MATH 30440, SEC 01, SPRING 2010

HOMEWORK 3 SOLUTIONS

4.6) First, find λ.

$$\int_0^\infty \lambda e^{-\frac{x}{100}}\,dx = \lambda\left[-100\,e^{-\frac{x}{100}}\right]_0^\infty$$

$$= 100\lambda$$

Since $\int$ should equal 1, $\lambda = \frac{1}{100}$.

X = time to breakdown.

$$P(50 \le X \le 150) = \int_{50}^{150} \frac{1}{100} e^{-\frac{x}{100}}\,dx$$

$$= \left[-e^{-\frac{x}{100}}\right]_{50}^{150}$$

$$= -.223 + .606$$

$$= .383$$

$$P(X \le 100) = \int_0^{100} \frac{1}{100} e^{-\frac{x}{100}}\,dx = \left[-e^{-\frac{x}{100}}\right]_0^{100}$$

$$= 1 - \frac{1}{e} = .63\ldots$$

4.7) First, P(a particular tube has to be replaced within 150 hours)

$$= \int_{100}^{150} \frac{100}{x^2}\,dx = \left[-\frac{100}{x} \right]_{100}^{150}$$

$$= -\frac{2}{3} + 1 = \frac{1}{3}$$

Next, P(exactly 2 out of 5) =

$$\binom{5}{2} \left(\tfrac{1}{3}\right)^2 \left(\tfrac{2}{3}\right)^3 = .3292$$

P(remaining 3 don't)

probability that the chosen 2 need to be replaced

ways of choosing the particular 2

4.9) 5! ways to test the 5 transistors in order

Possible values for N_1 : 1, 2, 3

(first defective must have appeared by 3^{th} test)

Possible values for N_2 : 1, 2, 3

Joint mass table:

N_2 \ N_1	1	2	3
1	.3	.2	.1
2	.2	.1	0
3	.1	0	0

$$P(N_1=1, N_2=1) = \frac{3 \times 2 \times 3!}{5!} = .3$$

(3 ways to choose first transistor, since must be defective; 2 ways to choose second, 3! ways for the rest)

$$P(N_1=1, N_2=2) = \frac{3 \times 2 \times 2 \times 2!}{5!} = .3$$

(3 ways to choose first (def), 2 ways to choose second (good), 2 ways to choose third (def), 2! ways for rest)

$$P(N_1=1, N_2=3) = \frac{3 \times 2 \times 1 \times 2 \times 1}{5!} = .1$$

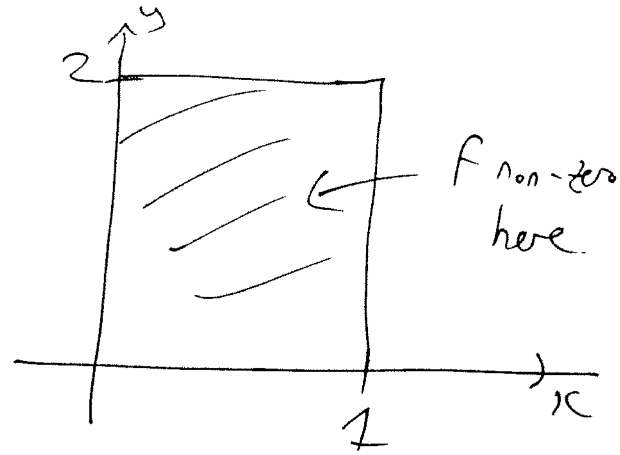
$$P(N_1=2, N_2=1) = \frac{2 \times 3 \times 2 \times 2!}{5!} = .2$$

$$P(N_1=2, N_2=2) = \frac{2 \times 3 \times 1 \times 2 \times 1}{5!} = .1$$

$$P(N_1=3, N_2=1) = \frac{2 \times 1 \times 3 \times 2 \times 1}{5!} = .1$$

all others 0

4.10) a)



$$\int_{x=0}^{1}\int_{y=0}^{2} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\, dx$$

$$= \frac{6}{7}\int_0^1 \left[x^2 y + \frac{xy^2}{4}\right]_0^2 dx$$

$$= \frac{6}{7}\int_0^1 \left[2x^2 + x\right] dx = \frac{6}{7}\left[\frac{2}{3}x^3 + \frac{x^2}{2}\right]_0^1$$

$$= \frac{6}{7} \times \frac{7}{6} = 1,$$

So f is a valid joint density.

