Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 6 — Solutions

1. **GW** 17.2

   Solution: a) $0, 1, 2, 3, \ldots$. It's Poisson because we're counting number of occurrences of an event, with different occurrences independent. We have $\lambda = 6.85$.

   b) $\approx .1487$ (Probability that Poisson with $\lambda = .685$ equal 7)

   c) $7 \times 6.85 = 47.95$.

   d) Since variance of Poisson is λ, standard deviation here is $\sqrt{47.95} \approx 6.92$.

   e) $\approx .9917$ (Probability that Poisson with $\lambda = .685$ greater than or equal to 2)

   f) Use Geometric with $p = .9917$; probability of three successive failures is $(1 - .9917)^3 \approx .00000057$.

2. **GW** 17.5

   Solution: a) Binomial with $n = 1,000,000$ and $p = 1/729,000$.

   b) $\Pr(X = 3) = \binom{1000000}{3}(1/729000)^3(728999/729000)^{999997}$.

   c) $np = 1000000/729000 \approx 1.3717$.

   d) Poisson(1.3717).

   e) $\Pr(X = 3) \approx (1.3717)^3/3!e^{-1.3717} \approx .1597$.

3. **GW** 17.7

   Solution: Expected number of wins per lifetime is 2 ($= 10000 \times (1/5000)$), so use $X \sim$ Poisson(2) as approximation. $\Pr(X \geq 1) \approx .865$.

4. **GW** 17.13

   Solution: Use $X \sim$ Poisson(2) to model number to arrive in next 10 minutes.

   a) $\Pr(X = 3) \approx .18$.

   b) $\Pr(X \geq 3) \approx .3233$.

   c) $\Pr(X = 3 | X \geq 3) = \Pr(X = 3, X \geq 3)/\Pr(X \geq 3) = \Pr(X = 3)/\Pr(X \geq 3)$. This is $.18/.3233 \approx .557$.

   d) $2 \times \$2.50 = \$5$.

e) $\text{Var}(2.5X) = (2.5)^2\text{Var}(X) = (2.5)^2 \times 2$, so standard deviation of $2.5X$ is $2.5 \times \sqrt{2} \approx 3.54$.

5. **GW** 17.22a

   Solution: X = number of men; $X \sim$ Poisson(2). Y = number of men; $Y \sim$ Poisson(2.5).

   $$\Pr(X = 1, Y = 2) = \Pr(X = 1)\Pr(Y = 2) \approx .27 \times .26 \approx .0702.$$

6. **GW** 18.10

   Solution: $\frac{\binom{10}{3}\binom{25}{2}}{\binom{35}{5}}$.

7. **GW** 18.18

   Solution: a) $X \sim$ Hypergeometric$(7, 12, 3)$.

   b) For each $k = 0, 1, 2, 3$, $\Pr(X = k) = \frac{\binom{7}{k}\binom{5}{3-k}}{\binom{12}{3}}$.

   c) $n\frac{M}{N} = 3 \times \frac{7}{12} = 1.75$.

8. **GW** 18.21

   Solution: Use $X \sim$ Hypergeometric$(13, 52, 13)$ to model number of Spades in your hand.

   a) $\sum_{k=7}^{13} \frac{\binom{13}{k}\binom{39}{13-k}}{\binom{52}{13}}$.

   b) $n\frac{M}{N} = 13 \times \frac{13}{52} = 3.25$.

9. **GW** 24.1

   Solution: a)

   $$\int_0^1 kx^2(1-x)^2\ dx = k\int_0^1 (x^2 - 2x^3 + x^4)\ dx = k[x^3/3 - x^4/2 + x^5/5]_0^1 = k/30.$$

   So $k = 30$.

   b) $\Pr(X \geq 3/4) = 30\int_{3/4}^1 x^2(1-x)^2\ dx$.

10. **GW** 24.4 a, b, c, g, h only

    Solution: a) $\Pr(3 \leq X \leq 6) = \int_3^6 (1/3)e^{-x/3}\ dx$.

    b) For $x < 0$, $F_X(x) = 0$. For $x \geq 0$,

    $$F_X(x) = \int_0^x \frac{1}{3}e^{-t/3}\ dt = [e^{-t/3}]_0^x = 1 - e^{-x/3}.$$

    c) $\Pr(X \geq 24) = 1 - F_X(24) = e^{-8}$.

    g) See figure 1 of figures page.

    h) See figure 2 of figures page.

11. **GW** 24.20

    Solution: Differentiate $F_X(x)$ to get the density:

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ (7/4)x & 0 \le x \le 1 \\ 0 & 1 < x < 7 \\ 1/8 & 7 \le x \le 8 \\ 0 & \text{if } 8 < x. \end{cases}$$

12. **GW** 24.23

    Solution: $\Pr(X < 2) = \int_0^2 x^2 e^{-x} \approx .6466$. To get this, use integration by parts twice.

