CHAPTER 4

41. $\sum_{i=7}^{10} \binom{10}{i} (1/2)^{10}$

42. $\binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + p^5 \geq \binom{3}{2} p^2(1-p) + p^3$

$\Leftrightarrow 6p^3 - 15p^2 + 12p - 3 \geq 0$

$\Leftrightarrow 6(p - 1/2)(p - 1)^2 \geq 0$

$\Leftrightarrow p \geq 1/2$

44. $\alpha \sum_{i=k}^{n} \binom{n}{i} p_1^i (1-p_1)^{n-i} + (1-\alpha) \sum_{i=k}^{n} \binom{n}{i} p_2^i (1-p_2)^{n-i}$

48. The probability that a package will be returned is $p = 1 - (.99)^{10} - 10(.99)^9(.01)$. Hence, if someone buys 3 packages then the probability they will return exactly 1 is $3p(1-p)^2$.

49. (a) $\frac{1}{2}\binom{10}{7}.4^7.6^3 + \frac{1}{2}\binom{10}{7}.7^7.3^3$

(b) $\dfrac{\frac{1}{2}\binom{9}{6}.4^7.6^3 + \frac{1}{2}.7^7.3^3}{.55}$

51. (a) $e^{-.2}$ (b) $1 - e^{-.2} - .2e^{-.2} = 1 - 1.2e^{-.2}$

Since each letter has a small probability of being a typo, the number of errors should approximately have a Poisson distribution.

53. (a) The probability that an arbitrary couple were both born on April 30 is, assuming independence and an equal chance of having being born on any given date, $(1/365)^2$. Hence, the number of such couples is approximately Poisson with mean $80{,}000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6}$.

(b) The probability that an arbitrary couple were born on the same day of the year is 1/365. Hence, the number of such couples is approximately Poisson with mean $80{,}000/365 \approx 219.18$. Hence, the probability of at least one such pair is $1 - e^{-219.18} \approx 1$.

54. (a) $e^{-2.2}$ (b) $1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$

55. $\frac{1}{2}e^{-3} + \frac{1}{2}e^{-4.2}$

58: a) Binomial : .1488 Poisson $\lambda = .8$: .14378

b) .3151 $\lambda = 9.5$: .13

c) .3487 $\lambda = 1$: .3678

d) .6661 $\lambda = 1.8$: .0723

60. $P\{\text{beneficial} \mid 2\} = \dfrac{P\{2 \mid \text{beneficial}\}3/4}{P\{2 \mid \text{beneficial}\}3/4 + P\{2 \mid \text{not beneficial}\}1/4}$

$$= \frac{e^{-3}\frac{3^2}{2}\frac{3}{4}}{e^{-3}\frac{3^2}{2}\frac{3}{4} + e^{-5}\frac{5^2}{2}\frac{1}{4}}$$

61. $1 - e^{-1.4} - 1.4e^{-1.4}$

63. (a) $e^{-2.5}$

(b) $1 - e^{-2.5} - 2.5e^{-2.5} - \dfrac{(2.5)^2}{2}e^{-2.5} - \dfrac{(2.5)^3}{3!}e^{-2.5}$

65. (a) $1 - e^{-1/2}$

(b) $P\{X \geq 2 \mid X \geq 1\} = \dfrac{1 - e^{-1/2} - \frac{1}{2}e^{-1/2}}{1 - e^{-1/2}}$

(c) $1 - e^{-1/2}$

(d) $1 - \exp\{-500 - i)/1000\}$

71. (a) $\left(\dfrac{26}{38}\right)^5$

(b) $\left(\dfrac{26}{38}\right)^3 \dfrac{12}{38}$

72. $P\{\text{wins in } i \text{ games}\} = \dbinom{i-1}{3}(.6)^4(.4)^{i-4}$

73. Let N be the number of games played. Then

$P\{N = 4\} = 2(1/2)^4 = 1/8$, $\quad P\{N = 5\} = 2\dbinom{4}{1}(1/2)(1/2)^4 = 1/4$

$P\{N = 6\} = 2\dbinom{5}{2}(1/2)^2(1/2)^4 = 5/16$, $\quad P\{N = 7\} = 5/16$

$E[N] = 4/8 + 5/4 + 30/16 + 35.16 = 93/16 = 5.8125$

75: $P(X=k) = \dbinom{k+9}{k}\dfrac{1}{2^{k+10}}$, $k=0,1,2,\ldots$

locating k tails in first $k+9$ tosses (last must be heads)

78: $P(\text{selecting 2 white, 2 black initially})$

$$= \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}$$

So $P(\text{need } n \text{ attempts}) = \left(\frac{17}{35}\right)^{n-1}\left(\frac{18}{35}\right)$

