Math 30530, Fall 2009, homework 4 Solutions

Page 87:

- **1.** Let G be the event that Susan is guilty. Let L be the event that Robert will lie. The probability that Robert will commit perjury is

- **5.** **(a)** $\frac{6}{11} \times \frac{5}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = 0.00216.$

  (b) $\frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} \times \frac{1}{7} = 0.00216.$

- **7.** Let A_i be the event that the ith person draws the "you lose" paper. Clearly,

$$P(A_1) = \frac{1}{200},$$

$$P(A_2) = P(A_1^c A_2) = P(A_1^c)P(A_2 \mid A_1^c) = \frac{199}{200} \cdot \frac{1}{199} = \frac{1}{200},$$

$$P(A_3) = P(A_1^c A_2^c A_3) = P(A_1^c)P(A_2^c \mid A_1^c)P(A_3 \mid A_1^c A_2^c) = \frac{199}{200} \cdot \frac{198}{199} \cdot \frac{1}{198} = \frac{1}{200},$$

and so on. Therefore, $P(A_i) = 1/200$ for $1 \le i \le 200$. This means that it makes no difference if you draw first, last or anywhere in the middle. Here is Marilyn Vos Savant's intuitive solution to this problem:

> It makes no difference if you draw first, last, or anywhere in the middle. Look at it this way: Say the robbers make everyone draw at once. You'd agree that everyone has the same change of losing (one in 200), right? Taking turns just makes that same event happen in a slow and orderly fashion. Envision a raffle at a church with 200 people in attendance, each person buys a ticket. Some buy a ticket when they arrive, some during the event, and some just before the winner is drawn. It doesn't matter. At the party the end result is this: all 200 guests draw a slip of paper, and, regardless of when they look at the slips, the result will be identical: one will lose. You can't alter your chances by looking at your slip before anyone else does, or waiting until everyone else has looked at theirs.

- **9.** For $1 \le n \le 39$, let E_n be the event that none of the first $n-1$ cards is a heart or the ace of spades. Let F_n be the event that the nth card drawn is the ace of spades. Then the event of "no heart before the ace of spades" is $\bigcup_{n=1}^{39} E_n F_n$. Clearly, $\{E_n F_n,\ 1 \le n \le 39\}$ forms a sequence of mutually exclusive events. Hence

$$P\Big(\bigcup_{n=1}^{39} E_n F_n\Big) = \sum_{n=1}^{39} P(E_n F_n) = \sum_{n=1}^{39} P(E_n)P(F_n \mid E_n)$$
$$= \sum_{n=1}^{39} \frac{\binom{38}{n-1}}{\binom{52}{n-1}} \times \frac{1}{53-n} = \frac{1}{14},$$

a result which is not unexpected.

Page 96 :

- **1.** $\frac{1}{2} \times 0.05 + \frac{1}{2} \times 0.0025 = 0.02625.$

- **3.** $\frac{1}{3}(0.75) + \frac{1}{3}(0.68) + \frac{1}{3}(0.47) = 0.633.$

- **9.** $(0.50)(0.04) + (0.30)(0.02) + (0.20)(0.04) = 0.034.$

- **21.** Let E be the event that the third number falls between the first two. Let A be the event that the first number is smaller than the second number. We have that

$$P(E \mid A) = \frac{P(EA)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Intuitively, the fact that $P(A) = 1/2$ and $P(EA) = 1/6$ should be clear (say, by symmetry). However, we can prove these rigorously. We show that $P(A) = 1/2$; $P(EA) = 1/6$ can be proved similarly. Let B be the event that the second number selected is smaller than the first number. Clearly $A = B^c$ and we only need to show that $P(B) = 1/2$. To do this, let B_i be the event that the first number drawn is i, $1 \le i \le n$. Since $\{B_1, B_2, \dots, B_n\}$ is a partition of the sample space,

$$P(B) = \sum_{i=1}^{n} P(B \mid B_i)P(B_i).$$

Now $P(B \mid B_1) = 0$ because if the first number selected is 1, the second number selected cannot be smaller. $P(B \mid B_i) = \dfrac{i-1}{n-1}$, $1 \le i \le n$ since if the first number is i, the second number must be one of $1, 2, 3, \dots, i-1$ if it is to be smaller. Thus

$$P(B) = \sum_{i=1}^{n} P(B \mid B_i)P(B_i) = \sum_{i=2}^{n} \frac{i-1}{n-1} \cdot \frac{1}{n} = \frac{1}{(n-1)n} \sum_{i=2}^{n} (i-1)$$
$$= \frac{1}{(n-1)n}\big[1 + 2 + 3 + \cdots + (n-1)\big] = \frac{1}{(n-1)n} \cdot \frac{(n-1)n}{2} = \frac{1}{2}.$$

Page 105 :

- **1.** $\dfrac{(3/4)(0.40)}{(3/4)(0.40) + (1/3)(0.60)} = \dfrac{3}{5}.$

- **3.** Let G and I be the events that the suspect is guilty and innocent, respectively. Let A be the event that the suspect is left-handed. Since $\{G, I\}$ is a partition of the sample space, we can use Bayes' formula to calculate $P(G \mid A)$, the probability that the suspect has committed the crime in view of the new evidence.

$$P(G \mid A) = \frac{P(A \mid G)P(G)}{P(A \mid G)P(G) + P(A \mid I)P(I)} = \frac{(0.85)(0.65)}{(0.85)(0.65) + (0.23)(0.35)} \approx 0.87.$$

- **9.** $$\frac{(0.15)(0.25)}{(0.15)(0.25)+(0.85)(0.75)}=0.056.$$

- **11.** $$\frac{1\left(\frac{1}{6}\right)}{\sum_{i=0}^{5}\left[\binom{1000-i}{100}\Big/\binom{1000}{100}\right]\left(\frac{1}{6}\right)}=0.21.$$

- **12.** Let A be the event that the wallet originally contained a \$2 bill. Let B be the event that the bill removed is a \$2 bill. The desired probability is given by

$$P(A \mid B)=\frac{P(B \mid A)P(A)}{P(B \mid A)P(A)+P(B \mid A^c)P(A^c)}$$
$$=\frac{1\times\frac{1}{2}}{1\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}}=\frac{2}{3}.$$