

# Introduction to Probability, Fall 2013

## Math 30530 Section 01

### Homework 1 — Solutions

1. Chapter 1, problem 1.

**Solution:**  $A = \{2, 4, 6\}$  and  $B = \{4, 5, 6\}$ , so:

$$A \cup B = \{2, 4, 5, 6\}, \quad (A \cup B)^c = \{1, 3\}.$$

On the other hand,

$$A^c = \{1, 3, 5\}, \quad B^c = \{1, 2, 3\}, \quad A^c \cap B^c = \{1, 3\}.$$

So  $(A \cup B)^c = A^c \cap B^c$  in this case.

Also:

$$A \cap B = \{4, 6\}, \quad (A \cap B)^c = \{1, 2, 3, 5\}.$$

On the other hand,

$$A^c = \{1, 3, 5\}, \quad B^c = \{1, 2, 3\}, \quad A^c \cup B^c = \{1, 2, 3, 5\}.$$

So  $(A \cap B)^c = A^c \cup B^c$  in this case.

2. Chapter 1, problem 2 (**Note:** I don't expect you to remember these kinds of equations. The point of this exercise is just to highlight that the same event can often be expressed in lots of different ways).

**Solution:** (a) This part I'll do very formally. First we show that  $A^c \subseteq (A^c \cap B) \cup (A^c \cap B^c)$ . Suppose  $x \in A^c$ . If  $x \in B$ , then  $x \in A^c \cap B$ . If  $x \in B^c$ , then  $x \in A^c \cap B^c$ . One of  $x \in B$ ,  $x \in B^c$  must happen, and we've just shown that either way,  $x \in (A^c \cap B) \cup (A^c \cap B^c)$ . So  $A^c \subseteq (A^c \cap B) \cup (A^c \cap B^c)$ .

Now we show that  $(A^c \cap B) \cup (A^c \cap B^c) \subseteq A^c$ . Suppose  $x \in (A^c \cap B) \cup (A^c \cap B^c)$ . Then EITHER  $x \in A^c \cap B$ , in which case in particular  $x \in A^c$ , OR  $x \in A^c \cap B^c$ , in which case again in particular  $x \in A^c$ . So either way  $x \in A^c$ . This shows that  $(A^c \cap B) \cup (A^c \cap B^c) \subseteq A^c$ . We've shown that

$$A^c \subseteq (A^c \cap B) \cup (A^c \cap B^c) \quad \text{AND} \quad (A^c \cap B) \cup (A^c \cap B^c) \subseteq A^c.$$

The only way this could happen is if  $A^c = (A^c \cap B) \cup (A^c \cap B^c)$ .

Now onto (b), which I'll just talk through. The left-hand side,  $(A \cap B)^c$ , is all elements of the universe that are not in BOTH  $A$  AND  $B$ . Something can not be in both  $A$  and  $B$  in three distinct (mutually exclusive) ways: EITHER it's not in  $A$ , but is in  $B$  (so its in  $A^c \cap B$ ), OR it's not in  $A$ , and not in  $B$  (so its in  $A^c \cap B^c$ ), OR it is in  $A$ , but is not in  $B$  (so its in  $A \cap B^c$ ). So another way of describing all elements of the universe that are not in both  $A$  and  $B$  is to say that they are all the elements that are either in  $A^c \cap B$ , or in  $A^c \cap B^c$ , or in  $A \cap B^c$ . In other words:

$$(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c).$$

Either approach (formal or by words) is fine as long as you understand what you are doing. Another possible approach is through drawing a Venn diagram; we'll see that in Question 16.

Now onto (c).  $A = \{1, 3, 5\}$  and  $B = \{1, 2, 3\}$  so

$$A \cap B = \{1, 3\}, \quad (A \cap B)^c = \{2, 4, 5, 6\},$$

and

$$A^c \cap B = \{2\}, \quad A^c \cap B^c = \{4, 6\}, \quad A \cap B^c = \{5\}, \quad (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) = \{2, 4, 5, 6\};$$

so  $(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c)$  are equal in this case.