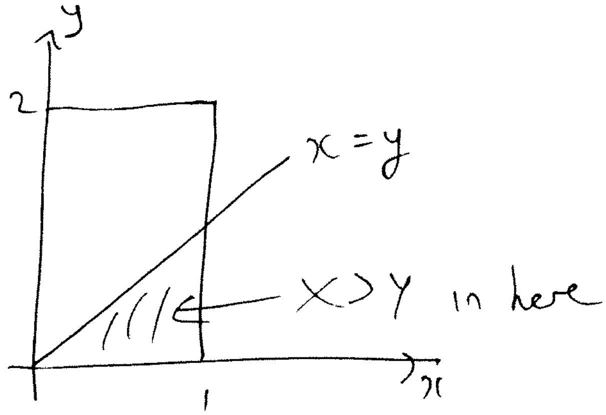
b) $$f_X(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$$

$$= \frac{6}{7}\left[x^2 y + \frac{xy^2}{4}\right]_0^2$$

$$= \frac{6}{7}\left(2x^2 + x\right), \quad \text{if } 0 \le x \le 1;$$

$f_X(x) = 0$ otherwise

c)



$$P(X > Y) = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\, dx = \frac{15}{56}.$$

4.11) $\max\{X_1, X_2, \dots, X_n\}$ is function which picks out largest of $X_1, X_2, \dots, X_n$.

Since each $X_i \in (0,1)$, $0 \le M \le 1$ always,

So $F_M(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases}$

For $0 \le x \le 1$:

$$\begin{aligned} F_M(x) &= P(M \le x) \\ &= P(\max\{X_1, \dots, X_n\} \le x) \\ &= P(X_1 \le x \text{ and } X_2 \le x \text{ and } \dots \text{ and } X_n \le x) \\ &= P(X_1 \le x)\, P(X_2 \le x) \dots P(X_n \le x) \end{aligned}$$

(using independence)

$= x^n \quad \left[\begin{array}{c} \text{Since } X_i \text{ is uniform on } (0,1), \\ P(X_i \le x) = \frac{x}{1} = x \end{array} \right]$

Get density of M by differentiating :

$$f_M(x) = \begin{cases} 0 & \text{if } x \le 0,\ x \ge 1 \\ n x^{n-1} & \text{if } 0 \le x \le 1 . \end{cases}$$