82: $P(\text{he } \cancel{\text{rejects}} \text{ accepts a given lot}) =$

$P(\text{all 4 randomly chosen components non-defective}) =$

$(.9)^4 = .6561$

$P(\text{he rejects}) = 1 - .6561 = .3449$

So on average, 34.49% of lots rejected

THEORETICAL EXERCISES

10. $$E[1/(X+1)] = \sum_{i=0}^{n} \frac{1}{i+1} \frac{n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} \frac{n!}{(n-i)!\,(i+1)!} p^i (1-p)^{n-i}$$

$$= \frac{1}{(n+1)p} \sum_{i=0}^{n} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i}$$

$$= \frac{1}{(n+1)p} \sum_{j=1}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j}$$

$$= \frac{1}{(n+1)p} \left[1 - \binom{n+1}{0} p^0 (1-p)^{n+1-0}\right]$$

$$= \frac{1}{(n+1)p} [1 - (1-p)^{n+1}]$$

15: For even n: $(q = 1-p)$

$$P(\text{even \# heads}) = P(0) + P(2) + \dots + P(n)$$
$$= \binom{n}{0} p^0 q^n + \binom{n}{2} p^2 q^{n-2} + \dots + \binom{n}{n} p^n q^0$$
$$= \sum_{i=0}^{n/2} \binom{n}{2i} p^{2i} q^{n-2i}$$

Binomial thm: $(p+q)^n = \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots$

$$(q-p)^n = \binom{n}{0}(-p)^0 q^n + \binom{n}{1}(-p)^1 q^{n-1} + \binom{n}{2}(-p)^2 q^{n-2} + \dots$$
$$= \binom{n}{0} p^0 q^n - \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots$$

So $(p+q)^n + (q-p)^n = 2 \sum_{i=0}^{n/2} \binom{n}{2i} p^{2i} q^{n-2i}$,

and $P(\text{even \# heads}) = \frac{1}{2}\left((p+q)^n + (q-p)^n\right) = \frac{1}{2}\left(1 + (q-p)^n\right)$

Similar for odd n.

16:

$$\frac{P(X=i)}{P(X=i+1)} = \frac{\lambda^i}{i!} e^{-\lambda} \Big/ \frac{\lambda^{i+1}}{(i+1)!} e^{-\lambda} = \frac{i+1}{\lambda}$$

when $i+1 \le \lambda$, ratio is ≤ 1, and Poisson probabilities are increasing

when $i+1 > \lambda$, ration is > 1, and Poisson probabilities are decreasing.

17. (a) If X is binomial (n, p) then, from exercise 15,

$$\begin{aligned} P\{X \text{ is even}\} &= [1 + (1-2p)^n]/2 \\ &= [1 + (1 - 2\lambda/n)^n]/2 \quad \text{when } \lambda = np \\ &\to (1 + e^{-2\lambda})/2 \quad \text{as } n \text{ approaches infinity} \end{aligned}$$

(b) $P\{X \text{ is even}\} = e^{-\lambda}\sum_n \lambda^{2n}/(2n)! = e^{-\lambda}(e^{\lambda} + e^{-\lambda})/2$

21. (i) 1/365
(ii) 1/365
(iii) 1 The events, though independent in pairs, are not independent.

26: Prove by induction on n

$n=0$: LHS is $e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$

RHS is $\frac{1}{0!}\int_\lambda^\infty e^{-x}\,dx = e^{-\lambda}$ ✓

$n>0$: $\frac{1}{n!}\int_\lambda^\infty e^{-x}x^n\,dx = \frac{1}{n!}\left[-e^{-x}x^n\right]_\lambda^\infty + \frac{1}{n!}\int_\lambda^\infty e^{-x}\,n\,x^{n-1}\,dx$

$u = x^n$
$dv = e^{-x}dx$
$du = nx^{n-1}dx$
$v = -e^{-x}$

$$= e^{-\lambda}\frac{\lambda^n}{n!} + \frac{1}{(n-1)!}\int_\lambda^\infty e^{-x}x^{n-1}\,dx$$

$$= e^{-\lambda}\frac{\lambda^n}{n!} + \sum_{i=0}^{n-1} e^{-\lambda}\frac{\lambda^i}{i!}$$

↑ induction

$$= \sum_{i=0}^{n} e^{-\lambda}\frac{\lambda^i}{i!}$$ ✓

27. $P\{X = n+k \mid X > n\} = \dfrac{P\{X = n+k\}}{P\{X > n\}}$

$$= \frac{p(1-p)^{n+k-1}}{(1-p)^n}$$

$$= p(1-p)^{k-1}$$