3. Chapter 1, problem 5.

**Solution:** Let  $G$  be event of student being a genius, and  $C$  the event of student loving chocolate. Know:  $\Pr(G) = .6$ ,  $\Pr(C) = .7$ ,  $\Pr(G \cap C) = .4$ . Want:  $\Pr((G \cup C)^c)$ .

First use  $\Pr(G \cup C) = \Pr(G) + \Pr(C) - \Pr(G \cap C)$  to get  $\Pr(G \cup C) = .6 + .7 - .4 = .9$ . Then use  $\Pr((G \cup C)^c) = 1 - \Pr(G \cup C)$  to get  $\Pr((G \cup C)^c) = 1 - .9 = .1$ .

4. Chapter 1, problem 6

**Solution:** Let  $p_o$  be the common probability for all the odd faces, and  $p_e$  the common probability for all the even faces. Since all probabilities add to 1, have

$$3p_o + 3p_e = 1.$$

We also have, from the problem,

$$p_e = 2p_o.$$

Solving this system, get  $p_o = 1/9$  and  $p_e = 2/9$ , and so

$$\Pr(\{1, 2, 3\}) = 2p_o + p_e = 4/9.$$

5. (a) Chapter 1, problem 7.

**Solution:** I'm assuming that the numbers on the dice are 1, 2, 3 and 4. The sample space consists of all vectors of finite length, whose last co-ordinate is a 2 or 4, and with all the other co-ordinates being either a 1 or a 3, TOGETHER WITH all infinite vectors with all the co-ordinates being either a 1 or a 3 (this last part of the sample space encodes all the ways of NEVER getting an even number — unlikely but conceivable, so should be in the sample space).

- (b) Let  $E_n$  denote the event that the experiment described in Chapter 1, problem 7 ends after exactly  $n$  rolls.

- i. What points of the sample space are contained in  $E_n$ ?

**Solution:** All vectors of length  $n$ , whose last co-ordinate is a 2 or 4, and with all the other co-ordinates being either a 1 or a 3.

- ii. What points of the sample space are contained in  $(\cup_{n=1}^{\infty} E_n)^c$ ?

**Solution:**  $(\cup_{n=1}^{\infty} E_n)$  describes all the outcomes where the experiment ends in some finite time; so when we take the complement, we are left with all the outcomes where the experiment does not end in finite time. That is, it's the set of all infinite vectors with all the co-ordinates being either a 1 or a 3.

6. Chapter 1, problem 8 (so that everyone is using the same notation: say that  $A$ ,  $B$  and  $C$  are the three opponents, and that you know that you can beat  $A$  with probability  $p$ ,  $B$  with probability  $q$  and  $C$  with probability  $r$ ; assume that  $p > q > r$ ).

**Solution:** Suppose you play  $A$ , then  $B$ , then  $C$ . Your probability of beating all three is  $pqr$ . Your probability of beating  $A$  and  $B$  and losing to  $C$  is  $pq(1-r)$ , and your probability of losing to  $A$  and beating  $B$  and  $C$  is  $(1-p)qr$ . These are the three distinct ways of winning at least two in a row, so your probability of winning at least 2 in a row in this case is:

$$ABC \longrightarrow pqr + pq(1-r) + (1-p)qr.$$

If you play  $C$  first, then  $B$ , then  $A$ , the probability is:

$$CBA \longrightarrow rqp + rq(1-p) + (1-r)qp,$$

which it is easy to see is exactly the same as the probability for  $A$ , then  $B$ , then  $C$ .

By the same reasoning, we have:

$$BAC \longrightarrow qpr + qp(1-r) + (1-q)pr,$$

which is exactly the same as  $CAB$ , and

$$BCA \longrightarrow qrp + qr(1-p) + (1-q)rp,$$

which is exactly the same as  $BAC$ .