4.12) a) $f_X(x) = \int_0^\infty x e^{-x} e^{-y}\, dy$

$$= x e^{-x} \left[-e^{-y} \right]_0^\infty$$

$$= x e^{-x}, \quad \underline{\text{if}}\ x \ge 0;\ 0 \text{ if } x \le 0.$$

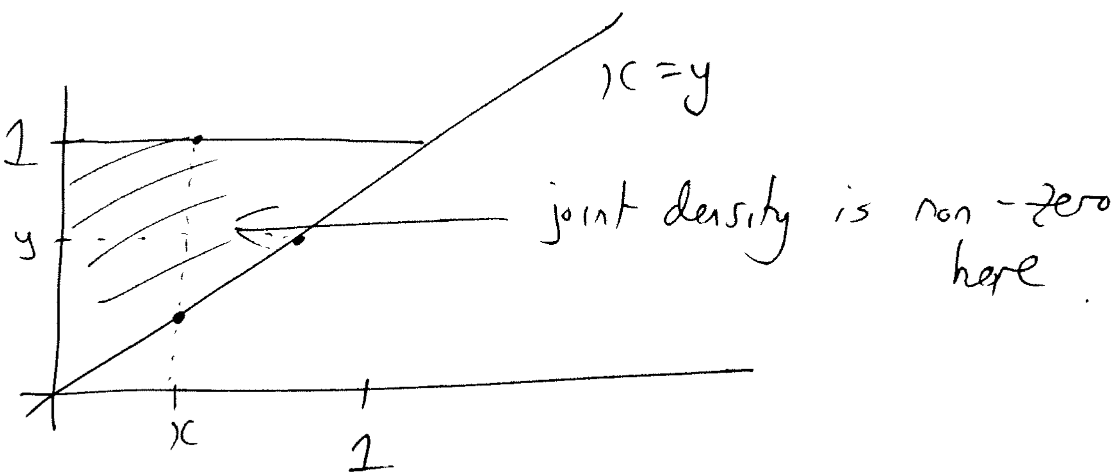
b) $f_Y(y) = e^{-y} \underbrace{\int_0^\infty x e^{-x}\, dx}$

$= 1$; either by integration by parts, or since we know that xe^{-x} is a density, (part a)), so integrates to 1

$$= e^{-y};\ \text{if } y \ge 0 \quad \left(\begin{array}{c} \text{if } y < 0, \\ f_Y(y) = 0 \end{array} \right)$$

c) Product of individual densities = joint density, So yes, X, Y are independent.

4.13)



a) $f_X(x) = 0$ if $x < 0$, $x > 1$.

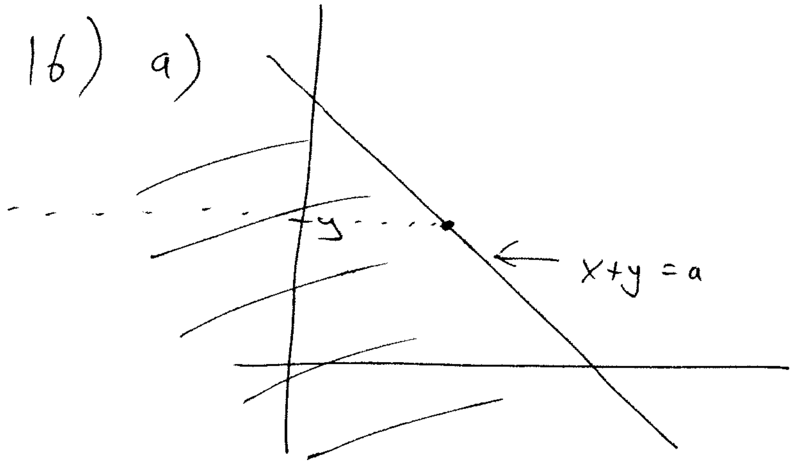
For $0 \le x \le 1$,

$$f_X(x) = \int_{y=x}^{y=1} 2\,dy = 2 - 2x$$

b) $$f_Y(y) = \begin{cases} 0 & \text{if } y < 0,\ y > 1 \\ \int_{x=0}^{x=y} 2\,dx = 2y, & 0 \le y \le 1 \end{cases}$$

c) Product of individual densities $\neq$ joint density, So no, X, Y not independent.

4.16) a)



