Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 3 — Solutions

1. You are in a room with five doors, and your host tells you that behind two randomly chosen doors he has placed a prize (all choices of two doors equally likely). You open the five doors, one after another, from left to right. Let X be the random variable that measures the number of doors you open *after* seeing the first prize you see, but *before* seeing the second prize.

   (a) What are the possible values that X can take on? **Solution**: 0 (the two prize doors are side-by-side), 1, 2 or 3 (the largest possible value, achieved when the prizes are behind the first and fifth door).

   (b) Compute the mass function of X, and plot it. **Solution**: There are 10 possible, equally likely sample points in the experiment where the status of each door is recorded in the order seen (with a P for prize, and an N for no prize). I list the 10 outcomes here, with (in parenthesis after each one) the value of the random variable X:

   PPNNN (0), PNPNN (1), PNNPN (2), PNNNP (3), NPPNN (0), NPNPN (1), NPNNP (2), NNPPN (0), NNPNP (1), NNNPP (0)

   We can now easily read off the mass function $p_X(x)$:

$$p_X(x) = \begin{cases} .4 & \text{if } x = 0, \\ .3 & \text{if } x = 1, \\ .2 & \text{if } x = 2, \\ .1 & \text{if } x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

   See figure 1 of the figures page for the plot of p_X.

   (c) Plot the Cumulative Distribution Function of X. **Solution**: See figure 2 of the figures page for the plot of F_X.

   (d) What is the probability that X is at least 2? **Solution**: $\Pr(X \geq 2) = \Pr(X = 2 \text{ or } X = 3) = .2 + .1 = .3$.

2. **GW** 6.2

   Solution: If we assume an ideal weighing scale that can measure arbitrary masses, then X is continuous with all non-negative reals as possible values. If we assume that the scale can only measure to, for example, the nearest ounce, then X is discrete, with all fractions of the form $n/16$, n a whole non-negative number, as possible values.

Y is discrete, with possible values $0, 1, 2, \ldots$. (0? I'm including this because, I guess, an empty basket can be called a "basket of onions" (or a "basket of kittens", or ...)).

3. **GW** 6.5

   Solution: X, Y and Z are all discrete, with possible values $0, 1, 2, \ldots$. (Technically we could put a cap on the max value of X, Y, Z, by considering memory capacity & compression rates; for the purposes of modeling the question in the homework, it seems easiest to let arbitrarily large values be taken on.)

4. **GW** 6.8

   Solution: If we assume an ideal timer that can measure times arbitrarily finely, then X and Y are continuous with all non-negative reals as possible values (as with the last question, there are practical limits to the values X and Y can take, but it's easier to ignore this).

   If we assume that the timer can only measure to, for example, the nearest minute, then X and Y are discrete, with all whole non-negative numbers as possible values.

   Z is discrete, with possible values $0, 1, 2, \ldots$. (0? I hope not).

5. **GW** 6.10

   Solution: If we assume an ideal timer that can measure times arbitrarily finely, then X and Y are continuous with all non-negative reals as possible values (as with previous questions, there are practical limits to the values X and Y can take, but it's easier to ignore this).

   If we assume that the timer can only measure to, for example, the nearest second, and we choose seconds as our units, then X and Y are discrete, with all whole non-negative numbers as possible values.

6. **GW** 6.11

   Solution: X is discrete, with $1, 2, 3, \ldots$ as possible values (0 is not a possibility!). Practically, X cannot take a value large than, say, 7,000,000,000.

   If we assume an ideal timer that can measure times arbitrarily finely, then Y is continuous with all *positive* reals as possible values (a friend request cannot be accepted in 0 seconds!) (as with previous questions, there are practical limits to the values Y can take, at both ends, but again easier to ignore this). You might choose to add "∞" to the possible values Y can take - maybe you never have a request accepted ...