13. What value of c, if any, makes the following a probability density function?

$$f(x) = \begin{cases} c(1 - x^2) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

    Solution: No value of c works: if $c = 0$ then $\int_{-\infty}^{\infty} f(x)\,dx = 0$; if $c > 0$ then $f(x)$ is negative for $1 < x < 2$; and if $c < 0$ then $f(x)$ is negative for $0 < x < 1$;

14. I get *on average* 3 colds per year. A new drug, which is believed to be effective for 20% of the population, will halve the number of colds I get on average (if effective; it will do nothing otherwise). During the course of the year in which I take the drug, I get just one cold. How likely is it that I am among the 20% for whom the drug is effective?

    Solution: By Bayes' formula

$$\Pr(\text{Drug effective } (E) | \text{I get one cold } (O)) =$$

$$\frac{\Pr(O|E)\Pr(E)}{\Pr(O|E)\Pr(E) + \Pr(O|\text{Drug not effective } (N))\Pr(N)}.$$

    We know $\Pr(E) = .2$ and $\Pr(N) = .8$. To calculate $\Pr(O|E)$, note that if the drug is effective, then the number of colds I get should be modeled by a Poisson rv with $\lambda = 1.5$; so $\Pr(O|E) = (3/2)e^{-3/2} \approx .335$. By the same reasoning, $\Pr(O|N) = 3e^{-3} \approx .149$. So

$$\Pr(\text{Drug effective } (E) | \text{I get one cold } (O)) = \frac{(.335)(.2)}{(.335)(.2) + (.149)(.8)} \approx .36.$$

15. Stenographer Jones makes on average 2.5 errors per page of court proceedings that he transcribes. Yesterday, he transcribed 60 pages of proceedings.

    (a) On average how many errors do I expect to see in total in the 60 pages?

        Solution: $2.5 \times 60 = 150$.

(b) On average, on how many pages will I expect to see 4 or more errors?

Solution: Let X be number of of mistakes per page; X is a Poisson with $\lambda = 2.5$, so $\Pr(X \geq 4) \approx .242$; thus on average I expect to see $150 \times .242 \approx 36.3$ such pages.

(c) How likely is it that the fifth page I come to is the first with 4 or more errors?

Solution: This is the probability that a Geometric with parameter $\approx .242$ takes vales 5; so it's $\approx (1 - .242)^4(.242) \approx .08$.

(d) Errors cost money, so Jones has to pay the office pool 25c for every error he has made over the 60 pages. How much does he expect to pay, and what's the probability that he has to pay more than his expected value?

Solution: Since he expects to make 150 errors, he expects to pay $150 \times 25c =$ \$37.50. Let X be the number of errors Jones makes over 60 pages. Because he makes on average 150, we should model X by a Poisson with $\lambda = 150$. So for the last part, we want $\Pr(X \geq 151)$ with $X \sim \text{Poisson}(150)$. This is $\approx .478$.

16. Assume that each time the New York Yankees play a baseball game, they have a probability .012 of having their pitcher throw a complete-game shutout, all games independent. During the course of a season, they play 162 games. Let X be the number of games in which their pitcher throws a complete-game shutout.

(a) Calculate, to 4 decimal places, the *exact* probabilities of X taking each of the values 0, 1, 2, 3 and 4. (For this you will definitely want to use some kind of calculator, such as the ones linked to at the beginning of the list of problems).

Solution: Exactly, $X \sim \text{Binomial}(162, .012)$, and the required probabilities are:

$$\begin{aligned}
\Pr(X = 0) &\approx .1415 \\
\Pr(X = 1) &\approx .2783 \\
\Pr(X = 2) &\approx .2721 \\
\Pr(X = 3) &\approx .1763 \\
\Pr(X = 4) &\approx 0851.
\end{aligned}$$

(b) Approximating X by a Poisson random variable, *estimate* to 4 decimal places the probabilities of X taking each of the values 0, 1, 2, 3 and 4.

Solution: Approximately, $X \sim \text{Poisson}(162 \times 012\ (= 1.944))$, and the required probabilities are:

$$\begin{aligned}
\Pr(X = 0) &\approx .1431 \\
\Pr(X = 1) &\approx .2782 \\
\Pr(X = 2) &\approx .2705 \\
\Pr(X = 3) &\approx .1753 \\
\Pr(X = 4) &\approx .0852.
\end{aligned}$$

(c) Ponder on the two sets of answers. Do you like the Poisson as an approximation to the Binomial in this case?

Solution: This is a matter of taste and how accurate you want your answers to be; but the approximations look pretty good to me.