Math 30530, Fall 2013, Homework 4 supplementary solutions

Chapter 2, Problems 1 and 2

Solution to Problem 2.1. Let X be the number of points the MIT team earns over the weekend. We have

$$\begin{aligned}
\mathbf{P}(X=0) &= 0.6\cdot 0.3 = 0.18,\\
\mathbf{P}(X=1) &= 0.4\cdot 0.5\cdot 0.3 + 0.6\cdot 0.5\cdot 0.7 = 0.27,\\
\mathbf{P}(X=2) &= 0.4\cdot 0.5\cdot 0.3 + 0.6\cdot 0.5\cdot 0.7 + 0.4\cdot 0.5\cdot 0.7\cdot 0.5 = 0.34,\\
\mathbf{P}(X=3) &= 0.4\cdot 0.5\cdot 0.7\cdot 0.5 + 0.4\cdot 0.5\cdot 0.7\cdot 0.5 = 0.14,\\
\mathbf{P}(X=4) &= 0.4\cdot 0.5\cdot 0.7\cdot 0.5 = 0.07,\\
\mathbf{P}(X>4) &= 0.
\end{aligned}$$

Solution to Problem 2.2. The number of guests that have the same birthday as you is binomial with $p = 1/365$ and $n = 499$. Thus the probability that exactly one other guest has the same birthday is

$$\binom{499}{1}\frac{1}{365}\left(\frac{364}{365}\right)^{498} \approx 0.3486.$$

Chapter 2, Problem 3

Solution to Problem 2.3. (a) Let L be the duration of the match. If Fischer wins a match consisting of L games, then $L-1$ draws must first occur before he wins. Summing over all possible lengths, we obtain

$$\mathbf{P}(\text{Fischer wins}) = \sum_{l=1}^{10}(0.3)^{l-1}(0.4) = 0.571425.$$

(b) The match has length L with $L < 10$, if and only if $(L-1)$ draws occur, followed by a win by either player. The match has length $L = 10$ if and only if 9 draws occur. The probability of a win by either player is 0.7. Thus

$$p_L(l) = \mathbf{P}(L=l) = \begin{cases} (0.3)^{l-1}(0.7), & l = 1,\ldots,9,\\ (0.3)^9, & l = 10,\\ 0, & \text{otherwise}.\end{cases}$$

Chapter 2, Problem 5 --- the description of the role of the second slot was very confusing here, so I'm only giving the solution to the relatively clear part of the problem: k packets generated, buffer accepts the first b of them and discards the rest.

Solution to Problem 2.5. (a) Let X be the number of packets stored at the end of the first slot. For $k < b$, the probability that $X = k$ is the same as the probability that k packets are generated by the source:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots, b-1,$$

while

$$p_X(b) = \sum_{k=b}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = 1 - \sum_{k=0}^{b-1} e^{-\lambda}\frac{\lambda^k}{k!}.$$

(b) The probability that some packets get discarded during the first slot is the same as the probability that more than b packets are generated by the source, so it is equal to

$$\sum_{k=b+1}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!},$$

or

$$1 - \sum_{k=0}^{b} e^{-\lambda}\frac{\lambda^k}{k!}.$$

Chapter 2, Problem 6

Solution to Problem 2.6. We consider the general case of part (b), and we show that $p > 1/2$ is a necessary and sufficient condition for $n = 2k+1$ games to be better than $n = 2k-1$ games. To prove this, let N be the number of Celtics' wins in the first $2k-1$ games. If A denotes the event that the Celtics win with $n = 2k+1$, and B denotes the event that the Celtics win with $n = 2k-1$, then

$$\mathbf{P}(A) = \mathbf{P}(N \geq k+1) + \mathbf{P}(N = k) \cdot \big(1 - (1-p)^2\big) + \mathbf{P}(N = k-1) \cdot p^2,$$

$$\mathbf{P}(B) = \mathbf{P}(N \geq k) = \mathbf{P}(N = k) + \mathbf{P}(N \geq k+1),$$

and therefore

$$\begin{aligned}
\mathbf{P}(A) - \mathbf{P}(B) &= \mathbf{P}(N = k-1) \cdot p^2 - \mathbf{P}(N = k) \cdot (1-p)^2 \\
&= \binom{2k-1}{k-1} p^{k-1}(1-p)^k p^2 - \binom{2k-1}{k} (1-p)^2 p^k (1-p)^{k-1} \\
&= \frac{(2k-1)!}{(k-1)!\,k!} p^k (1-p)^k (2p-1).
\end{aligned}$$

It follows that $\mathbf{P}(A) > \mathbf{P}(B)$ if and only if $p > \frac{1}{2}$. Thus, a longer series is better for the better team.

Chapter 2, Problem 7b)

(b) In case (1), we have

$$p_X(1) = \mathbf{P}(K_1) = \frac{2}{10},$$
$$p_X(2) = \mathbf{P}(K_1^c)\mathbf{P}(K_2 \,|\, K_1^c) = \frac{8}{10} \cdot \frac{2}{9},$$
$$p_X(3) = \mathbf{P}(K_1^c)\mathbf{P}(K_2^c \,|\, K_1^c)\mathbf{P}(K_3 \,|\, K_1^c \cap K_2^c) = \frac{8}{10} \cdot \frac{7}{9} \cdot \frac{2}{8} = \frac{7}{10} \cdot \frac{2}{9}.$$

Proceeding similarly, we see that the PMF of X is

$$p_X(x) = \frac{2 \cdot (10 - x)}{90}, \qquad x = 1, 2, \ldots, 10.$$

Consider now an alternative line of reasoning to derive the PMF of X. If we view the problem as ordering the keys in advance and then trying them in succession, the probability that the number of trials required is x is the probability that the first $x - 1$ keys do not contain either of the two correct keys and the xth key is one of the correct keys. We can count the number of ways for this to happen and divide by the total number of ways to order the keys to determine $p_X(x)$. The total number of ways to order the keys is 10! For the xth key to be the first correct key, the other key must be among the last $10 - x$ keys, so there are $10 - x$ spots in which it can be located. There are 8! ways in which the other 8 keys can be in the other 8 locations. We must then multiply by two since either of the two correct keys could be in the xth position. We therefore have $2 \cdot 10 - x \cdot 8!$ ways for the xth key to be the first correct one and

$$p_X(x) = \frac{2 \cdot (10 - x)8!}{10!} = \frac{2 \cdot (10 - x)}{90}, \qquad x = 1, 2, \ldots, 10,$$

as before.

In case (2), X is again a geometric random variable with $p = 1/5$.

Chapter 2, Problem 10

Solution to Problem 2.10. Using the expression for the Poisson PMF, we have, for $k \geq 1$,

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\lambda^k \cdot e^{-\lambda}}{k!} \cdot \frac{(k-1)!}{\lambda^{k-1} \cdot e^{-\lambda}} = \frac{\lambda}{k}.$$

Thus if $k \leq \lambda$ the ratio is greater or equal to 1, and it follows that $p_X(k)$ is monotonically increasing. Otherwise, the ratio is less than one, and $p_X(k)$ is monotonically decreasing, as required.