7. **GW** 6.17

   Solution: There are 52 times 51 ways to pick two cards, if I care about the order of the cards. Since I don't care about the order ($(7\heartsuit, 6\clubsuit)$ is the same to me as $(6\clubsuit, 7\heartsuit)$), I have to halve this number to get 1326 equally-likely outcomes to my experiment.

a) there are 40 times 39 *ordered* pairs with no face-cards, so removing order, there are 780 possibilities; $\Pr(X = 0) = 780/1326 \approx .59$.

b) there are 12 times 40 plus 40 times 12 ordered pairs with one face-card (the first multiplication counts those in which the face card is the first card, the second counts those in which the face card is the second), so removing order, there are 480 possibilities; $\Pr(X = 1) = 480/1326 \approx .36$.

c) there are 12 times 11 ordered pairs with two face-cards, so removing order, there are 66 possibilities; $\Pr(X = 2) = 66/1326 \approx .05$.

Reality check: the three probabilities add to 1!

8. **GW** 7.1

   Solution: a) If we record that he eats the snack the same day as purchase with an "e" and that he does not eat the snack the same day as purchase with a "d", then the sample space is all lists of length 4, with all entries in the list coming from the alphabet $\{e, d\}$:

$$S = \{(x_1, x_2, x_3, x_4) | \text{each } x_i \text{ is either "}e\text{" or "}d\text{"}\}.$$

   b) There are 16 possible outcomes, occurring with variable probabilities: the one outcome with 4 e's has probability $.7^4$; each of the four outcomes with one d has probability $.3 \times .7^3$; each of the six outcomes with two d's has probability $.3^2 \times .7^2$; each of the four outcomes with three d's has probability $.3^3 \times .7$; the one outcome with 4 d's has probability $.3^4$. This leads to the following mass function:

$$p_X(x) = \begin{cases} .0081 & \text{if } x = 0, \\ .0756 & \text{if } x = 1, \\ .2646 & \text{if } x = 2, \\ .4116 & \text{if } x = 3, \\ .2401 & \text{if } x = 4, \\ 0 & \text{otherwise.} \end{cases}$$

   c) The CDF can be calculated directly from the mass function:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ .0081 & \text{if } 0 \le x < 1, \\ .0081 + .0756 = .0837 & \text{if } 1 \le x < 2, \\ .0837 + .2646 = .3483 & \text{if } 2 \le x < 3, \\ .3483 + .4116 = .7599 & \text{if } 3 \le x \le 4, \\ .7599 + .2401 = 1 & \text{if } x \ge 4. \end{cases}$$

9. **GW** 7.4

   Solution: a)

$$p_X(x) = \begin{cases} .13 & \text{if } x = 1, \\ .28 & \text{if } x = 2, \\ .18 & \text{if } x = 3, \\ .41 & \text{if } x = 4, \\ 0 & \text{otherwise.} \end{cases}$$

b) See figure 3 of the figures page

c)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ .13 & \text{if } 1 \le x < 2, \\ .41 & \text{if } 2 \le x < 3, \\ .59 & \text{if } 3 \le x < 4, \\ 1 & \text{if } x \ge 4. \end{cases}$$

d) See figure 3 of the figures page

10. **GW** 7.5

Solution: a) X can take on values $0, 1, 2, 3, 4, \ldots$, and can also take on the value ∞ (if no-one ever completes the challenge). The mass function is

$$p_X(x) = \begin{cases} (.9)^{x-1}(.1) & \text{if } x = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by the geometric sum formula, $.1 + (.9)(.1) + (.9)^2(.1) + (.9)^3(.1) + \ldots = 1$, so this accounts for all the available probability. So we should assign probability 0 to $x = \infty$, and as such might as well not include it in the list of possible values.

b) See figure 4 of the figures page

c) This is awkward to write down. One way is to say:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ \sum_{k=1}^{[x]} (.9)^{x-1}(.1) & \text{if } x \ge 1. \end{cases}$$

Here I'm using $[x]$ to denote the largest integer that is no bigger than x (the "integer part" of x). The sum can be simplified, but there isn't much value in doing the simplification. A clearer picture is provided by the plot, shown below.

d) See figure 5 of the figures page

11. **GW** 7.8 (a, b only)

Solution: a) The sum of the probabilities given is $8c$, so $c = 1/8$.

b) See figure 6 of the figures page

12. **GW** 7.10

Solution: We determine the mass by seeing where the CDF jumps, and get

$$p_X(x) = \begin{cases} .3 & \text{if } x = 2, \\ .5 & \text{if } x = 4, \\ .15 & \text{if } x = 6, \\ .05 & \text{if } x = 8, \\ 0 & \text{otherwise.} \end{cases}$$

