Introduction to Probability, Fall 2013

Math 30530 Section 01

Homework 6 — solutions

1. Let X be a Poisson random variable with parameter λ. Show that the variance of X is λ.

   Solution:

$$
\begin{aligned}
E(X^2) &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \left(k(k-1)+k\right) \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \left(\sum_{k=0}^{\infty} \left(k(k-1)-k\right) \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \right) \\
&= e^{-\lambda} \left(\sum_{k=2}^{\infty} \left(k(k-1)\right) \frac{\lambda^k}{k!} + \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} \right) \\
&= e^{-\lambda} \left(\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \right) \\
&= e^{-\lambda} \left(\lambda^2 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \\
&= e^{-\lambda} \left(\lambda^2 e^{\lambda} + \lambda e^{\lambda} \right) \\
&= \lambda^2 + \lambda.
\end{aligned}
$$

   So, since $E(X) = \lambda$, $\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

2. I roll two dice. Alice looks at the maximum of the two numbers that come up, and calls this X. Bob looks at the minimum, and calls it Y (for example, if the roll leads to a 3 and a 5, then $X = 5$ and $Y = 3$; if the roll leads to two 6's, then $X = Y = 6$).

   (a) Write down the joint mass function of X and Y in a 6 by 6 table.

   Solution: Values of X go down, values of Y go across. For clarity, I have written only the numerator of each entry; the denominator is always 36.

	1	**2**	**3**	**4**	**5**	**6**
1	1	0	0	0	0	0
2	2	1	0	0	0	0
3	2	2	1	0	0	0
4	2	2	2	1	0	0
5	2	2	2	2	1	0
6	2	2	2	2	2	1

For example, the $X = 4$, $y = 3$ entry is 2 (out of 36), since there exactly 2 ways of getting max of 4 and min of 3: first dice 4, second dice 3, or first dice 3, second dice 4.

(b) After the roll, Bob pays Alice $X^2 - Y^2$ dollars. Calculate the expected number of dollars that Alice receives.

Solution: Sum over all pairs (x, y) with $1 \leq x \leq 6$ and $1 \leq y \leq 6$; the thing we are summing is $x^2 - y^2$ weighted with the (x, y) entry of the table (divided by 36). Answer is $245/18 \approx \$13.61$.

3. Four students, 1, 2, 3 and 4, take a make-up quiz, which is made up of 5 parts, i,ii,iii,iv and v. 1 answers only parts i,ii and iii. 2 and 3 both answer only parts i, iv and v. 4 answers all 5 parts. This means that between the four students, there are 14 question-parts completed. I pick one of these 14 at random to grade first, and I record the following pair of numbers: X, which is the number of the student whose quiz I have chosen, and Y, which is the part number of the answer I am about to grade (I record $Y = 1$ if it is part i, $Y = 2$ if it is part ii, etc.).

(a) Write down the joint mass function of X and Y in a table.

Solution: Values of X go down, values of Y go across. For clarity, I have written only the numerator of each entry; the denominator is always 14.

	1	**2**	**3**	**4**	**5**
1	1	1	1	0	0
2	1	0	0	1	1
3	1	0	0	1	1
4	1	1	1	1	1

(b) Find the marginal mass function of X **using the table**.

Solution: Summing across the rows, we find that $p_X(1) = p_X(2) = p_X(3) = 3/14$ and $p_X(4) = 5/14$; all other values of $p_X(x)$ are 0.

(c) Find the marginal mass function of Y **using the table**.

Solution: Summing down the columns, we find that $p_Y(1) = 4/14$, $p_Y(2) = p_X(3) = 2/14$ and $p_Y(4) = p_Y(5) = 3/14$; all other values of $p_Y(y)$ are 0.

4. Chapter 2, problem 24

Solution:

Solution to Problem 2.24. (a) There are 21 integer pairs (x, y) in the region

$$R = \{(x, y) \mid -2 \le x \le 4,\ -1 \le y - x \le 1\},$$

so that the joint PMF of X and Y is

$$p_{X,Y}(x, y) = \begin{cases} 1/21, & \text{if } (x, y) \text{ is in } R, \\ 0, & \text{otherwise.} \end{cases}$$

For each x in the range $[-2, 4]$, there are three possible values of Y. Thus, we have

$$p_X(x) = \begin{cases} 3/21, & \text{if } x = -2, -1, 0, 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of X is the midpoint of the range $[-2, 4]$:

$$\mathbf{E}[X] = 1.$$

The marginal PMF of Y is obtained by using the tabular method. We have

$$p_Y(y) = \begin{cases} 1/21, & \text{if } y = -3, \\ 2/21, & \text{if } y = -2, \\ 3/21, & \text{if } y = -1, 0, 1, 2, 3, \\ 2/21, & \text{if } y = 4, \\ 1/21, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of Y is

$$\mathbf{E}[Y] = \frac{1}{21} \cdot (-3 + 5) + \frac{2}{21} \cdot (-2 + 4) + \frac{3}{21} \cdot (-1 + 1 + 2 + 3) = 1.$$

(b) The profit is given by

$$P = 100X + 200Y,$$

so that

$$\mathbf{E}[P] = 100 \cdot \mathbf{E}[X] + 200 \cdot \mathbf{E}[Y] = 100 \cdot 1 + 200 \cdot 1 = 300.$$

5. Chapter 2, problem 26

Solution:

Solution to Problem 2.26. (a) The possible values of the random variable X are the ten numbers $101, \ldots, 110$, and the PMF is given by

$$p_X(k) = \begin{cases} \mathbf{P}(X > k-1) - \mathbf{P}(X > k), & \text{if } k = 101, \ldots 110, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\mathbf{P}(X > 100) = 1$ and for $k = 101, \ldots 110$,

$$\begin{aligned} \mathbf{P}(X > k) &= \mathbf{P}(X_1 > k, X_2 > k, X_3 > k) \\ &= \mathbf{P}(X_1 > k)\,\mathbf{P}(X_2 > k)\,\mathbf{P}(X_3 > k) \\ &= \frac{(110 - k)^3}{10^3}. \end{aligned}$$

It follows that

$$p_X(k) = \begin{cases} \dfrac{(111-k)^3 - (110-k)^3}{10^3}, & \text{if } k = 101, \ldots 110, \\ 0, & \text{otherwise.} \end{cases}$$

(An alternative solution is based on the notion of a CDF, which will be introduced in Chapter 3.)

(b) Since X_i is uniformly distributed over the integers in the range $[101, 110]$, we have $\mathbf{E}[X_i] = (101 + 110)/2 = 105.5$. The expected value of X is

$$\mathbf{E}[X] = \sum_{k=-\infty}^{\infty} k \cdot p_X(k) = \sum_{k=101}^{110} k \cdot p_x(k) = \sum_{k=101}^{110} k \cdot \frac{(111-k)^3 - (110-k)^3}{10^3}.$$

The above expression can be evaluated to be equal to 103.025. The expected improvement is therefore 105.5 - 103.025 = 2.475.

Alternate approach to part a): How many ways can I write down three numbers in a row, all between 101 and 110, with the smallest of the three being $100 + \ell$? EITHER all three numbers are $100 + \ell$ (one way) OR exactly two of them are $100 + \ell$ (three ways to choose which two, $10 - \ell$ ways to choose last number bigger than $100 + \ell$) OR exactly one of them is $100 + \ell$ (three ways to choose which one, $(10 - \ell)^2$ ways to choose last two numbers both bigger than $100 + ell$). So the total number of ways is

$$1 + 3(10 - \ell) + 3(10 - \ell)^2.$$

Since there are 1000 possible ways to choose the three numbers, this shows that, for $1 \leq \ell \leq 10$,

$$\Pr(\min\{X_1, X_2, X_3\} = 100 + \ell) = \frac{1 + 3(10 - \ell) + 3(10 - \ell)^2}{1000}.$$

This looks different from the solution above, but after a little algebra you will see that it is the same.

6. Back to Alice and Bob: Alice repeatedly rolls a dice until the first time that she rolls a 6, and lets X be the number of attempts it took her. Independently, Bob repeatedly rolls a pair of dice until the first time that he sees a pair of 6's, and lets X be the number of attempts it took her.

   (a) Which pairs (x, y) are such that there is a non-zero probability that $X = x$ and simultaneously $Y = y$?

   Solution: All pairs (x, y) with $x > 0$ and $y > 0$ (x, y both integers) are possible.

   (b) For each such pair (x, y), calculate $\Pr(X = x, Y = y)$ (i.e., the value of the joint mass function $p_{X,Y}(x, y)$).

   Solution: Here and in the remaining parts, I'll use p for 1/6 and q for 1/36, to make the writing easier. By independence of Bob and Alice's rolls,

   $$p_{X,Y}(x, y) = \Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y) = (1 - p)^{x-1} p (1 - q)^{y-1} q.$$

(c) Use the joint mass function to calculate the probability that Alice and Bob finish their experiments after the same number of trials (that is, that $X = Y$).

Solution: This is asking for $p_{X,Y}(1,1) + p_{X,Y}(2,2) + p_{X,Y}(3,3) + \ldots$. Using the sum of a geometric series formula, then,

$$\begin{aligned}
\Pr(X = Y) &= pq + (1-p)(1-q)pq + (1-p)^2(1-q)^2pq + (1-p)^3(1-q)^3pq + \ldots \\
&= \frac{pq}{1-(1-p)(1-q)}.
\end{aligned}$$

Substituting in our values for p, q, get $\Pr(X = Y) = 1/41 \approx .02439$.

(d) Use the joint mass function to calculate the probability that Alice finishes her experiment before Bob does (that is, that $X < Y$).

Solution: Suppose Alice finishes in exactly k rolls (a probability $(1-p)^{k-1}p$ event). Then the event $X < Y$ is the same as the event that Bob failed to finish within k rolls, which is a probability $(1-q)^k$ event. So, using law of total probability and the sum of a geometric series,

$$\begin{aligned}
\Pr(X < Y) &= \sum_{k=1}^{\infty} \Pr(X < Y | X = k) \Pr(X = k) \\
&= (1-q)p + (1-q)^2(1-p)p + (1-q)^3(1-p)^2p + (1-q)^4(1-p)^3p + \ldots \\
&= \frac{(1-q)p}{1-(1-q)(1-p)}.
\end{aligned}$$

Substituting in our values for p, q, get $\Pr(X = Y) = 35/41 \approx .853659$.
Reality check: the same argument gives $P(Y < X) = ((1-p)q)/(1-(1-q)(1-p)) = 5/41$, so $\Pr(X = Y) + \Pr(X < Y) + \Pr(Y < X) = 1$, as of course it should.