Chapter 7

Problems

1. Let $X = 1$ if the coin toss lands heads, and let it equal 0 otherwise. Also, let Y denote the value that shows up on the die. Then, with $p(i, j) = P\{X = i, Y = j\}$

$$E[\text{return}] = \sum_{j=1}^{6} 2jp(1, j) + \sum_{j=1}^{6} \frac{j}{2} p(0, j)$$

$$= \frac{1}{12}(42 + 10.5) = 52.5/12$$

3. If the first win is on trial N, then the winnings is $W = 1 - (N - 1) = 2 - N$. Thus,

(a) $P(W > 0) = P(N = 1) = 1/2$
(b) $P(W < 0) = P(N > 2) = 1/4$
(c) $E[W] = 2 - E[N] = 0$

5. The joint density of the point (X, Y) at which the accident occurs is

$$f(x, y) = \frac{1}{9}, -3/2 < x, y < 3/2$$
$$= f(x) f(y)$$

where

$$f(a) = 1/3, -3/2 < a < 3/2.$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-3/2, 3/2)$ Therefore,

$$E[|X| + |Y|] = 2\int_{-3/2}^{3/2} \frac{1}{3} x\,dx = \frac{4}{3}\int_{0}^{3/2} x dx = 3/2.$$

7) a) $P(\text{particular item chosen by } A) = \frac{C\binom{9}{2}}{C\binom{10}{3}} = .3$

$P(\text{particular item chosen by both } A \text{ and } B) = .3^2 = .09$ (independence)

X = # items chosen by both

$X = X_1 + \dots + X_{10}$ where $X_i = \begin{cases} 1 & \text{if } A, B \text{ both choose it} \\ 0 & \text{o wise} \end{cases}$

$E(X) = \sum_{i=1}^{10} E(X_i) = 10 \times .09 = .9$

b) Using same reasoning as before:

P(particular item <u>not</u> chosen by either) $= .7^2 = .49$

So E(# items chosen by neither) $= 10 \times .49 = 4.9$

c) P(particular item chosen by one but not other)

$= 2 \times .3 \times .7 = .42$

So E(# items chosen by one but not other)

$= 4.2$

11. Let X_i equal 1 if a changeover occurs on the i^{th} flip and 0 otherwise. Then

$$E[X_i] = P\{i-1 \text{ is } H, i \text{ is } T\} + P\{i-1 \text{ is } T, i \text{ is } H\}$$
$$= 2(1-p)p, \quad i \geq 2.$$

$$E[\text{number of changeovers}] = E\left[\sum X_i\right] = \sum_{i=1}^{n} E[X_i] = 2(n-1)(1-p)$$

16. $$E[X] = \int_{y>x} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

17. Let I_i equal 1 if guess i is correct and 0 otherwise.

(a) Since any guess will be correct with probability $1/n$ it follows that

$$E[N] = \sum_{i=1}^{n} E[I_i] = n/n = 1$$

(b) The best strategy in this case is to always guess a card which has not yet appeared. For this strategy, the i^{th} guess will be correct with probability $1/(n-i+1)$ and so

$$E[N] = \sum_{i=1}^{n} 1/(n-i+1)$$

(c) Suppose you will guess in the order 1, 2, …, n. That is, you will continually guess card 1 until it appears, and then card 2 until it appears, and so on. Let J_i denote the indicator variable for the event that you will eventually be correct when guessing card i; and note that this event will occur if among cards 1 thru i, card 1 is first, card 2 is second, …, and card i is the last among these i cards. Since all $i!$ orderings among these cards are equally likely it follows that

$$E[J_i] = 1/i! \text{ and thus } E[N] = E\left[\sum_{i=1}^{n} J_i\right] = \sum_{i=1}^{n} 1/i!$$

30. $$E[(X-Y)]^2 = \text{Var}(X-Y) = \text{Var}(X) + \text{Var}(-Y) = 2\sigma^2$$

$j=1$

33. (a) $E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4 = \text{Var}(X) + E^2[X] + 4E[X] + 4 = 14$

(b) $\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 45$

36. Let $X_i = \begin{cases} 1 & \text{roll } i \text{ lands on } 1 \\ 0 & \text{otherwise} \end{cases}$, $Y_i = \begin{cases} 1 & \text{roll } i \text{ lands on } 2 \\ 0 & \text{otherwise} \end{cases}$

$$\text{Cov}(X_i, Y_j) = E[X_i Y_j] - E[X_i]E[Y_j]$$
$$= \begin{cases} -\frac{1}{36} & i = j \text{ (since } X_i Y_j = 0 \text{ when } i = j \\ \frac{1}{36} - \frac{1}{36} = 0 & i \neq j \end{cases}$$

$$\text{Cov} \sum_i X_i, \sum_j Y_j = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$
$$= -\frac{n}{}$$

