Math 30530: Introduction to Probability, Fall 2013

Midterm Exam II

Solutions

1. (a) Is there a value of c that makes the following function a density function? (Either find c, or explain why there isn't such a c).

$$f(x) = \begin{cases} c(x^2 - 3x + 2) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Solution: $\int_0^3 (x^2 - 3x + 2)\, dx = 3/2$, so the only viable choice for c would be $c = 2/3$. But, $f(x) = (2/3)(x^2 - 3x + 2)$ takes some negative values between 0 and 3 (in fact it is negative for all x, $1 < x < 2$). So this choice of c forced by the normalization rule will not do, and there is no c that works.

(b) A random variable X has density function

$$f(x) = \begin{cases} \frac{9-x^2}{18} & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $E(X)$ and $\text{Var}(X)$.

Solution:

$$E(X) = \int_0^3 x \left(\frac{9 - x^2}{18} \right) dx = \frac{9}{8},$$

$$E(X^2) = \int_0^3 x^2 \left(\frac{9 - x^2}{18} \right) dx = \frac{9}{5},$$

so

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{171}{320}.$$

2. (a) The lifetime of a car's fan belt is exponentially distributed with $\lambda = 1/30$ (units are thousands of miles). After 22 thousand miles of operation, my car's fan belt is still working. What is the probability that the fan belt will break down between 35 thousand and 45 miles of operation?

Solution: Let $X \sim$ exponential(1/30) be lifetime of fan belt. We want

$$\Pr(35 \leq X \leq 45 \mid X > 22)$$

(a **conditional** probability). By memorylessness of the exponential, this is the same as

$$\Pr(13 \leq X \leq 23) = \int_{13}^{23} \frac{1}{30} e^{-x/30}\, dx \approx .1838.$$

(b) The lifetime of a certain manufacturer's lightbulbs is known to be exponentially distributed. In a test of a large number of lightbulbs, 40% of them burned out at or after 50 weeks of continuous use. Find the value of x such that 95% of all lightbulbs from this manufacturer have a lifetime of at least x, measured in weeks of continuous use.

Solution: Let $X \sim \text{exponential}(\lambda)$ be the lifetime of a lightbulb. We know

$$\Pr(X > 50) = .4,$$

from which we get

$$\int_{50}^{\infty} \lambda e^{-\lambda x}\, dx = .4.$$

This is same as

$$e^{-50\lambda} = .4$$

or

$$\lambda = \frac{\ln .4}{-50} \approx .0183.$$

We want to find that x such that

$$\Pr(X > x) = .95,$$

i.e.,

$$\int_{x}^{\infty} \lambda e^{-\lambda t}\, dt = .95.$$

This is same as

$$e^{-x\lambda} = .95$$

or

$$x = \frac{\ln .95}{-\lambda} = 50\frac{\ln .95}{\ln .4} \approx 2.8.$$

3. (a) A box contains a large number of nails whose lengths are normally distributed with mean 4 cm and standard deviation .1 cm. 10 nails are chosen at random from the batch. What is the probability that exactly 5 of them will be between 3.9 cm and 4.1 cm in length?

Solution: The probability that a single nail is between 3.9 and 4.1 is $\Pr(3.9 < X < 4.1)$ where $X \sim \text{Normal}(4, (.1)^2)$, which is the same as $\Pr(-1 \leq Z \leq 1)$, approximately .68. So the probability that exactly 5 of the nails satisfy the length condition is approximately

$$\binom{10}{5}(.68)^5(.32)^5 \approx .12.$$

(b) A nail manufacturer has a machine that produces nails with mean length 4 cm, with a standard deviation that can be set to any value σ cm. What value should he set σ to be, to ensure that only 10% of all nails produced are longer than 4.1 cm?

Solution: Let X be the length of a nail; $X \sim \text{Normal}(4, \sigma^2)$. We want σ so that $\Pr(X > 4.1) = .1$. Normalizing, this is the same as $\Pr(Z > .1/\sigma) = .1$. From a standard normal table we find $\Pr(Z > 1.28) = .1$, so $.1/\sigma = 1.28$, or $\sigma \approx .078$.

(c) Which is more likely: for a normal random variable to be within half a standard deviation of its mean, or for it to be more than one standard deviation away from its mean?

Solution: From a standard normal table we find

$$\Pr(|Z| < .5) = \Pr(-.5 < Z < .5) = \Phi(.5) - \Phi(-.5) = \Phi(.5) - (1 - \Phi(.5)) = 2\Phi(.5) - 1 \approx .38,$$

while

$$\Pr(|Z| > 1) = 2\Pr(Z > 1) = 2(1 - \Phi(1)) \approx .32,$$

so it is more likely to be within a half standard deviation of mean than be more than one standard deviation away.

4. Let the random variables X and Y have joint density

$$f(x, y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } x > 0 \text{ and } 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal densities f_X and f_Y.

Solution: In general, $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$. Applying here, we get

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that to fully specify the marginal densities, we need to give their values for **all** possible inputs (including those intervals where the densities are 0).

(b) Are X and Y independent? Explain.

Solution: Because $f_X(x)f_Y(y) = f(x, y)$ for all x, y, they are independent.

(c) Write down (no need to evaluate) an integral whose value is $P\{X < 2Y\}$.

Solution: After drawing a diagram, we see that the region of the plane where $x < 2y$ and the joint density is non-zero is the triangle bounded by $(0, 0)$, $(0, 2)$ and $(4, 2)$. This leads to the following possible double integrals:

$$P(X < 2Y) = \int_0^2 \int_0^{2y} e^{-x}/2\, dxdy = \int_0^4 \int_{x/2}^2 e^{-x}/2\, dydx.$$

5. X is a uniform random variable that takes values between 1 and e^3. Let $Y = \ln X$.

(a) Find the CDF of Y.

Solution: Range of possible values of Y is $\ln 1 \le y \le \ln e^3$ or $0 \le y \le 3$. For each such y, have

$$\begin{aligned} \Pr(Y \le y) &= \Pr(X \le e^y) \\ &= \frac{e^y - 1}{e^3 - 1} \end{aligned}$$

(the last line using the density of X). So

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{e^y - 1}{e^3 - 1}, & \text{if } 0 \le y \le 3 \\ 1, & \text{if } y > 3. \end{cases}$$

Notice that we have to specify the CDF for **all** possible values of the input.

(b) Find the density function of Y.

Solution: We differentiate the distribution function to get:

$$f_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{e^y}{e^3 - 1}, & \text{if } 0 \le y \le 3 \\ 0, & \text{if } y > 3. \end{cases}$$

(c) Calculate the expected value of Y.

Solution:

$$E(Y) = \int_0^3 y \left(\frac{e^y}{e^3 - 1} \right) \, dy = \frac{2e^3 + 1}{e^3 - 1}$$

(this requires integration by parts).