We want to show that the *BAC* probability (putting the weakest player, the one you are most likely to beat, in the middle) (which, as we've calculated, is the same as *CAB* probability) is the largest among the three different probabilities that we have calculated. I.e., we want

$$qpr + qp(1 - r) + (1 - q)pr > pqr + pq(1 - r) + (1 - p)qr,$$

which is the same as

$$qp - qpr + pr - qpr > pq - pqr + qr - pqr$$

which is the same as

$$pr > qr$$

which is the same as

$$p > q,$$

which we know is true. By similar reasoning,

$$qpr + qp(1 - r) + (1 - q)pr > qrp + qr(1 - p) + (1 - q)rp$$

reduces to

$$p > r,$$

which we know is true.

7. Chapter 1, problem 9 (a) only).

**Solution:** We first show that  $A = (A \cap S_1) \cup (A \cap S_2) \cup \dots \cup (A \cap S_n)$ . We could do this very formally, or just talk through it. Talking through it, we could say: Suppose  $x \in A$ . Then of course  $x$  appears on the left-hand side. Does it appear on the right-hand side? Yes: because  $\Omega = S_1 \cup \dots \cup S_n$ ,  $x$  must appear in one of the  $S$ 's, say  $S_i$ . That puts  $x \in A \cap S_i$ , so puts it in the set described on the right-hand side. Now suppose  $x \notin A$ . Then of course  $x$  doesn't appear on the left-hand side. Does it appear on the right-hand side? No: if it did, it would be in one of the  $A \cap S_i$ 's, so it would be in  $A$ , but it's not. So the right-hand side is a way of describing exactly everything in  $A$ , as we wanted.

Next, notice that  $A \cap S_1, A \cap S_2, \dots, A \cap S_n$  are all disjoint: if  $x \in A \cap S_i$ , and  $x \in A \cap S_j$ , for some different  $S_i, S_j$ , then  $x \in S_i$  and  $x \in S_j$ ; but that's impossible since the  $S$ 's are disjoint.

Now we can apply Axiom 2 of the axioms of Probability (or rather, the slight extension of it that we proved in class), to conclude that since  $A = (A \cap S_1) \cup (A \cap S_2) \cup \dots \cup (A \cap S_n)$  and the terms unioned on the right-hand side are disjoint, we have

$$\Pr(A) = \Pr(A \cap S_1) + \Pr(A \cap S_2) + \dots + \Pr(A \cap S_n).$$

8. Chapter 1, problem 10.

**Solution:** Because  $A = (A \cap B) \cup (A \cap B^c)$ , and  $A \cap B$ ,  $A \cap B^c$  are disjoint, we have

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c).$$

Similarly

$$\Pr(B) = \Pr(A \cap B) + \Pr(A^c \cap B).$$

So, adding,

$$\Pr(A) + \Pr(B) = \Pr(A \cap B) + \Pr(A \cap B^c) + \Pr(A \cap B) + \Pr(A^c \cap B).$$

Rearranging,

$$\Pr(A \cap B^c) + \Pr(A^c \cap B) = \Pr(A) + \Pr(B) - 2\Pr(A \cap B).$$

But now notice that  $A \cap B^c$ ,  $A^c \cap B$  are disjoint, so

$$\Pr(A \cap B^c) + \Pr(A^c \cap B) = \Pr((A \cap B^c) \cup (A^c \cap B)).$$

Putting these last two together we get

$$\Pr((A \cap B^c) \cup (A^c \cap B)) + \Pr(A) + \Pr(B) - 2\Pr(A \cap B).$$

9. A bag has three balls in it: 1 red, 1 blue and 1 green. Here's an experiment: pick a ball at random, note its color, return it to the bag, then again pick a ball at random, and note its color.

(a) Write down a sample space for this experiment.

**Solution:** One possibility is

$$\Omega = \{RR, RB, RG, BR, BB, BG, GR, GB, GG\}$$

where, for example,  $RG$  indicates the outcome "red ball drawn first, green ball second".

(b) Calculate the probability that the two colors observed are the same.

