Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 2 — Solutions

- **1.2.1 8d**: See solutions to homework 1 for this one
- **1.2.1 11a**: See solutions to homework 1 for this one
- **1.2.2 1b**: In row 1 of the adjacency matrix of C_{2k}, there are 1's in positions 2 and $2k$, and zeroes everywhere else. In row i, for $2 \leq i \leq 2k-1$, there are 1's in positions $i-1$ and $i+1$, and zeroes everywhere else. In row $2k$, there are 1's in positions 1 and $2k-1$, and zeroes everywhere else. The specific example of $k=3$ is shown below.

$$A(C_6) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

  In row 1 of the adjacency matrix of C_{2k+1}, there are 1's in positions 2 and $2k+1$, and zeroes everywhere else. In row i, for $2 \leq i \leq 2k$, there are 1's in positions $i-1$ and $i+1$, and zeroes everywhere else. In row $2k+1$, there are 1's in positions 1 and $2k$, and zeroes everywhere else. The specific example of $k=3$ is shown below.

$$A(C_7) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- **1.2.2 1c**: The adjacency matrix of $K_{m,n}$ (with the labeling as described in the question) is a block matrix. Divide the $m+n$ by $m+n$ matrix into four blocks: m by m in the Northwest, m by n in the Northeast, n by m in the Southwest, n by n in the Southeast. Put an all zeroes matrix in the first and fourth of these blocks, and an all 1's matrix in the second and third. The case $m = 4$, $n = 3$ is shown below.

$$A(K_{4,3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- **1.2.2 1d**: $A(K_n)$ is an n by n matrix with a zero in every position down the main diagonal, and a one everywhere else. The case $n = 6$ is shown below.

$$A(K_6) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

- **1.2.2 2**: We count the number of walks of length 3 from i and j in K_4; this is the i-j entry of A^3. For $i = j$, the only such walks are cycles, and for each i there are 6 of them. For $i \neq j$, we have paths of length 3 (there are 2 of them for each such i, j); walks that begin by traversing an edge ik, then ki, then ij, for $k \neq j$ (there are 2 of them); walks that begin by traversing ij, then jk, then kj, for $k \neq j$ (there are 2 of them); and finally the walk $ijij$, for a total of 7. So we get

$$A^3 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$$

- **1.2.2 4a**: The i-i entry of A^3 counts the number of walks of length 3 from i to i. Such walks must be cycles, and so triangles. But each such triangle is counted twice as a walk (the walks $ijki$ and $ikji$ count the same triangle). So dividing the i-i entry of A^3 by 2 counts the number triangles involving i.

- **1.2.2 4b**: Summing the numbers $(A^3)_{ii}/2$, we count all the triangles, but each one gets counted 3 times (the triangle ijk gets counted once each in the ith, jth and kth term of the sum). So dividing this sum by 3, that is, dividing the trace of A^3 by 6, counts the number of triangles in total.

- **1.2.2 5**: Vertices 1 and 5 are at even distance apart in C_{10}, whichever way we choose to go around the cycle, so there will be *no* walks of even length that start at 1 and even at 5. To prove this formally, we could prove the following statement by induction: "the vertex we are at in C_{10}, after a walk of length n starting from vertex 1, has an even label if n is odd and an odd label if n is even". The case $n = 0$ is clear. For $n > 0$ and even, after $n-1$ steps we are (by induction) at an even vertex, so after n steps at an odd vertex; for $n > 0$ and odd, after $n-1$ steps we are (by induction) at an odd vertex, so after n steps at an even vertex.

If follows that the number of walks of length 2009 starting at 1 and ending at 5 is zero, and so this is the $(1,5)$ entry of A^{2009}.

There's something more general going on here: If G is bipartite, with partite sets X and Y, then every step of a walk takes you either from X to Y or from Y to X; so all even-length walks must start and end in the same partite set; and all odd-length walks must end in a different partite set to the one they started in.

- **1.3.1 3**: T is a tree, so has no odd cycles (it has no cycles of any kind), so by the basic theorem concerning bipartite graphs, it must be bipartite!

  A more direct proof might go by induction on n: for $n = 2$, the assertion is clear. For $n > 2$, find a leaf v, and remove it. The resulting graph is a tree on $n-1$ vertices, so by induction is bipartite. If it has bipartition $X \cup Y$ with v's unique neighbor being in X, get a bipartition $X' \cup Y'$ of the original tree by setting $X' = X$ and $Y' = Y \cup \{v\}$; it's a quick check that this bipartition witnesses that the original tree is bipartite.

