Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 4 — Solutions

1. **GW** 8.2

   Solution: For $x \geq 1, y \geq 1$ (and integers),

   $$\Pr(X = x, Y = y) = (.99)^{x-1}(.01)(.97)^{y-1}(.03)$$

   (here using independence of X and Y). For all other values of x, y, $\Pr(X = x, Y = y) = 0$.

2. **GW** 8.3

   Solution: For integers $x, y \geq 1$ (the only possible values that X, Y can take),

   $$\Pr(X = x, Y = y) = \Pr(\{TTT \ldots TTHTTTT \ldots TTH\}) = (1/2)^{x+y}$$

   (here there are $x - 1$ tails in the first string of tails, and $y - 1$ in the other). On the other hand,

   $$\Pr(X = x) = \Pr(\{TTT \ldots TTH\}) = (1/2)^x$$

   and

   $$\Pr(Y = y) = \Pr(\{TTT \ldots TTH\}) = (1/2)^y.$$

   The reason for this last is that once we start looking for the second head, it is exactly the same as starting over (and looking for a first head on a fresh sequence of tosses). So

   $$\Pr(X = x, Y = y) = \Pr(X = x) \Pr(Y = y),$$

   and so X and Y *are* independent.

3. **GW** 8.7 (for this one, also find the joint mass of X and Y)

   Solutions: The possible values for X are 0, 1 and 2; the possible values for Y are 0,1,2. So we need to compute 9 probabilities to specify the joint mass. Three of them are easy:

   $$\Pr(X = 2, Y = 2) = 0; \qquad \Pr(X = 2, Y = 1) = 0; \qquad \Pr(X = 1, Y = 2) = 0,$$

since 10's and face cards are disjoint, and so any of the above pairs would require at least 3 cards. For the other values, we have

$$\Pr(X = 0, Y = 0) = \frac{\binom{36}{2}}{\binom{52}{2}} = \frac{36 \times 35}{52 \times 51} \approx .475$$

since to get 0 and 0, we need to select our pair of cards from among the 36 non-face, non-10 cards. For a similar reason,

$$\Pr(X = 2, Y = 0) = \frac{\binom{12}{2}}{\binom{52}{2}} \approx .049, \qquad \Pr(X = 0, Y = 2) = \frac{\binom{4}{2}}{\binom{52}{2}} \approx .004.$$

We also have

$$\Pr(X = 1, Y = 0) = \frac{12 \times 36}{\binom{52}{2}} \approx .325, \qquad \Pr(X = 0, Y = 1) = \frac{4 \times 36}{\binom{52}{2}} \approx .108$$

and

$$\Pr(X = 1, Y = 1) = \frac{12 \times 4}{\binom{52}{2}} \approx .036,$$

since (for example) to get $X = 1$, $Y = 0$ we have to choose one of 12 face cards, and one of 36 non-face, non-10 cards.

X and Y are clearly not independent: for example, $\Pr(X = 2, Y = 2) = 0$ but $\Pr(X = 2)$ and $\Pr(Y = 2)$ are non-zero, so $\Pr(X = 2, Y = 2) \neq \Pr(X = 2)\Pr(Y = 2)$.

4. **GW** 8.10 (part a) only)

   Solution: We have to specify 16 possible values; X and Y can both be 1, 2, 3, or 4. Since X is the minimum and Y the maximum, whenever $x > y$ among these numbers we have $p_{X,Y}(x, y) = 0$. The only outcome that leads to $X = Y = 1$ is $(1, 1)$, so this probability is 1/16, and the same for $X = Y = 2, 3$ or 4. The two outcomes that lead to $X = 1, Y = 2$ are $(1, 2)$ and $(2, 1)$, so this probability is $2/16 = 1/8$, and the same for $X = x, Y = y$ for all other $x < y$. So we get:

   $$p_{X,Y}(x, y) = 1/16 \text{ if } x = y = 1, 2, 3 \text{ or } 4,$$

   $$p_{X,Y}(x, y) = 1/8 \text{ if } (x, y) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4) \text{ or } (3, 4),$$