**Solution:** A reasonable assumption here is that all 9 of the outcomes of  $\Omega$  are equally likely, that is, that we have a discrete uniform probability model. Let  $E$  be the event that the two balls drawn have the same color, so  $E = \{RR, GG, BB\}$ , and  $\Pr(E) = 3/9 = 1/3$ .

10. A system has 5 components, each of which is either working or failed. An experiment consists of observing the 5 components, and recording the status of each one. Record an outcome of this experiment by a vector  $(x_1, x_2, x_3, x_4, x_5)$ , with  $x_i = 0$  if component  $i$  is failed, and  $x_i = 1$  if it is working.

- (a) How big is the sample space?

**Solution:** There are 2 choices for  $x_1$ . Whatever we choose for  $x_1$ , there are 2 choices for  $x_2$ ; so there are  $2 \times 2 = 4$  choices for  $(x_1, x_2)$ . Whatever we choose for  $(x_1, x_2)$ , there are 2 choices for  $x_3$ ; so there are  $2 \times 4 = 8$  choices for  $(x_1, x_2, x_3)$ . Continuing in this way, we get a total of  $2 \times 2 \times 2 \times 2 \times 2 = 32$  choices for  $(x_1, x_2, x_3, x_4, x_5)$ .

- (b) How many outcomes in the event “Component 1 is working, and component 4 is failed”?

**Solution:** We must have  $x_1 = 1$  and  $x_4 = 0$ . There are 2 choices for  $x_2$ , then whatever we choose for  $x_2$ , there are 2 choices for  $x_3$ , then whatever we choose for  $x_3$ , there are 2 choices for  $x_5$ , leading to  $2 \times 2 \times 2 = 8$  choices for  $(x_2, x_3, x_5)$ . So there are 8 outcomes in the event under discussion.

- (c) Suppose that the system works if EITHER components 1 and 2 are both working OR components 3 and 4 are both working OR components 1, 3 and 5 are working. Write down all the outcomes in the event “The system is working”.

**Solution:** The easiest approach is to list all 32 possible outcomes, and check which ones satisfy at least one of the three conditions (1, 2 working, or 3, 4 working, or 1, 3, 5 working). There are 15 of them in all.

11.  $A$  and  $B$  are mutually exclusive events with  $\Pr(A) = .3$  and  $\Pr(B) = .5$ .

- (a) What is the probability that either  $A$  or  $B$  occurs?

**Solution:**  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = .3 + .5 - 0 = .8$  (the “0” for  $\Pr(A \cap B)$  is because  $A$  and  $B$  are mutually exclusive).

- (b) What is the probability that  $A$  occurs but  $B$  does not?

**Solution:** Once  $A$  occurs, we know for certain that  $B$  does not (because  $A$  and  $B$  are mutually exclusive), so “ $A$  occurs but  $B$  does not” is the same as “ $A$  occurs”; the probability is .3.

- (c) What is the probability that both  $A$  and  $B$  occur?

**Solution:** This is 0 because  $A$  and  $B$  are mutually exclusive events.

12. Put a rook at a random square on an 8 by 8 chessboard (all squares equally likely). Put a second rook down on a random different square (all free squares equally likely); keep going, always putting a new rook down on a randomly chosen square that doesn't already have a rook on it (all free squares equally likely). After having put down 8 rooks, what is the probability that no two rooks are attacking each other (that is, on the same row of same column as each other)?

**Solution:** How many outcomes in the sample space,

$\Omega$  = all ways of places rooks numbered 1 through 8 on distinct squares of the chessboard?

