the "heartamard product" of F and G. Show that if F and G are rational, then so is $F \heartsuit G$. Moreover, if F is rational and G is algebraic, then $F \heartsuit G$ is algebraic.

6.13. a. [3−] Let $k \in \mathbb{P}$, and define $\eta = \sum_{n\geq 0} \binom{kn}{n} x^n$. Example 6.2.7 shows that η is a root of the polynomial

$$P(y) = k^k x y^k - (y-1)[(k-1)y+1]^{k-1}.$$

Find (as fractional series) the other $k-1$ roots of the polynomial $P(y)$. Deduce that $P(y)$ is irreducible (as a polynomial over $\mathbb{C}(x)$).

b. [3−] Find the discriminant of $P(y)$.

6.14. [3−] Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by

$$f(i-1, j) - 2f(i, j) + f(i+1, j-1) = 0 \tag{6.53}$$

for all $(i, j) \in \mathbb{N} \times \mathbb{N} - \{(0,0)\}$, with the initial conditions $f(0,0) = 1$ and $f(i,j) = 0$ if $i < 0$ or $j < 0$. Thus $f(i, 0) = 2^{-i}$, $f(0,1) = \frac{1}{4}$, $f(1,1) = \frac{1}{4}$, etc. Find the generating function $F(x, y) = \sum_{i,j\geq 0} f(i,j) x^i y^j$.

6.15. [2+] Let $f, g, h \in K[[x]]$ with $h(0) = 0$. Find a polynomial $P(f, g, h, x)$ so that

$$\operatorname{diag} \frac{1}{1 - sf(st) - tg(st) - h(st)} = \frac{1}{\sqrt{P}},$$

where diag is in the variable x.

6.16. [5−] Let $f(n)$ be the number of paths from $(0,0)$ to (n,n) using the steps $(1,0)$, $(0,1)$, and $(1,1)$; and let $g(n)$ be the number of paths from $(0,0)$ to (n,n) using any elements of $\mathbb{N}^2 - \{(0,0)\}$ as steps. It is immediate from equations (6.27) and (6.30) that $g(n) = 2^{n-1} f(n)$, $n > 0$. Is there a combinatorial proof?

6.17. a. [2+] Let S be a subset of $\mathbb{N} \times \mathbb{N} - \{(0,0)\}$ such that (i) every element of S has the form (n,n), $(n+1, n)$, or $(n, n+1)$, and (ii) $(n, n+1) \in S$ if and only if $(n+1, n) \in S$. Let $g(n)$ be the number of paths from $(0,0)$ to (n,n) using steps from S. Let $h(n)$ be the number of such paths that never go above the line $y = x$. (Let $g(0) = h(0) = 1$.) Define $G(x) = \sum_{n\geq 0} g(n) x^n$, $H(x) = \sum_{n\geq 0} h(n) x^n$, and $K(x) = \sum_{(n,n)\in S} x^n$. Show that

$$H(x) = \frac{2}{1 - K(x) + G(x)^{-1}}.$$

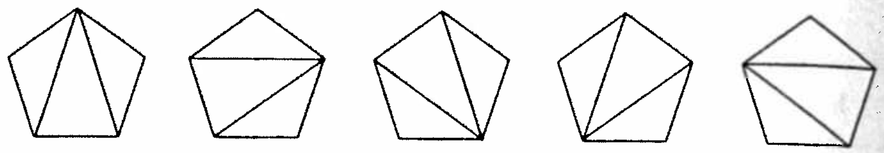
b. [2−] Compute $H(x)$ explicitly when $S = \{(0,1), (1,0), (1,1)\}$ and deduce that in this case $h(n)$ is the Schröder number r_n, thus confirming Exercise 6.39(j).

c. [3−] Give a *combinatorial* proof that when $S = \{(0,1), (1,0), (1,1)\}$ and $n \geq 2$, then $h(n)$ is twice the number of ways to dissect a convex $(n+2)$-gon with any number of diagonals that don't intersect in their interiors.

6.18. [3] Let S be a subset of $\mathbb{N} \times \mathbb{N} - \{(0,0)\}$ such that $\sum_{(m,n)\in S} x^m y^n$ is rational, e.g., S is finite or cofinite. Let $f(n)$ be the number of lattice paths from $(0,0)$ to (n,n) with steps from S that never go above the line $y = x$. Show that $\sum_{n\geq 0} f(n) x^n$ is algebraic.

