Math 30530: Introduction to Probability, Fall 2013

Midterm Exam I

Practice question solutions

1. Two fair coins are tossed at the same time. If the two coins show the same face (two heads or two tails) we win, and if they show different faces we lose. Let A be the event that the first coin comes up heads, B the event that the second coin comes up heads, and C the event that we win. Which of the following statements is FALSE? (**NOTE**: It may be that more than one of them is false; you should identify ALL the false statements.)

   (a) A and B are independent

   (b) A and C are not independent

   (c) B and C are independent

   (d) $\Pr(A|C) = 1/4$.

   Solution: $\Pr(A) = \Pr(B) = \Pr(C) = 1/2$. $\Pr(A \cap B) = \Pr(A \cap C) = \Pr(B \cap C) = 1/4$. So each of the pairs $(A, B), (A, C), (B, C)$ are independent; this makes the first and third statements true and the second false. Also, $\Pr(A|C) = \Pr(A) = 1/2$, making the fourth statement false. So: the second and fourth statements are the false ones.

2. I have a biased coin that comes up Heads with probability $1/3$. I toss the coin 5 times independently, and count the total number of Heads I get. Let X be this number. Which of the following is a correct expression for $\Pr(\text{first toss is a head}|X = \text{ either 1 or 5})$?

   (a) $\frac{\frac{1}{3}\left(\frac{2}{3}\right)^4}{5\cdot\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}$

   (b) $\frac{\frac{1}{3}\left(\frac{2}{3}\right)^4}{\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}$

   (c) $\frac{\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}{5\cdot\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}$

   (d) $\frac{1}{5}$.

   Solution: First we want (for the denominator of the conditional probability) the probability that $X = 1$ or 5; since $X \sim \text{Binomial}(5, 1/3)$, that's

   $$5 \cdot \frac{1}{3}\left(\frac{2}{3}\right)^4 + \left(\frac{1}{3}\right)^5.$$

   Then (for the denominator of the conditional probability) we want the probability that the first toss is a head AND (either $X = 1$ or $X = 5$); this is the same as (first toss is a head AND

$X = 1$) OR (first toss is a head AND $X = 5$). The probability of (first toss is a head AND $X = 1$) is

$$\frac{1}{3}\left(\frac{2}{3}\right)^4$$

and the probability of (first toss is a head AND $X = 5$) is

$$\left(\frac{1}{3}\right)^5.$$

These events are disjoint, so the probability that the first toss is a head AND (either $X = 1$ or $X = 5$) is

$$\frac{1}{3}\left(\frac{2}{3}\right)^4 + \left(\frac{1}{3}\right)^5.$$

Using the definition of conditional probability, we see that the correct option is the third one.

3. In how many ways can 7 people be seated in a row if, among the 7, there are a group of 4 who insist on being seated together, and among that group of four, there are 2 who insist on sitting side-by-side?

   (a) 576
   (b) 288
   (c) 504
   (d) 252.

   Solution: There are 4 ways to place the block of 4 (first through fourth, second through fifth, third through sixth, fourth through seventh). Wherever they are placed, there are three ways to identify the two spots among the four occupied by the two who insist on sitting side-by-side (first and second, second and third, third and fourth within block). There are then two ways to order those two, two ways to order the other two from the gang of four, and six (3!) ways to order the remaining three, leading to

   $$4 \times 3 \times 2 \times 2 \times 6 = 288$$

   arrangements (option 2).

4. To get a driving licence, I need to pass a written test. Each time I take the written test, I pass it with probability 1/5, all attempts independent. I take the test twice, and fail it each time. Let Y be the total number of *additional* times I have to take the test until I finally pass it. Then

   (a) $E(Y) = 3$
   (b) $E(Y) = 2.5$
   (c) $E(Y) = 5$
   (d) $E(Y)$ cannot be calculated from the given information.

   Solution: Because successive trials are independent, information about previous trials doesn't change future probabilities, and so Y is just Geometric(1/5), with expectation $1/(1/5) = 5$ (option 3).

5. I roll two fair dice, and let X be the sum of the two numbers that come up, and Y the difference between the larger number and the smaller number (so, for example, if the numbers are 3 and 5, then $X = 8$ and $Y = 2$). Which of these is the value of the joint mass function $p_{X,Y}(5,3)$?

   (a) 1/18
   (b) 1/26
   (c) 0
   (d) 1/54.

   Solution: There are just two ways to get a sum of 5 and a difference of 3: 4 and 1 or 1 and 4; so $p_{X,Y}(5,3) = 2/36 = 1/18$ (option 1).

