Math 30530, Fall 2009, Homework 6 Solutions

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- **1.** F, the distribution functions of X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/15 & \text{if } 1 \leq x < 2 \\ 3/15 & \text{if } 2 \leq x < 3 \\ 6/15 & \text{if } 3 \leq x < 4 \\ 10/15 & \text{if } 4 \leq x < 5 \\ 1 & \text{if } x \geq 5. \end{cases}$$

- **4.** Let p be the probability mass function of X. We have

x	-2	2	4	6
$p(x)$	1/2	1/10	13/45	1/9

- **11.** The set of possible values of X is $\{2, 3, 4, \dots\}$. For $n \geq 2$, $X = n$ if and only if either all of the first $n-1$ bits generated are 0 and the nth bit generated is 1, or all of the first $n-1$ bits generated are 1 and the nth bit generated is 0. Therefore, by independence,

$$P(X = n) = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}, \quad n \geq 2.$$

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- **2.** Let X be the fine that the citizen pays on a random day. Then

$$E(X) = 25(0.60) + 0(0.40) = 15.$$

Therefore, it is much better to park legally.

- **3.** The expected value of the winning amount is

$$30\left(\frac{4000}{2,000,000}\right) + 800\left(\frac{500}{2,000,000}\right) + 1,200,000\left(\frac{1}{2,000,000}\right) = 0.86.$$

Considering the cost of the ticket, the expected value of the player's *gain* in one game is $-1 + 0.86 = -0.14$.

• **5.** Let X be the net gain in one play of the game. The set of possible values of X is $\{-8, -4, 0, 6, 10\}$. The probabilities associated with these values are

$$p(-8) = p(0) = \frac{1}{\binom{5}{2}} = \frac{1}{10}, \quad p(-4) = \frac{\binom{2}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{4}{10},$$

and $p(6) = p(10) = \dfrac{\binom{2}{1}}{\binom{5}{2}} = \dfrac{2}{10}$. Hence

$$E(X) = -8 \cdot \frac{1}{10} - 4 \cdot \frac{4}{10} + 0 \cdot \frac{1}{10} + 6 \cdot \frac{2}{10} + 10 \cdot \frac{2}{10} = \frac{4}{5}.$$

Since $E(X) > 0$, the game is not fair.

• **9.** **(a)** $\displaystyle\sum_{i=-2}^{2} p(x) = \frac{9}{27} + \frac{4}{27} + \frac{1}{27} + \frac{4}{27} + \frac{9}{27} = 1.$

(b) $E(X) = \sum_{x=-2}^{2} xp(x) = 0, \quad E(|X|) = \sum_{x=-2}^{2} |x|p(x) = 44/27,$

$E(X^2) = \sum_{x=-2}^{2} x^2 p(x) = 80/27$. Hence

$$E(2X^2 - 5X + 7) = 2(80/27) - 5(0) + 7 = 349/27.$$

• **11.** $p(x)$ the probability mass function of X is given by

x	-3	0	3	4
$p(x)$	3/8	1/8	1/4	1/4

Hence

$$E(X) = -3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{5}{8},$$

$$E(X^2) = 9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{77}{8},$$

$$E(|X|) = 3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{23}{8},$$

$$E(X^2 - 2|X|) = \frac{77}{8} - 2\left(\frac{23}{8}\right) = \frac{31}{8},$$

$$E(X|X|) = -9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{23}{8}.$$

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- **1.** On average, in the long run, the two businesses have the same profit. The one that has a profit with lower standard deviation should be chosen by Mr. Jones because he's interested in steady income. Therefore, he should choose the first business.

- **3.** $E(X) = \sum_{x=-3}^{3} xp(x) = -1$, $E(X^2) = \sum_{x=-3}^{3} x^2 p(x) = 4$. Therefore, $\text{Var}(X) = 4-1 = 3$.

- **5.** By straightforward calculations,

$$E(X) = \sum_{i=1}^{N} i \cdot \frac{1}{N} = \frac{1}{N} \cdot \frac{N(N+1)}{2} = \frac{N+1}{2},$$

$$E(X^2) = \sum_{i=1}^{N} i^2 \cdot \frac{1}{N} = \frac{1}{N} \cdot \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6},$$

$$\text{Var}(X) = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} = \frac{N^2-1}{12},$$

$$\sigma_X = \sqrt{\frac{N^2-1}{12}}.$$

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- **1.** Let X_1 be the number of TV sets the salesperson in store 1 sells and X_2 be the number of TV sets the salesperson in store 2 sells. We have that $X_1^* = (10-13)/5 = -0.6$ and $X_2^* = (6-7)/4 = -0.25$. Therefore, the number of TV sets the salesperson in store 2 sells is 0.6 standard deviations below the mean, whereas the number of TV sets the salesperson in store 2 sells is 0.25 standard deviations below the mean. So Mr. Norton should hire the salesperson who worked in store 2.

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- **1.** $\binom{8}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^4 = 0.087.$

- **3.** $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = 0.054.$

- **8.** $\sum_{i=0}^{8}\binom{15}{i}(0.8)^i(0.2)^{15-i} = 0.142.$

11. We know that $p(x)$ is maximum at $[(n+1)p]$. If $(n+1)p$ is an integer, $p(x)$ is maximum at $[(n+1)p] = np + p$. But in such a case, some straightforward algebra shows that

$$\binom{n}{np+p} p^{np+p}(1-p)^{n-np-p} = \binom{n}{np+p-1} p^{np+p-1}(1-p)^{n-np-p+1},$$

implying that $p(x)$ is also maximum at $np + p - 1$.

15. The maximum occurs at $k = [11(0.45)] = 4$. The maximum probability is

$$\binom{10}{4}(0.45)^4(0.55)^6 = 0.238.$$

19. The expected value of the expenses if sent in one parcel is

$$45.20 \times 0.07 + 5.20 \times 0.93 = 8.$$

The expected value of the expenses if sent in two parcels is

$$(23.30 \times 2)(0.07)^2 + (23.30 + 3.30)\binom{2}{1}(0.07)(0.93) + (6.60)(0.93)^2 = 9.4.$$

Therefore, it is preferable to send in a single parcel.

26. **(a)** A four-engine plane is preferable to a two-engine plane if and only if

$$1 - \binom{4}{0}p^0(1-p)^4 - \binom{4}{1}p(1-p)^3 > 1 - \binom{2}{0}p^0(1-p)^2.$$

This inequality gives $p > 2/3$. Hence a four-engine plane is preferable if and only if $p > 2/3$. If $p = 2/3$, it makes no difference.

(b) A five-engine plane is preferable to a three-engine plane if and only if

$$\binom{5}{5}p^5(1-p)^0 + \binom{5}{4}p^4(1-p) + \binom{5}{3}p^3(1-p)^2 > \binom{3}{2}p^2(1-p) + p^3.$$

Simplifying this inequality, we get $3(p-1)^2(2p-1) \geq 0$ which implies that a five-engine plane is preferable if and only if $2p - 1 \geq 0$. That is, for $p > 1/2$, a five-engine plane is preferable; for $p < 1/2$, a three-engine plane is preferable; for $p = 1/2$ it makes no difference.