Math 30530: Introduction to Probability, Fall 2009

Midterm Exam 1

Monday October 12

Name: SOLUTIONS

Instructor: David Galvin

This exam contains 7 problems on 8 pages (including the front cover).
It is open-book, open notes. You may use a calculator.
Show all your work on the paper provided.
The honor code is in effect for this exam.

Scores

Question	Score	Out of
1		10
2		12
3		9
4		9
5		9
6		10
7		6
Total		65

GOOD LUCK !!!

1. Let $\mathcal{P}_2$ be the set of all subsets of the set $\{1,2\}$ (so $\mathcal{P}_2 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$). Consider the experiment of choosing two *distinct* elements from $\mathcal{P}_2$, order not mattering, with all choices of two elements equally likely.

   (a) (3 points) List all the elements of the sample space.

   $\emptyset, \{1\}$ / $\emptyset, \{2\}$ / $\emptyset, \{1,2\}$ /
   $\{1\}, \{2\}$ / $\{1\}, \{1,2\}$ / $\{2\}, \{1,2\}$ (6 in all)

   (b) (2 points) Compute the probability that $\{1\}$ is one of the chosen elements.

   $\frac{3}{6} = .5$

   (c) (2 points) Compute the probability that the two chosen elements are complements of each other.

   $\frac{2}{6} = .33\ldots$

   (d) (3 points) Consider the events

   $$A = \{\{1\} \text{ is one of the chosen elements}\}$$
   $$B = \{\text{the chosen elements are complements of each other}\}.$$

   Are A and B independent? Justify.

   $P(A) = .5$
   $P(B) = .33\ldots$
   $P(AB) = \frac{1}{6} = .166\ldots = .5 \times .33\ldots$

   YES, independent.

2. A fair coin (one that shows heads and tails each with probability 1/2) is tossed repeatedly until the first time that the same face comes up twice in a row. Let X be the random variable that counts the number of tosses needed until this happens.

(a) (2 points) What are the possible values that X can take?

$X = 2, 3, 4, 5, \ldots$

(b) (3 points) Compute the mass function of X.

Two ways to get k : HTHT... (ending with repeat)
THTH... (" " ")

Each probability $(\frac{1}{2})^k$

So $P(X=k) = 2(\frac{1}{2})^k, \quad k = 2, 3, 4, \ldots$

(c) (3 points) What is the probability that it will take more than 3 tosses to first see the same face coming up twice in a row?

$$P(X>3) = 1 - P(2) - P(3)$$
$$= 1 - (\tfrac{1}{2} + \tfrac{1}{4}) = \tfrac{1}{4}$$

(d) (4 points) Compute the expectation of X.

$$E(X) = \sum_{k=2}^{\infty} k(\tfrac{1}{2})^{k-1} = \frac{d}{dx}\left[x^2 + x^3 + \cdots\right]\Big|_{x=\frac{1}{2}}$$
$$= \frac{d}{dx}\left[\frac{x^2}{1-x}\right]\Big|_{x=\frac{1}{2}}$$
$$= \frac{(1-x)2x + x^2}{(1-x)^2}\Big|_{x=\frac{1}{2}}$$
$$= 3$$

3. Let A and B be events with $P(A) = x$, $P(B) = y$ and $P(AB) = z$.

(a) (4 points) Write an expression for the probability that neither A nor B occurs, in terms of x, y and z.

$$P(A^c B^c) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(AB)]$$
$$= 1 - x - y + z$$

(b) (5 points) Suppose that the events A and B are independent. Show that the events A^c and B^c are also independent.

If A, B ind, $P(AB) = P(A)P(B)$, $z = xy$

$$P(A^c B^c) = 1 - x - y + z$$
$$= 1 - x - y + xy$$
$$= (1-x)(1-y)$$
$$= P(A^c)P(B^c).$$

4. 30 snowblowers, of which 7 have defects, are sold to a hardware store. The store manager inspects 6 of the snowblowers randomly.

(a) (5 points) What is the probability that he finds at least one defective snowblower?

$$P(\text{at least one}) = 1 - P(\text{none})$$

$$= 1 - \frac{\binom{23}{6}}{\binom{30}{6}} = .82999\ldots$$

(b) (4 points) What is the probability that he finds exactly two defective snowblowers? (For this part, you can leave your answer in terms of $\binom{n}{r}$'s.)

$$\frac{\binom{7}{2}\binom{23}{4}}{\binom{30}{6}}$$

5. A student takes a multiple choice test in which there are four options for each question. For each question to which the student is not certain of the answer, she makes a guess by choosing an answer at random from the options, each option equally likely. Based on her previous exam results, the professor estimates that the student will be certain of the answer on 70% of the questions.

(a) (4 points) The student picks a random question to start with. Assuming the professor's estimate is correct, what is the probability that she gets the question right?

$C = \{\text{Correct}\}$

$K = \{\text{Knows answer for certain}\}$

$D = \{\text{doesn't know answer}\}$

$$P(C) = P(C|K)P(K) + P(C|D)P(D)$$

$$= 1 \times .7 + .25 \times .3 = .775$$

(b) (5 points) The professor picks a random question to grade, and notices that the student answered it correctly. What is the probability that she was certain of the answer?

$$P(K|C) = \frac{P(CK)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|D)P(D)}$$

$$= \frac{.7}{.775} = .90\ldots$$

6. 8 fair dice are rolled. Let X be the number of dice that land on 6.

(a) (3 points) What is the expectation and variance of X?

X is Binomial, $n = 8$, $p = \frac{1}{6}$

$$E(X) = np = \frac{8}{6} = \frac{4}{3}$$

$$Var(X) = np(1-p) = \frac{40}{36} = \frac{10}{9}$$

(b) (3 points) What is the probability that X is at most 2?

$$P(X \leq 2) = P(0) + P(1) + P(2)$$

$$= \binom{8}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^8 + \binom{8}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^7 + \binom{8}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$$

$$= .8651\ldots$$

(c) (4 points) The first dice that was rolled landed on 6. Given this information, what is now the probability that X is at most 2?

$$P(X \leq 2 \mid \text{first is } 6) = P(Y \leq 1)$$

where Y is Binomial, $n = 7$, $p = \frac{1}{6}$

(collapsing sample space)

$$= \binom{7}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^7 + \binom{7}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^6$$

$$= .6697\ldots$$

7. (6 points) Your brother and father challenge you to a chess tournament. To win the challenge, you have to win two games *in a row* in a series of at most 3 games. EITHER you play a game with your brother first, then your father, and then (if necessary) your brother again, OR you play a game with your father first, then your brother, and then (if necessary) your father again.

You know from experience that you beat your brother with probability p_b and beat your father with probability p_f, where $p_b < p_f$ (so you are more likely to beat your father). All games are independent of each other. To maximize your chances of winning the challenge, which of the two formats should you choose: brother, father, brother, or father, brother, father? Justify your answer.

Playing B, F, B :

Win if <u>either</u> win first two (prob $p_b p_F$)

<u>or</u> win second, third after losing first

(prob $(1-p_b) p_F p_b$)

So $P(\text{Winning}) = p_b p_F + (1-p_b) p_F p_b$

$= p_b p_F (2 - p_b)$

Playing F, B, F :

$P(\text{Winning}) = p_b p_F (2 - p_F)$ (Symmetry)

Since $p_b < p_F$, $p_b p_F (2-p_b) > p_b p_F (2-p_F)$,

So better to play Brother, Father, Brother.