6.19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 66 sets S_i, (a) ≤ i ≤ (nnn), given below. We illustrate undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

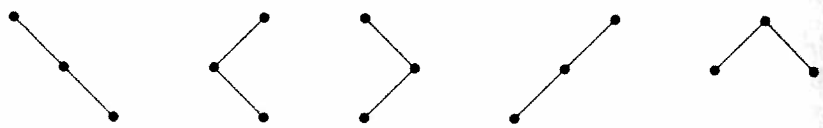
a. Triangulations of a convex $(n + 2)$-gon into n triangles by $n - 1$ diagonals that do not intersect in their interiors:



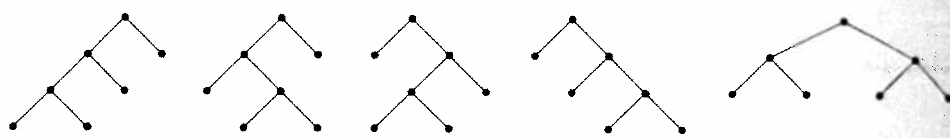
b. Binary parenthesizations of a string of $n + 1$ letters:

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

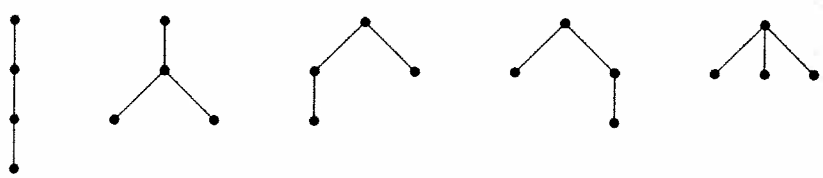
c. Binary trees with n vertices:



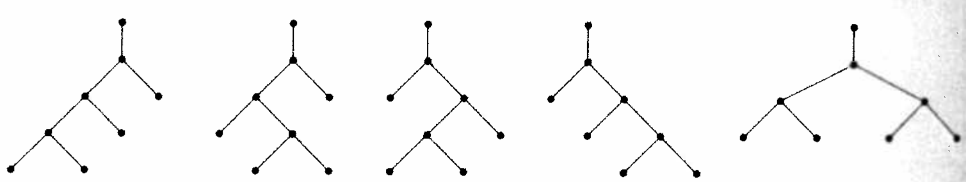
d. Plane binary trees with $2n + 1$ vertices (or $n + 1$ endpoints):



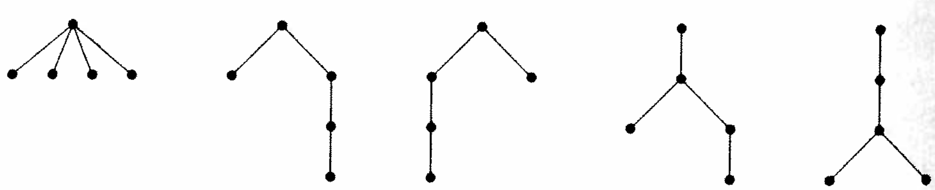
e. Plane trees with $n + 1$ vertices:



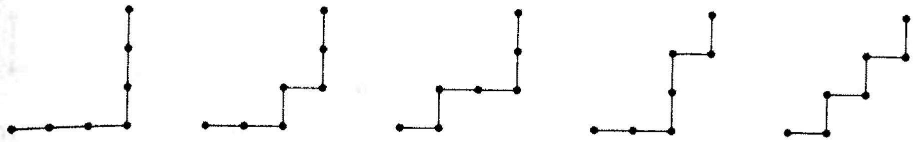
f. Planted (i.e., root has degree one) trivalent plane trees with $2n + 2$ vertices:



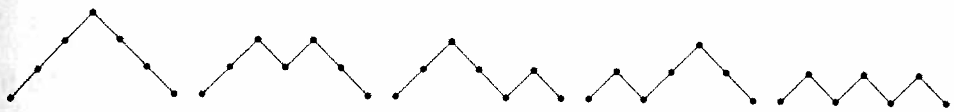
g. Plane trees with $n + 2$ vertices such that the rightmost path of each subtree of the root has even length:



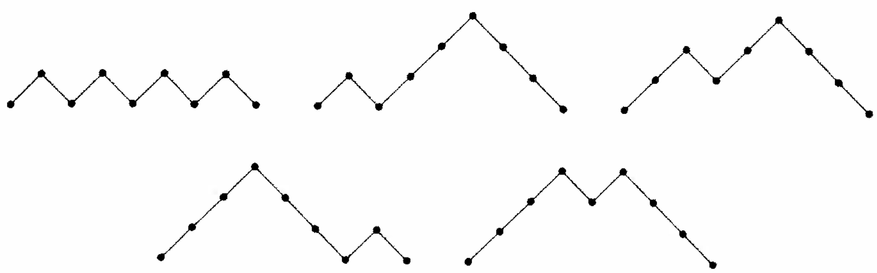
h. Lattice paths from $(0, 0)$ to (n, n) with steps $(0, 1)$ or $(1, 0)$, never rising above the line $y = x$:



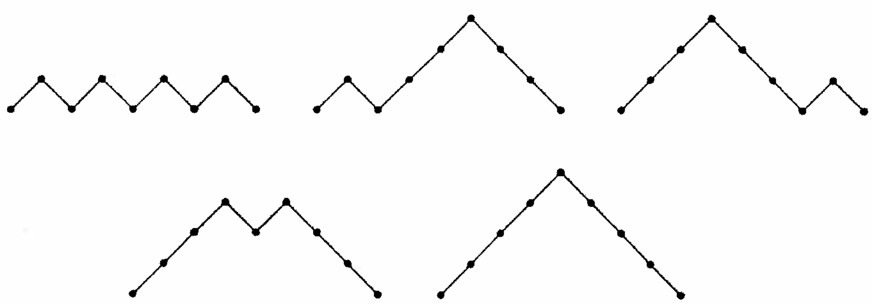
i. Dyck paths from $(0, 0)$ to $(2n, 0)$, i.e., lattice paths with steps $(1, 1)$ and $(1, -1)$, never falling below the x-axis:



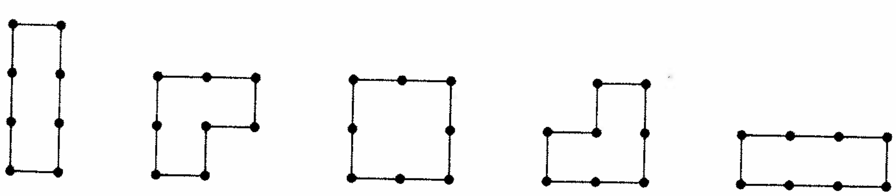
j. Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n+2, 0)$ such that any maximal sequence of consecutive steps $(1, -1)$ ending on the x-axis has odd length:



k. Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n + 2, 0)$ with no peaks at height two

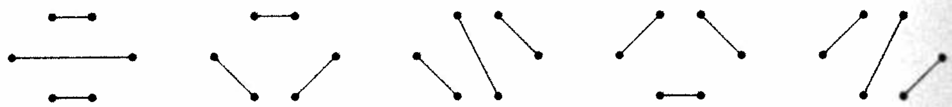


l. (Unordered) pairs of lattice paths with $n + 1$ steps each, starting at $(0, 0)$, using steps $(1, 0)$ or $(0, 1)$, ending at the same point, and only intersecting at the beginning and end:



m. (Unordered) pairs of lattice paths with $n - 1$ steps each, starting at $(0, 0)$, using steps $(1, 0)$ or $(0, 1)$, ending at the same point, such that one path never rises above the other path:

n. n nonintersecting chords joining $2n$ points on the circumference of a circle:



o. Ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points:

p. Ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right):

q. Ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right):

r. Sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below):

$$111--- \quad 11-1-- \quad 11--1- \quad 1-11-- \quad 1-1-1-$$

s. Sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$:

$$111 \quad 112 \quad 113 \quad 122 \quad 123$$

t. Sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq 2i$:

$$12 \quad 13 \quad 14 \quad 23 \quad 24$$

u. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$:

$$000 \quad 001 \quad 010 \quad 011 \quad 012$$

v. Sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative:

$$0,0 \quad 0,1 \quad 1,-1 \quad 1,0 \quad 1,1$$

w. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$:

$$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$$

x. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $0 \leq a_i \leq n - i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$, then $j - i > a_i - a_j$:

$$000 \quad 010 \quad 100 \quad 200 \quad 110$$

y. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$:

$$123 \quad 133 \quad 223 \quad 323 \quad 333$$

z. Sequences $a_1, a_2, \ldots, a_n$ of integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r \leq j - 1$:

$$111 \quad 112 \quad 113 \quad 121 \quad 123$$

aa. Equivalence classes B of words in the alphabet $[n-1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n-1]$ satisfying $|i - j| \geq 2$:

$$\{\emptyset\} \quad \{1\} \quad \{2\} \quad \{12\} \quad \{21\}$$

(For $n = 4$ a representative of each class is given by $\emptyset$, 1, 2, 3, 12, 21, 13, 23, 32, 123, 132, 213, 321, 2132.)