   and

   $$p_{X,Y}(x, y) = 0 \text{ for all other values of } x, y.$$

5. **GW** 9.7

   Solution: The probability that he gets it right on the first try is 1/10. The probability that he gets it right on the second try is $(9/10)(1/9) = 1/10$ (he gets it wrong on the first try with probability 9/10 and then, given that he got it wrong on the first try, he gets it right on the second with probability 1/9, since there are now 9 channels left to try and one right one). By the same reasoning, the probability that he gets it right on the third try is $(9/10)(8/9)(1/8) = 1/10$. This cancelation of fractions continues: the probability that he gets it right on the kth try is $1/10$, for all $k = 1, 2, \ldots, 10$. So the expected number of attempts is

$$\sum_{k=1}^{10} \frac{k}{10} = 5.5.$$

6. **GW** 9.8

   Solution: The probability that I get k cashews, for each $k = 0, 1, 2, 3$, is

   $$\Pr(C = k) = \frac{\binom{30}{k}\binom{45}{3-k}}{\binom{75}{3}}.$$

   The denominator here counts the total number of ways of selecting 3 nuts from 75. The numerator counts the number of ways of selecting k cashews from among the 30 cashews, and $3 - k$ non-cashews from among the non-cashews. So

   $$E(C) = \sum_{k=0}^{3} \frac{k\binom{30}{k}\binom{45}{3-k}}{\binom{75}{3}} = 1.2.$$

   We will revisit this example in class, as an instance of the *hypergeometric random variable*.

7. **GW** 9.10

   They win all three games, and so gain 300 fans, with probability $(.7)^3 = .343$.

   They win exactly two games, and so gain 190 fans, with probability $3(.7)^2(.3) = .441$.

   They win exactly one game, and so gain 80 fans, with probability $3(.7)(.3)^2 = .189$.

   They win exactly zero games, and so gain -30 fans, with probability $(.3)^3 = .027$.

   The expected number of fans gained is

   $$300 \times .343 + 190 \times .441 + 80 \times .189 - 30 \times .027 = 201.$$

8. **GW** 9.18

   Solution: We saw in class that if you a repeated a trial independently until success has occurred, each time with success probability p, then the expected number of trials until success is $1/p$. Here $p = .1$, so expectation is 10.

9. **GW** 10.6

   Solution: Let X_i be the random variable that is 1 if the ith rabbit goes down the ith hole, and zero otherwise. We have $\Pr(X_i = 1) = 1/6$, so $E(X_i) = 1/6$. The total number of rabbits to down hole 1 is $X = x_1 + \ldots + X_{30}$, so $E(X) = 30 \times (1/6) = 5$, by linearity of expectation.

10. **GW** 10.15 (part a) only)

    Solution: $\Pr(A_j) = (5/6)^{j-1}$ (to need j or more rolls, we must have failed to get a 4 on each of the first $j-1$ rolls). So $E(X_j) = (5/6)^{j-1}$. Since $X = X_1 + X_2 + \ldots$, by linearity of expectation we have

$$E(X) = 1 + (5/6) + (5/6)^2 + \ldots = \frac{1}{1-(5/6)} = 6.$$

11. **GW** 10.23

    Solution: Let X_i be the random variable that is 1 if the ith students is in both the same lab as Bob, and the same lecture. We have $\Pr(X_i = 1) = (19/119)(59/119)$. To see this, note that once Bob has been put in lab 1 (say) there are 19 spots remaining in that lab, and 119 people to fill them; so there is a 19/119 probability of a particular person joining him in that lab; and once Bob has been put in lecture 1 (say) there are 59 spots remaining in that lab, and 119 people to fill them; so there is a 59/119 probability of a particular person joining him in that lecture; we multiply the two probabilities since the labs and lectures are built independently. So $E(X_i) = (19/119)(59/119)$. The total number of students who are in both Bob's lab and lecture is $X = x_1 + \ldots + X_{119}$, so

$$E(X) = 119 \times \frac{19 \times 59}{119 \times 119} \approx 9.42.$$

12. I'm on *Who wants to be a Millionaire*, looking at the $250,000 question. My pot of money is currently $100,000. If I get the $250,000 question right, my pot goes to $250,000, and I get the chance to look at the $500,000 question; if I get that right, my pot goes to $500,000, and I get the chance to look at the $1,000,000 question; if I get that right, my pot goes to $1,000,000, and the game stops. If ever I get a question wrong, my pot goes down to $25,000 and the game stops. At any point, I can also choose to not answer a question; in that case, I keep the pot I have at that point, and the game stops.