There are 64 choices for where to put rook 1, then (regardless of where rook 1 goes) 63 choices for where to put rook 2, then (regardless of where rooks 1 and 2 goes) 62 choices for where to put rook 3, and so on down to 57 choices for the placement of rook 8 (regardless of where rooks 1 through 7 go). This means that

$$|\Omega| = 64 \times 63 \times 62 \times 61 \times 60 \times 59 \times 58 \times 57.$$

It's a reasonable assumption, given the description of the problem, that all of these outcomes are equally likely, that is, that we have a discrete uniform probability model. Let  $E$  be the event that no two rooks are attacking each other. How big is  $E$ ? There are 64 choices for where to put rook 1. In  $E$ , this rules out not one location for rook 2, but rather 15 locations (the place we put rook 1, and everything in its row and column). An easier way to say this is that once rook 1 is placed, if we don't want to place a rook in the same row or column as rook 1, then rook 2 must go in the 7 by 7 chessboard (49 squares, same as  $64 - 15$ ) obtained from the regular chessboard by deleting the row and column of rook 1. Once rook 2 has been placed somewhere on the 7 by 7 board, if we don't want to place another rook in the same row or column as either rook 1 or rook 2, then rook 3 must go in the 6 by 6 chessboard (36 squares) obtained from the regular chessboard by deleting the row and column of rook 1, and the row and column of rook 2. Carrying on this way, we find that there are 25 choices for placement of rook 4, 16 choices for placement of rook 5, 9 choices for placement of rook 6, 4 choices for placement of rook 7, and 1 choice for placement of rook 8, leading to

$$|E| = 64 \times 49 \times 36 \times 25 \times 16 \times 9 \times 4 \times 1.$$

Since we are in a discrete uniform probability model,

$$\Pr(E) = \frac{|E|}{|\Omega|} = \frac{64 \times 49 \times 36 \times 25 \times 16 \times 9 \times 4 \times 1}{64 \times 63 \times 62 \times 61 \times 60 \times 59 \times 58 \times 57} \approx 9.1 \times 10^{-6}.$$

13. There are 5 hotels in downtown South Bend. Last night, 3 travellers arrived from distant La Porte, Indiana. Each of the three picked a hotel at random from among the five to check-in to (each hotel equally likely). The three travellers didn't communicate with each other about their check-in plans.

(a) What is the probability that all three stayed at different hotels?

**Solution:** The sample space for this experiment consists of all triples  $(a, b, c)$  where each of  $a, b, c$  is a number from among 1, 2, 3, 4, 5 (so, for example, outcome  $(1, 1, 1)$  means that all three stayed in hotel number 1). All sample points are equally likely (it seems from the problem description that we should assume this), so we are in a discrete uniform probability model. We have

$$|\Omega| = 5 \times 5 \times 5 = 125$$

(the first traveller has 5 choices, so does the second, and so does the third). Let  $E$  be the event that the three choose different hotels. We have

$$|E| = 5 \times 4 \times 3 = 60$$

(the first traveller has 5 choices, but once the first has made his choice, the second only has 4 choices, since he must stay in a different hotel from the first; then the third has 3 choices). So  $\Pr(E) = 60/125 = 12/25 = .48$ .

(b) What is the probability that all three stayed at the same hotel?

**Solution:** Let  $F$  be the event that all three choose the same hotel. We have  $|F| = 5$  (the outcomes in  $E$  are  $(1, 1, 1)$ ,  $(2, 2, 2)$ ,  $(3, 3, 3)$ ,  $(4, 4, 4)$  and  $(5, 5, 5)$ ), so  $\Pr(F) = 5/125 = 1/25 = .04$ .

14. Let  $E$ ,  $F$  and  $G$  be three events. Using  $\cup$ ,  $\cap$  and  $^c$ , find expressions (such as  $F \cap G^c$ ) for each of the following events:

(a) Only  $E$  occurs.

**Solution:** There are many possible answers for this and all the other parts. One is  $E \cap F^c \cap G^c$ .

(b) At least one of  $E$ ,  $F$  and  $G$  occurs.

**Solution:**  $E \cup F \cup G$ .

(c) All three of  $E$ ,  $F$  and  $G$  occur.

**Solution:**  $E \cap F \cap G$ .

(d) At most one of  $E$ ,  $F$  and  $G$  occur.

**Solution:** I'll do this as "either none occur, or only  $E$ , or only  $F$ , or only  $G$ ":

$$(E \cup F \cup G)^c \cup (E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G).$$

(e) Exactly two of  $E$ ,  $F$  and  $G$  occur.