bb. Partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_1 \leq n - 1$ (so the diagram of λ is contained in an $(n-1) \times (n-1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$:

$$(0,0) \quad (1,0) \quad (1,1) \quad (2,1) \quad (2,2)$$

cc. Permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$:

$$112233 \quad 112332 \quad 122331 \quad 123321 \quad 122133$$

dd. Permutations $a_1 a_2 \cdots a_{2n}$ of the set $[2n]$ such that: (i) $1, 3, \ldots, 2n-1$ appear in increasing order, (ii) $2, 4, \ldots, 2n$ appear in increasing order, and (iii) $2i - 1$ appears before $2i$, $1 \leq i \leq n$:

ee. Permutations $a_1a_2\cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations:

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

ff. Permutations $a_1a_2\cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations):

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

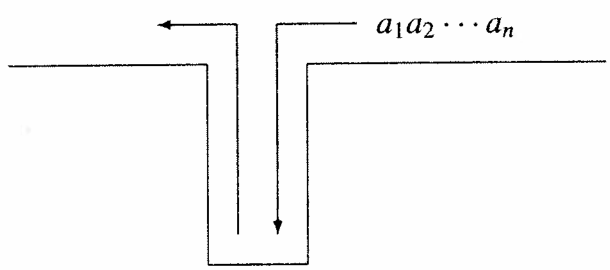
gg. Permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \ldots, 2n) \cdot w$ has $n + 1$ cycles:

$$(1, 2, 3, 4, 5, 6)(1, 2)(3, 4)(5, 6) = (1)(2, 4, 6)(3)(5)$$
$$(1, 2, 3, 4, 5, 6)(1, 2)(3, 6)(4, 5) = (1)(2, 6)(3, 5)(4)$$
$$(1, 2, 3, 4, 5, 6)(1, 4)(2, 3)(5, 6) = (1, 3)(2)(4, 6)(5)$$
$$(1, 2, 3, 4, 5, 6)(1, 6)(2, 3)(4, 5) = (1, 3, 5)(2)(4)(6)$$
$$(1, 2, 3, 4, 5, 6)(1, 6)(2, 5)(3, 4) = (1, 5)(2, 4)(3)(6)$$

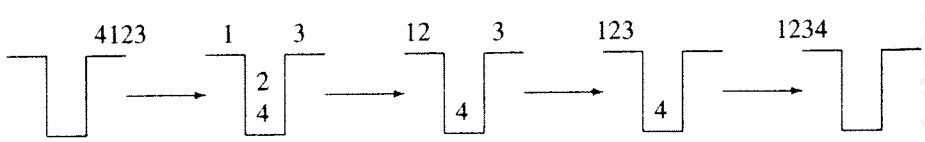
hh. Pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n + 1$ cycles, and $uv = (1, 2, \ldots, n)$:

$$(1)(2)(3) \cdot (1, 2, 3) \quad (1, 2, 3) \cdot (1)(2)(3) \quad (1, 2)(3) \cdot (1, 3)(2)$$
$$(1, 3)(2) \cdot (1)(2, 3) \quad (1)(2, 3) \cdot (1, 2)(3)$$

ii. Permutations $a_1a_2\cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset) = \emptyset$. If $w = unv$ is a sequence of distinct integers with largest term n, then $S(w) = S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w) = w$:



For example,

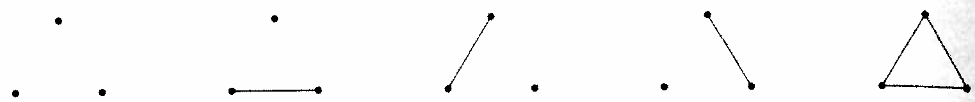


$$123 \quad 132 \quad 213 \quad 312 \quad 321$$

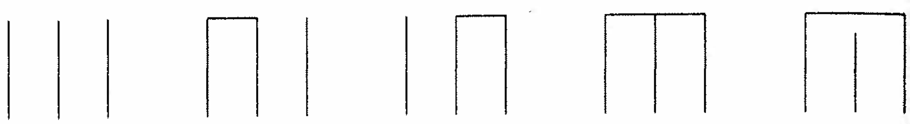
pp. Noncrossing partitions of $[n]$, i.e., partitions $\pi = \{B_1, \dots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$:

123 12–3 13–2 23–1 1–2–3

qq. Partitions $\{B_1, \dots, B_k\}$ of $[n]$ such that if the numbers $1, 2, \dots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \dots, B_k$ are pairwise disjoint:



rr. Noncrossing Murasaki diagrams with n vertical lines:



ss. Noncrossing partitions of some set $[k]$ with $n + 1$ blocks, such that any two elements of the same block differ by at least three:

1–2–3–4 14–2–3–5 15–2–3–4 25–1–3–4 16–25–3–4

tt. Noncrossing partitions of $[2n + 1]$ into $n + 1$ blocks, such that no block contains two consecutive integers:

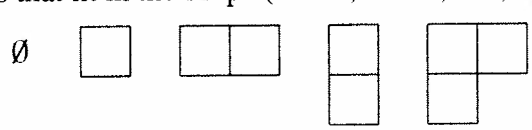
137–46–2–5 1357–2–4–6 157–24–3–6
17–246–3–5 17–26–35–4

uu. *Nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$:

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

vv. Young diagrams that fit in the shape $(n - 1, n - 2, \dots, 1)$:



ww. Standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n-1)$):

123	124	125	134	135
456	356	346	256	246

or

123	124	125	134	135
45	35	34	25	24

xx. Pa sq

yy. C er

zz. C (n

aaa. Li

bbb. O

ccc. O ad

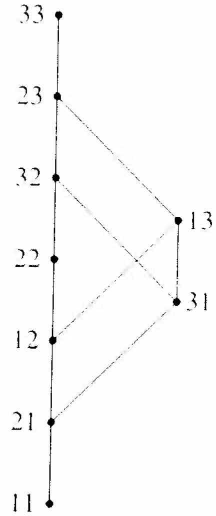


Figure 6-5. A poset with $C_4 = 14$ order ideals.

xx. Pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows:

$$(123, 123) \quad \begin{pmatrix} 12 & 12 \\ 3 \;, & 3 \end{pmatrix} \quad \begin{pmatrix} 12 & 13 \\ 3 \;, & 2 \end{pmatrix} \quad \begin{pmatrix} 13 & 12 \\ 2 \;, & 3 \end{pmatrix} \quad \begin{pmatrix} 13 & 13 \\ 2 \;, & 2 \end{pmatrix}$$

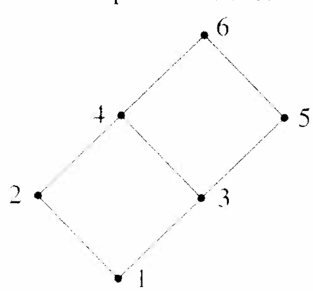
yy. Column-strict plane partitions of shape $(n-1, n-2, \ldots, 1)$, such that each entry in the i-th row is equal to $n-i$ or $n-i+1$:

$$\begin{matrix} 33 \\ 2 \end{matrix} \quad \begin{matrix} 33 \\ 1 \end{matrix} \quad \begin{matrix} 32 \\ 2 \end{matrix} \quad \begin{matrix} 32 \\ 1 \end{matrix} \quad \begin{matrix} 22 \\ 1 \end{matrix}$$

zz. Convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m), such that if $(i, j) \in S$ then $0 < i - j < n$:

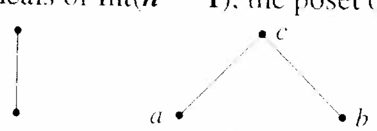
$$\emptyset \quad \{(1,0)\} \quad \{(2,0)\} \quad \{(1,0),(2,0)\} \quad \{(2,0),(2,1)\}$$

aaa. Linear extensions of the poset $\mathbf{2} \times \boldsymbol{n}$:



123456
123546
132456
132546
135246

bbb. Order ideals of $\mathrm{Int}(\boldsymbol{n} - \mathbf{1})$, the poset of intervals of the chain $\boldsymbol{n} - \mathbf{1}$:



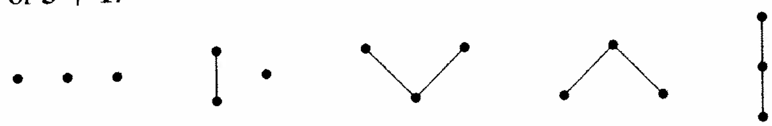
2 Int(**2**)