6. (a) A sample space Ω is partitioned into 5 disjoint pieces: $\Omega = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$. State Bayes' formula for calculating $\Pr(S_1|E)$ knowing $\Pr(E|S_i)$ and $\Pr(S_i)$ for each i.

   Solution: $\Pr(S_1|E) =$

$$\frac{\Pr(E|S_1)\Pr(S_1)}{\Pr(E|S_1)\Pr(S_1) + \Pr(E|S_2)\Pr(S_2) + \Pr(E|S_3)\Pr(S_3) + \Pr(E|S_4)\Pr(S_4) + \Pr(E|S_5)\Pr(S_5)}.$$

   (b) When I go away for a week, I ask my neighbor to regularly water my plants. I know from experience that she is 90% likely to actually do this. If my plants are regularly watered, they stay healthy with probability .9. If my plants are not regularly watered, they stay healthy with probability .2.

   i. What is the probability that my plants will be healthy when I return from my trip?

   Solution: Let H be the event that my plants are healthy, and W the event that my neighbor came by to water them. Then

$$\Pr(H) = \Pr(H|W)\Pr(W) + \Pr(H|W^c)\Pr(W^c) = (.9)(.9) + (.2)(.1) = .83.$$

   ii. I come home from my trip, and find that my plants are not healthy. What is the probability that my neighbor failed to come by to regularly water them?

   Solution: By definition of conditional probability,

$$\Pr(W^c|H^c) = \frac{\Pr(W^c \cap H^c)}{\Pr(H^c)} = \frac{\Pr(H^c|W^c)\Pr(W^c)}{\Pr(H^c)} = \frac{(.8)(.1)}{.17} \approx .48.$$

7. The file cabinet in my office has 7 drawers, three of which contain some pens. I open drawers at random (never opening the same drawer twice) until I find a drawer with pens in it. Let X be the number of drawers that I open.

   (a) Find the mass function of X.

   Solution: The possible values for X are 1, 2, 3, 4 or 5. The mass function is

$$p_X(x) = \begin{cases} \frac{3}{7} = \frac{15}{35} & \text{if } x = 1 \\ \frac{4}{7}\frac{3}{6} = \frac{2}{7} = \frac{10}{35} & \text{if } x = 2 \\ \frac{4}{7}\frac{3}{6}\frac{3}{5} = \frac{6}{35} & \text{if } x = 3 \\ \frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{3}{4} = \frac{3}{35} & \text{if } x = 4 \\ \frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{1}{4}\frac{3}{3} = \frac{1}{35} & \text{if } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find the expected value of X.

Solution: $E(X) = 1(15/35) + 2(10/35) + 3(6/35) + 4(3/35) + 5(1/35) = 2$.

(c) Find the variance of X.

Solution: $E(X^2) = 1^2(15/35) + 2^2(10/35) + 3^2(6/35) + 4^2(3/35) + 5^2(1/35) = 5.2$, so $\mathrm{Var}(X) = 5.2 - (2)^2 = 1.2$.

8. In my hand I have five playing cards, specifically: the two of spades, the two of clubs, the three of hearts, the three of clubs and the four of diamonds. I choose two cards, one after the other, *with replacement*, and note the face values. Jack is interested in X, the larger of the two values noted, and Jill is interested in Y, the difference between the larger and the smaller values.

(a) Calculate the joint mass function $p_{X,Y}(x, y)$ of X and Y (a table of values is ok).

Solution: Here's a table of the joint mass values, with the possible values for Y (0,1,2) across the top, and those for X (2,3,4) along the side. For example, how can $Y = 1$ and $X = 3$? Only if the two cards chosen were a two and a three; the probability that first I select a two, and then a three, is $(2/5)(2/5) = 4/25$, and the probability that first I select a three, and then a two, is $(2/5)(2/5) = 4/25$, so the net probability of getting $X = 3$, $Y = 1$ is $8/25$.

	0	**1**	**2**
2	$\frac{4}{25}$	0	0
3	$\frac{4}{25}$	$\frac{8}{25}$	0
4	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{4}{25}$

(b) Given that $Y = 0$, what is the probability that $X = 2$?

Solution: $\Pr(X = 2|Y = 0) = \frac{\Pr(X=2,Y=0)}{\Pr(Y=0)} = \frac{4/25}{4/25+4/25+1/25} = 4/9$.

(c) Are the events $\{X = 3\}$ and $\{Y = 1\}$ independent?

Solution: $\Pr(X = 3) = (4/25) + (8/25) = 12/25$ and $\Pr(Y = 1) = (8/25) + (4/25) = 12/25$, so $\Pr(X = 3)\Pr(Y = 1) = 144/625$. But $\Pr(X = 3, Y = 1) = 8/25 \neq 144/625$, so these events are not independent.