**Solution:** I'll do the "or" of each of the three ways of getting exactly two to occur:

$$(E \cap F \cap G^c) \cup (E \cap F^c \cap G) \cup (E^c \cap F \cap G).$$

15. Use the axioms of probability to show that if  $E$  and  $F$  are any events, then

$$\Pr(E \cap F) \geq \Pr(E) + \Pr(F) - 1.$$

**Solution:** We showing in class, using the axioms, that

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F),$$



or

$$\Pr(E \cap F) = \Pr(E) + \Pr(F) - \Pr(E \cup F).$$

Notice that  $E \cup F \subseteq \Omega$ , so from another fact that we showed in class (from the axioms)

$$\Pr(E \cup F) \leq \Pr(\Omega) = 1$$

(the last equality from Axiom 3). Combing these last two equations we get

$$\Pr(E \cap F) \geq \Pr(E) + \Pr(F) - 1.$$

16. Use a Venn diagram to explain why the following relations between events are always true:

(a)  $E \cap F \subseteq E \subseteq E \cup F$ .

**Solution:**  $E \cap F \subseteq E$ : This just involves observing that in the typical Venn diagram drawing of  $E$  and  $F$  inside  $\Omega$ , the region corresponding to  $E \cap F$  (the football shaped region between  $E$  and  $F$ ) is completely contained in the region corresponding to  $E$ .

$E \subseteq E \cup F$ : This just involves observing that in the typical Venn diagram drawing of  $E$  and  $F$  inside  $\Omega$ , the region corresponding to  $E$  is completely contained in the region corresponding to  $E \cup F$  (the “infinity” ( $\infty$ ) shaped region that covers both  $E$  and  $F$ ).

(b) If  $E \subseteq F$  then  $F^c \subseteq E^c$ .

**Solution:** Knowing that  $E \subseteq F$ , we don’t draw a typical Venn diagram, with  $E$  and  $F$  intersecting; instead we put  $E$  completely inside of  $F$ . Then  $F^c$  is the region outside  $F$ , which is completely contained in  $E^c$ , which is the region outside  $E$ , together with the region between  $E$  and  $F$ .

17. Go to Random.org’s dice roller (<http://www.random.org/dice/>), and select two dice. Roll the two dice 50 times, and each time record the difference between the higher and the lower number rolled (it will be one of 0, 1, 2, 3, 4 or 5). Record your results in a table like the one below (your actual “times rolled” numbers may be different):

difference	0	1	2	3	4	5
times rolled	3	14	15	8	6	4

- (a) Let  $p_i$  be the probability that when you roll two dice, the difference between the two numbers you roll is exactly  $i$ . Use your data to empirically guess at the values of  $p_0, p_1, p_2, p_3, p_4$  and  $p_5$ .

**Solution:** The results that I got are given in the table above. They suggest:

$$p_0 = 3/50 = .06, \quad p_1 = .28, \quad p_2 = .3, \quad p_3 = .16, \quad p_4 = .12, \quad p_5 = .08.$$

- (b) Use your guesses to estimate the probability that when you roll two dice, the difference between the two numbers you roll is at least 3.

**Solution:**  $\Pr(\{3, 4, 5\}) = .16 + .12 + .08 = .36$ .

- (c) By listing all the events in the sample space of roll two dice, calculate  $p_i$  exactly for each  $i$ , and calculate the exact probability that the difference between the two numbers is at least 3.

**Solution:**

$$p_0 = 6/36, \quad p_1 = 10/36, \quad p_2 = 8/36, \quad p_3 = 6/36, \quad p_4 = 4/36, \quad p_5 = 2/36$$

(for example, of the 36 equally likely outcomes for (number on first dice, number on second), 4 of them  $((1, 5), (2, 6), (5, 1), (6, 2))$  give a difference of 4). So the exact answer for the probability of getting a difference of at least 3 is

$$\Pr(\{3, 4, 5\}) = 6/36 + 4/36 + 2/36 = 12/36 = .33\dots,$$

a little bit lower than the empirical estimate.