$\emptyset$, a, b, ab, abc

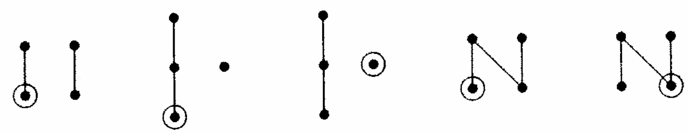
ccc. Order ideals of the poset A_n obtained from the poset $(\boldsymbol{n}-\mathbf{1}) \times (\boldsymbol{n}-\mathbf{1})$ by adding the relations $(i, j) < (j, i)$ if $i > j$ (see Figure 6-5 for the Hasse diagram of A_4):

$$\emptyset \quad \{11\} \quad \{11, 21\} \quad \{11, 21, 12\} \quad \{11, 21, 12, 22\}$$

ddd. Nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$:



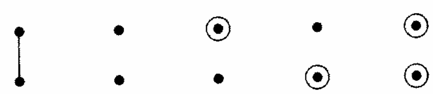
eee. Nonisomorphic $(n+1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element:



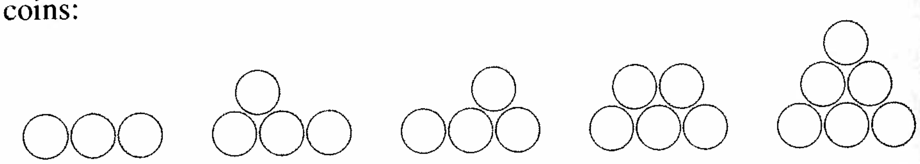
fff. Relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \le i < j < k \le n$ and iRk, then iRj and jRk (in the example below we write ij for the pair (i, j), and we omit the pairs ii):

$$\emptyset \quad \{12, 21\} \quad \{23, 32\} \quad \{12, 21, 23, 32\} \quad \{12, 21, 13, 31, 23, 32\}$$

ggg. Joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices:



hhh. Ways to stack coins in the plane, the bottom row consisting of n consecutive coins:



iii. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \ge 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors:

14321 13521 13231 12531 12341

jjj. n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0:

000 013 022 112 233

kkk. n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \ge j$ and there is a lattice path from $(0, 0)$ to (i, j) with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$ that lies entirely inside S:

$$\{(0,0), (1,0), (2,0)\} \quad \{(0,0), (1,0), (1,1)\} \quad \{(0,0), (1,0), (2,1)\}$$

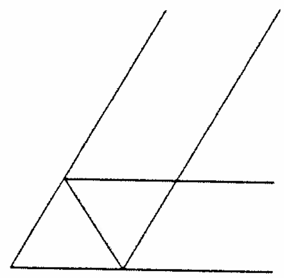
$$\{(0,0), (1,1), (2,1)\} \quad \{(0,0), (1,1), (2,2)\}$$

lll. Regions into which the cone $x_1 \ge x_2 \ge \cdots \ge x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \le i < j \le n$ (the diagram below shows the

```
1   1   1   1   1   1   1   1   1   1   1   1   1   1
  1   3   2   1   5   1   2   3   1   3   2   1   5
    2   5   1   4   4   1   5   2   2   5   1   4
      3   2   3   3   3   2   3   3   3   2   3
        1   5   2   2   5   1   4   4   1   5
          2   3   1   3   2   1   5   1   2
            1   1   1   1   1   1   1   1
```

Figure 6-6. The frieze pattern corresponding to the sequence (1, 3, 2, 1, 5, 1, 2, 3).

situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$):



mmm. Positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (necessarily with $n+1$ rows)

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n+2} & a_1 & a_2 & \cdots & a_{n-1} & \\ b_1 & b_2 & b_3 & \cdots & b_{n+2} & b_1 & \cdots & b_{n-2} & & \\ & & \vdots & & & & & & & \\ r_1 & r_2 & r_3 & \cdots & r_{n+2} & r_1 & & & & \\ 1 & 1 & 1 & \cdots & 1 & & & & & \end{array} \tag{6.54}$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru + 1$ (an example of such an array for $(a_1, \ldots, a_8) = (1, 3, 2, 1, 5, 1, 2, 3)$ (necessarily unique) is given by Figure 6-6):

12213 22131 21312 13122 31221

nnn. n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix}$$

is positive definite with determinant one:

131 122 221 213 312