13. **GW** 7.16 (a, b, c only)

    Solution: a) $-3, -1, 1, 3.$

    b)

$$p_X(x) = \begin{cases} 1/8 & \text{if } x = -3, \\ 3/8 & \text{if } x = -1, \\ 3/8 & \text{if } x = 1, \\ 1.8 & \text{if } x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

    c) See figure 7 of the figures page

14. **GW** 7.19 (a, b, c only)

    Solution: a) $X = 0, 1, 2, 3, 4, 5, 6$ are possible values

    b) X is binomial, with $n = 6$, $p = 1/13$ (or $4/52$), so

$$p_X(x) = \begin{cases} \binom{6}{x}(1/13)^x(12/13)^{6-x} & \text{if } x = 0, 1, 2, 3, 4, 5, 6, \\ 0 & \text{otherwise.} \end{cases}$$

    Specific values are as follows (rounded): $.6187, .3093, .0644, .0072, .0004, .00001, 0.$

    b) See figure 8 of the figures page

15. For each of the random variables that appear in 7.1, 7.4, 7.5 and 7.19, say which are binomial and which are not. In those cases where the random variable is binomial, say what n and p are; in those cases where the random variable is not binomial, explain why not.

    Solution:

    - 7.1: Binomial, with $n = 4, p = .7$
    - 7.4: Not binomial, because we are not counting number of success
    - 7.5: Not binomial, because (for example) the number of trials until the experiment stops is not fixed in advance
    - 7.19: Binomial, with $n = 6, p = 1/13$

16. **GW** 13.3 (a, b, c, e only)

    Solution: a) Success: student shows that they have done the homework by correctly solving the problem, $p = 6/300$

    b) Success: student shows that they have not done the homework by not correctly solving the problem, $p = 294/300$

    c) This is Bernoulli, because all we are interested in is success or failure; parameter is $6/300 = .02$

    e) $6/300$ or $.02$

17. **GW** 13.12

    Solution: $1/X$ is not well defined, because one possible value for X is 0, and $1/0$ is undefined.

18. **GW** 14.1 (a, b, c, d, e only)

    Solution:

    a) Success: as a candy passes by, noticing that it is purple, $p = .2$

    b) Failure: as a candy passes by, noticing that it is not purple, $p = .8$

    c) X is the number of purple candies that pass by, in a batch of 25

    d) This is binomial, because there are a fixed number of repetitions (25, so $n = 25$) of the same, independent trial. Here $p = .2$

    e) $\Pr(X = 5) = \binom{25}{5}(.2)^5(.8)^{20} = .196$

19. **GW** 14.4

    Solution: I think not; I presume that Gracie will keep asking until she hears from someone who has seen here dog. We (and she) don't know how many attempts this will take, so the number isn't fixed in advance.

20. **GW** 14.7

    Solution: Suppose we spin n times, and let X be the number of 1's rolled. $X \sim \text{Binomial}(n, .3)$, so $\Pr(X \geq 1) = \sum_{i=1}^{n} \binom{n}{i}(.3)^i(.7)^{n-i}$. We want to choose n so that it the least value for which this probability is greater than .95.

    A binomial calculator shows that $n = 8$ yields a probability of .9424, while $n = 9$ yields a probability of .9596; so we should spin 9 times.

    Alternately, we could say $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - .7^n$. At $n = 8$, this is .9424, too small, while at $n = 9$ it is .9596; so again we should spin 9 times.

21. **GW** 14.21

    Solution: a) Ignore this question! We haven't seen expected value yet.

    b) X is *not* binomial. Although there are a fixed number (7) of trials, they are *not* independent (and don't necessarily have the same success probabilities). For example, if the first card drawn is a ♡ (probability 1/4), then it is less likely that the second is (probability 12/51); if the first card drawn is *not* a ♡ (probability 3/4), then it is more likely that the second is (probability 13/51).

22. **GW** 14.26

    Solution: Let X be the number of males in a batch of 100,000; X is Binomial with $n = 100,000$ and $p = .02$, so the probability of 2100 or more males is

    $$\Pr(X \geq 2100) = \sum_{i=2100}^{100000} \binom{100000}{i}(.02)^i(.98)^{100000-i}.$$

    This doesn't look like much fun to calculate.