38. $E[XY] = \int_0^\infty \int_0^x y 2e^{-2x} dydx$

$= \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy = \frac{\Gamma(3)}{8} = \frac{1}{4}$

$E[X] = \int_0^\infty x f_x(x) dx,\ f_x(x) = \int_0^x \frac{2e^{-2x}}{x} dy\ = 2e^{-2x}$

$= \frac{1}{2}$

$E[Y] = \int_0^\infty y f_Y(y) dy,\ f_Y(y) = \int_0^\infty \frac{2e^{-2x}}{x} dx$

$= \int_0^\infty \int_y^\infty y \frac{2e^{-2x}}{x} dxdy$

$= \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x} dydx$

$= \int_0^\infty x e^{-2x} dx = \frac{1}{4}\int y e^{-2} dy = \frac{\Gamma(2)}{4} = \frac{1}{4}$

$\text{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

39. $\text{Cov}(Y_n, Y_n) = \text{Var}(Y_n) = 3\sigma^2$

$\text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2} + X_{n+3})$

$= \text{Cov}(X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2}) = \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2$

$\text{Cov}(Y_n, Y_{n+2}) = \text{Cov}(X_{n+2}, X_{n+2}) = \sigma^2$

$\text{Cov}(Y_n, Y_{n+j}) = 0$ when $j \geq 3$

75. X is Poisson with mean $\lambda = 2$ and Y is Binomial with parameters 10, 3/4. Hence

(a) $P\{X + Y = 2\} = P\{X = 0\}P\{Y = 2\} + P\{X = 1\}P\{Y = 1\} + P\{X = 2\}P\{Y = 0\}$

$= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + 2e^{-2}(1/4)^{10}$

(b) $P\{XY = 0\} = P\{X = 0\} + P\{Y = 0\} - P\{X = Y = 0\}$

$= e^{-2} + (1/4)^{10} - e^{-2}(1/4)^{10}$

(c) $E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \frac{3}{4} = 15$

Theoretical Exercises

1. Let $\mu = E[X]$. Then for any a

$$\begin{aligned}E[(X-a)^2 &= E[(X-\mu+\mu-a)^2] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2E[(x-\mu)(\mu-a)] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \\ &= E[(X-\mu)^2 + (\mu-a)^2\end{aligned}$$

2. $$\begin{aligned}E[\,|X-a|\, &= \int_{x<a}(a-x)f(x)dx + \int_{x>a}(x-a)f(x)dx \\ &= aF(a) - \int_{x<a}xf(x)dx + \int_{x>a}xf(x)dx - a[1-F(a)]\end{aligned}$$

Differentiating the above yields
derivative = $2af(a) + 2F(a) - af(a) - af(a) - 1$
Setting equal to 0 yields that $2F(a) = 1$ which establishes the result.

9. Let

$$I_j = \begin{cases}1 & \text{if a run of size } k \text{ begins at the } j^{\text{th}} \text{ flip} \\ 0 & \text{otherwise}\end{cases}$$

Then

$$\text{Number of runs of size } k = \sum_{j=1}^{n-k+1} I_j$$

$$\begin{aligned}E[\text{Number of runs of size } k &= E\left[\sum_{j=1}^{n-k+1} I_j\right] \\ &= P(I_1 = 1) + \sum_{j=2}^{n-k} P(I_j = 1) + P(I_{n-k+1} = 1) \\ &= p^k(1-p) + (n-k-1)p^k(1-p)^2 + p^k(1-p)\end{aligned}$$

19. $$\begin{aligned}\text{Cov}(X+Y, X-Y) &= \text{Cov}(X, X) + \text{Cov}(X, -Y) + \text{Cov}(Y, X) + \text{Cov}(Y, -Y) \\ &= \text{Var}(X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Var}(Y) \\ &= \text{Var}(X) - \text{Var}(Y) = 0.\end{aligned}$$

48. $\phi_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = e^{tb}E[e^{taX}] = e^{tb}\phi_X(ta)$

49. Let $Y = \log(X)$. Since Y is normal with mean μ and variance σ^2 it follows that its moment generating function is

$$M(t) = E[e^{tY}] = e^{\mu t + \sigma^2 t^2/2}$$

Hence, since $X = e^Y$, we have that

$$E[X] = M(1) = e^{\mu + \sigma^2/2}$$

and

$$E[X^2] = M(2) = e^{2\mu + 2\sigma^2}$$

Therefore,

$$\text{Var}(X) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$