    I estimate that I will have a 5% chance of knowing the answer to the $500,000 question, and a 1% chance of knowing the answer to the $1,000,000 question; in either of these cases, I will not guess the answer unless I know it, and I will answer it if I know it.

    Looking at the $250,000 question, I have a hunch about the right answer, and estimate that my hunch is correct with probability p.

    (a) What is the expected value of my pot, as a function of p, if I follow my hunch and give what I think is the right answer? (I'm asking here for expected value of the pot, when the whole game is over: if I answer the $250,000 question correctly, there is some chance that I might also answer the $500,000 question correctly, and even the $1,000,000 question.)

    Solution: I win:

- \$1,000,000 with probability $p \times .05 \times .01 = .0005p$
- \$500,000 with probability $p \times .05 \times .99 = .0495p$
- \$250,000 with probability $p \times .95 = .95p$
- \$25,000 with probability $1 - p$

So my expected pot at the end is $1000000 \times .0005p + 500000 \times .0495p + 25000 \times .95p + 25000 \times (1-p) = 237750p + 25000$.

(b) How sure should I be of my hunch (what's the value of p), to have an expected final pot of more than \$100,000 if I go for it on the question I'm looking at?

Solution: Solving $237750p + 25000 = 100000$ gives $p = .315...$. So I want to be 32% sure (or more) to make the risk of guessing work out (on average).

13. If X is a random variable that takes on values 0, 1, 2 etc. with probabilities p_0, p_1, p_2, etc. (and takes on no other values), show that

$$\mathbf{E}(X) = \Pr(X \geq 1) + \Pr(X \geq 2) + \Pr(X \geq 3) + \ldots = \sum_{k=1}^{\infty} \Pr(X \geq k).$$

Solution:

$$\begin{aligned} E(X) &= 0.\Pr(X=0) + 1.\Pr(X=1) + 2.\Pr(X=2) + \ldots \\ &= p_1 + 2p_2 + 3p_3 + \ldots \\ &= p_1 + \\ &\quad\; p_2 + p_2 + \\ &\quad\; p_3 + p_3 + p_3 + \\ &\quad\; \ldots . \end{aligned}$$

Summing down the first *column* of this triangular array gives $\Pr(X \geq 1)$; summing down the second column gives $\Pr(X \geq 2)$; the third gives $\Pr(X \geq 3)$, and so on. So

$$E(X) = \Pr(X \geq 1) + \Pr(X \geq 2) + \Pr(X \geq 3) + \ldots = \sum_{k=1}^{\infty} \Pr(X \geq k),$$

as claimed.

14. Three busses come into a bus depot at the same time. Bus A has 40 passengers, bus B has 45 and bus C has 60. Jack and Jill want to do an experiment to estimate the average number of passengers on an incoming bus. Jack picks a bus at random, each bus equally likely, and looks at the number of people in the chosen bus; let X be that number. Jill picks a random passenger from among all three busses, all passengers equally likely, and looks at the number of people who were on that passenger's bus; let Y be that number.

(a) Compute $\mathbf{E}(X)$ and $\mathbf{E}(Y)$.

Solution: $E(X) = 40.(1/3) + 45.(1/3) + 60.(1/3) = 48.333....$ $E(Y) = 40.(40/145) + 45.(45/145) + 60.(60/145) = 49.82....$

(b) Give a brief explanation as to why Jill's answer should be larger than Jack's, no matter what the number of busses or the number of passengers on each bus.

Solution: Jack is treating each of the three busses in the same way, and is just compute the average bus size. Jill is using a scheme that makes her more likely to focus on larger busses, since these have more passengers (and so are more likely to be chosen by Jill). So Jill, with a skew towards larger numbers, will get a larger answer.

15. Alice and Bob play the game of Two-finger Morra. Here's how they play: each closes their eyes and holds up either one or two fingers, and at the same time calls out their guess as to how many fingers their opponent is holding up. They then both open their eyes. If exactly one of the two have guessed correctly, (s)he wins a number of dollars equal to the sum of the number of fingers the two players are holding up. If both guess correctly, or neither does, no money changes hands. If both players play independently, and are equally likely to choose each of their four options (one finger up and guess one, one finger up and guess two, two fingers up and guess one, two fingers up and guess two), compute $\mathbf{E}(X)$, where X is how much Alice wins in a playing of the game.

