the statement holds for $n = 0$.

Now assume that $2^k > k$.

Hence,

$$2^{k+1} = (2)2^k > 2k = k + k \geq k + 1$$

Thus the statement holds for $n = k + 1$.

Therefore the statement is true by induction. ■

Inductive Proof Challenge

Prove the following:

$$2 * 6 * 10 * \dots * (4n - 2) = (2n)!/n!, \text{ for all } n$$

Conclusion

“Don't cry because it's over. Smile because it happened.”

-Dr. Seuss

A Few Words of Wisdom

- o You are now very well-equipped to handle almost any proof that comes your way.
- o Even so, you will likely get stuck in the future.

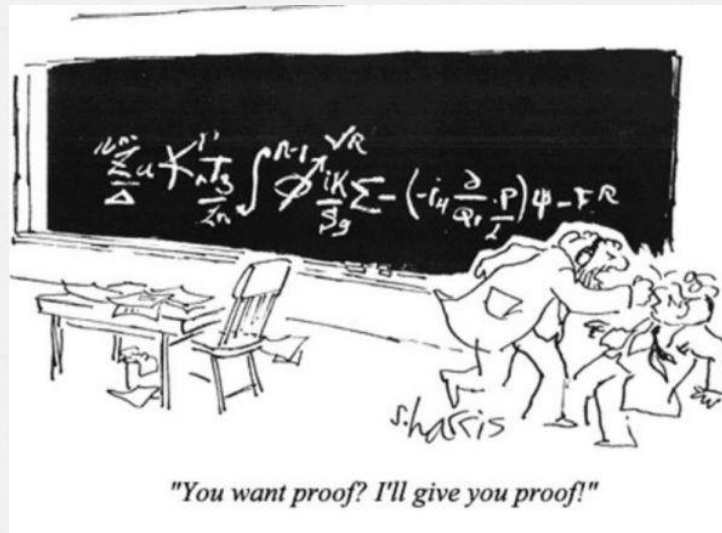
A Few Words of Wisdom

Remember these tips when you get stuck:

1. Start writing something!
 - o The epiphanies come when you start messing around with ideas.
2. Check to see if you've overlooked a theorem
 - o So much suffering and head-banging is prevented by simply re-skimming the chapter.
3. Try a fresh approach
 - o Sometimes our first ideas just aren't the right ones.

Conclusion

Now go show your professors that they can't intimidate you with proofs anymore!



Thanks for coming!