- **1.3.1 4**: If r and s are both at least 2, then $K_{r,s}$ is *not* a tree: if a, b are any two vertices in the partite set of size r, and c and d are any two vertices in the partite set of size s, then $a - c - b - d - a$ is a cycle.

  The contrapositive of this statement (specifically, "$r, s \geq 2$ implies $K_{r,s}$ not a tree") is: "$K_{r,s}$ is a tree implies at least one of r, s is 1", exactly as we had to show.

- **1.3.2 1a**: Suppose a forest has k components, and n vertices. If the k components have orders $n_1, \ldots, n_k$, then, since each component is a tree, the sizes of the components are $n_1 - 1, \ldots, n_k - 1$, and so the total size of the graph is $\sum_{j=1}^{k}(n_j - 1) = \left(\sum_{j=1}^{k} n_j\right) - k = n - k$. So a 10 vertex forest must have 9 or fewer edges, and none with 12 edges exists.

- **1.3.2 1d**: Following the reasoning of part a), a 14-vertex forest with exactly 13 edges is a tree. Many such exist; for example, a path on 14 vertices.

- **1.3.2 1e**: Following the reasoning of part a), a 14-vertex forest with exactly 12 edges must have exactly two components. Many such exist; for example, two paths each on 7 vertices, with no vertices in common.

- **1.3.2 2**: If T has an even number of edges, it must have an odd number of vertices (one more than the number of edges). If all degrees are odd, then the sum of the degrees is also odd (the sum of an odd number of odd numbers is odd); but we know that the sum of the degrees of any graph is even (twice the number of edges). So a tree with an even number of edges must have at least one even degree vertex. (Note that this argument even works for the tree with no edges.)

- **1.3.2 5**: There are two parts to this proof: showing that if G is a tree, there is a unique path between any two vertices, and showing that if there is a unique path between any two vertices, then G is a tree.

  Suppose G is a tree. Pick two vertices u and v. Because G is a tree, there is a path from u to v. Suppose that there is a second, different path. Let w be the last vertex at which the two paths (both started from u) agree. The two diverging paths must meet again (since they at least meet at v); let w' be the first vertex at which they meet again. Following w to w' along the first path, and then w' to w along the second path (backwards), gives a cycle in G, a contradiction. So for each u and v, there is a unique path connecting the two.

  Suppose that for any two vertices u and v, there is a unique path connecting the two. Then certainly G is connected. Suppose G has a cycle. Then for any two vertices along the cycle, there are at least two paths connecting the two vertices (just follow the path in two different directions). So G must be acyclic, and so a tree.

- **1.3.2 9**: Let $P = v_1 \dots v_k$ be a longest path in T. The neighbors of v_1 cannot include $v_2, v_3, \dots, v_k$, since then there would be a cycle in T. Nor can v_1 have a neighbor w that is not among $\{v_2, \dots, v_k\}$, for then $wv_1 \dots v_k$ would be a longer path. So v_2 is the only neighbor of v_1, and the degree of v_1 is 1, that is, v_1 is a leaf. Similarly, v_k is a leaf. And since $n \geq 2$, T has at least a path of length 1, and so $v_1 \neq v_k$; this gives at least 2 leaves.

- **1.3.2 12**: Suppose that none of the leaves have a common neighbor. Then on deletion of all the leaves, every vertex in the graph has degree at least 2 (the vertices which are not leaves, and not adjacent to a leaf, have degree at least two in the original graph, since they are not leaves, and their degree does not change after the deletion of leaves, since they are not adjacent to leaves; the vertices which are adjacent to a leaf drop their degree by 1, since we are assuming that no two leaves are adjacent to the same such vertex, and since these vertices started with degree at least 3, they end with degree at least 2). We know from a previous homework problem (1.1.2 5, from Homework 1) that a graph with minimum degree at least 2 must have a cycle, and so the original graph has a cycle. This is a contradiction, since the original graph was a tree. So some pair of leaves must have a common neighbor.

- **Extra problem**:

  - 1 is false in general. If T has no edges (0 is an even number!) then the only vertex has degree 0, so this is a counterexample to the statement. For all larger trees, however, the statement is true: all larger trees have at least two leaves, and each of these has odd degree.
  - 2 is very false in general: take, for example, the tree with one edge (and so both vertices having odd degree) for a counterexample.
  - 3 is true always: all trees with at least one edge have at least two leaves, and each of these has odd degree.