    Solution: The possible values for X are 4 (if both hold up two fingers, Alice guesses 2 and Bob guesses 1: one outcome out of 16, so probability 1/16); 3 (if Alice holds up two fingers, Bob one, Alice guesses 1 and Bob 1 OR if Alice holds up one finger, Bob two, Alice guesses 2 and Bob 1: two outcomes out of 16, so probability 2/16); 2 (if both hold up one finger, Alice guesses 1 and Bob guesses 2: one outcome out of 16, so probability 1/16); and 0 (everything else, probability $1 - (1/16) - (2/16) - (1/16) = 12/16$). So

$$E(X) = 4.(1/16) + 3.(2/16) + 2.(1/16) = 12/16 = \$.75.$$

16. Here's a strategy for roulette: bet \$1 on red (which has probability 18/38 of coming up). If red comes up, take your \$2 reward and quit. If red does not come up, make two more \$1 bets on red, then quit. Let X be the total net winnings at the end of this process. X has the following possible values: $+1$ (if red comes up on first roll, or if red does not come up on the first roll, but does on each of the other two), -1 (if red does not come up on the first roll, and comes up exactly once during the other two), and -3 (if red does not come up on the first roll, and does not on each of the other two either).

(a) Compute $\Pr(X > 0)$

Solution: We begin by calculating the mass function of X, based on the description:

$$p_X(x) = \begin{cases} (18/38) + (20/38)(18/38)(18/38) \approx .5918 & \text{if } x = 1 \\ (20/38)2(20/38)(18/38) \approx .2624 & \text{if } x = -1 \\ (20/38)^3 \approx .1458 & \text{if } x = -3 \\ 0 & \text{otherwise} \end{cases}$$

So $\Pr(X > 0) = .5918$ (greater than 1/2! Sounds good...)

(b) Compute $\mathbf{E}(X)$

Solution:

$$E(X) \approx .5918 - .2624 - 3(.1458) \approx -.108$$

(c) Does this seem like a reasonable strategy?

Solution: No; although you have a more than 50% chance of gaining money, on average (in the long run) you will lose money, because the expectation is negative.

17. Roger Federer and Andy Murray play a best-of-five set tennis match (stopping when one of the two has reached 3 sets), with each of the two equally likely to win each set, independently of the others. Let X be the number of sets they play.

(a) Compute $\mathbf{E}(X)$

Solution: First we compute the mass function. There are two ways that the match can go three sets: FFF or MMM, each with probability 1/8, so $\Pr(X = 3) = 1/4$. There are six ways that the match can go four sets: MFFF, FMFF, FFMF, FMMM, MFMM or MMFM, each with probability 1/16, so $\Pr(X = 4) = 3/8$. We could do the same for $\Pr(X = 5)$, or simply say $\Pr(X = 5) = 1 - \Pr(X = 3) - \Pr(X = 4) = 3/8$. So

$$E(X) = 3(1/4) + 4(3/8) + 5(3/8) = 33/8 = 4.125.$$

(b) If they play ten best-of-five-set matches, each independent of the others, what's the expected total number of sets played? (Justify your answer)

Solution: Let X_i be the number of sets in the ith match; the total number of sets played is $X_1 + X_2 + \ldots X_{10}$, and by linearity of expectation, the expected number of total sets is $E(X_1) + E(X_2) + \ldots E(X_{10}) = 10(33/8) = 41.25$.

18. Each week I have office hours. There is a probability .05 that my cell-phone will ring during my office hours, while I'm answering a student's question. The semester has 15 weeks. Let X be the number of weeks in which my phone rings during office hours while I'm answering a questions. (Assume weeks are independent of each other).

(a) What are the possible values for X?

Solution: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$.

(b) What is the probability that $X = 3$?

Solution: $\Pr(X = 3) = \binom{15}{3}(.05)^3(.95)^12 = .0307$.

(c) What is the expected value of X? (**Hint**: rather than using the direct definition of expectation, break X down into the sum of simpler random variables)

Solution: Write $X = X_1 + X_2 + \ldots + X_{15}$ where X_i is 1 if my phone goes off during the ith week, and 0 otherwise; $E(X_i) = .05$. By linearity, $E(X) = \sum_{i=1}^{15} E(X_i) = 15 * .05 = .75$.