$$P(X+Y \le a) = \iint_{\text{Shaded region}} f(x,y)\, dA$$

by independence

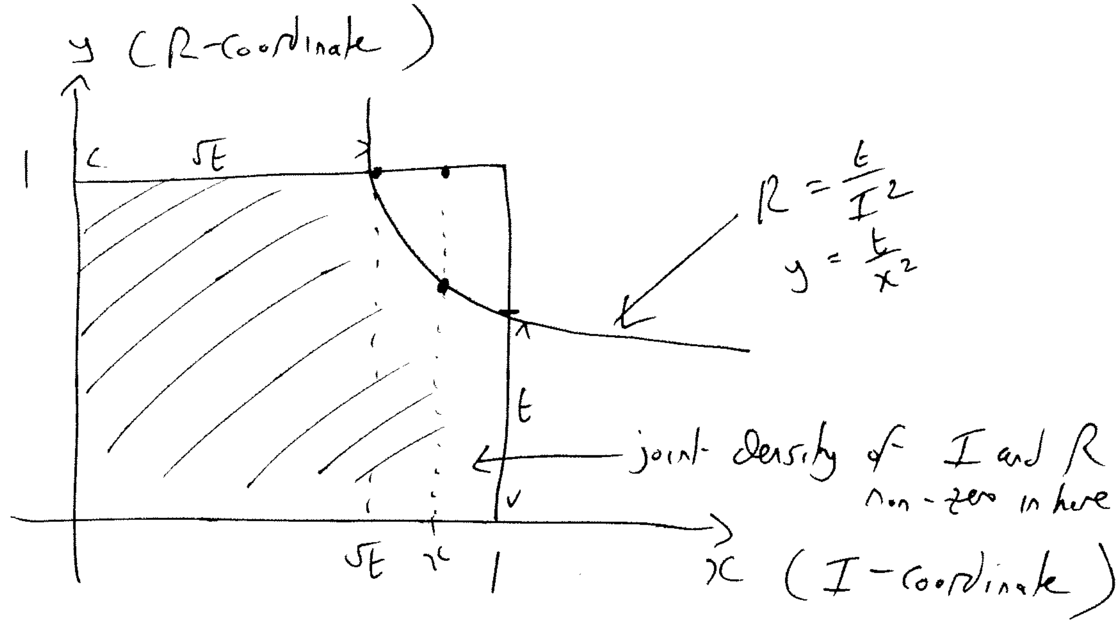
$$= \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{a-y} f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{a-y} f_X(x)\, dx \right] f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} F_X(a-y) f_Y(y)\, dy$$

definition of distribution function.

4.17)



Joint density of I and R:

$$f(x,y) = \begin{cases} 6x(1-x)\,2y & \text{inside } [0,1]\times[0,1] \\ 0 & \text{elsewhere} \end{cases}$$

(notice I use a different variable name for each random variable)

We want density of $W = I^2 R$

Step 1: Range of values for W is 0 to 1,

So $f_W(x) = 0$ for $x < 0$, $x > 1$

Step 2: Calculate distribution function of W, using joint density of I and R

For $0 \le t \le 1$:

$$F_W(t) = P(W \le t)$$
$$= P(I^2 R \le t)$$
$$= P\left(R \le \frac{t}{I^2}\right)$$
$$= \iint_{\substack{\text{shaded region on} \\ \text{previous page}}} f(x,y)\, dA$$
$$= 1 - \iint_{\substack{\text{unshaded region in the} \\ \text{top right hand corner}}} f(x,y)\, dA$$
$$= 1 - \int_{x=\sqrt{t}}^{1} \int_{y=\frac{t}{x^2}}^{1} 12x(1-x)y \, dy \, dx$$
$$= 1 - \int_{\sqrt{t}}^{1} \left[6x(1-x)y^2\right]_{\frac{t}{x^2}}^{1} dx$$
$$= 1 - \int_{\sqrt{t}}^{1} \left(6x - 6x^2 - \frac{6t}{x^3} + \frac{6t}{x^2}\right) dx$$
$$= 1 - \left[3x^2 - 2x^3 + \frac{3t}{x^2} - \frac{6t}{x}\right]_{\sqrt{t}}^{1}$$

$$= 1 - 3 \cancel{+2} \cancel{-3t} \cancel{+6t} + 3t - 2t^{\frac{3}{2}} + 3 - 6t^{\frac{1}{2}}$$

$$= 3 + 6t - 2t^{\frac{3}{2}} - 6t^{\frac{1}{2}}$$

Step 3 : Differentiate to get density :

For $0 \leq t \leq 1$,

$$\underline{f_w(t) = 6 - 3t^{\frac{1}{2}} - 3t^{-\frac{1}{2}}}$$