Math 30440 - Spring 2009

Outline solutions to final practice exam

1) Total time is $N(80, 25)$

Want $P(N(80,25) \leq 90) = P\left(Z \leq \frac{90-80}{\sqrt{25}}\right)$

$= P(Z \leq 2)$

2) Measuring in minutes, arrivals are on average every $\frac{60}{11}$ minutes, so this should be $\frac{1}{\lambda}$; we model T by an exponential RV with parameter $\frac{11}{60}$.

a) $P(T > 10) = \int_{10}^{\infty} f(x)\,dx = \int_{10}^{\infty} \frac{11}{60} e^{-\frac{11x}{60}}\,dx$

$= \left[-e^{\frac{-11x}{60}}\right]_{10}^{\infty}$

$= e^{-\frac{110}{60}}$

b) $P(T > 15 \mid T > 5) = P(T > 10) = e^{\frac{-110}{60}}$

(by memorylessness)

3) $E(X) = \int_1^\infty x \frac{1}{3x^4}\, dx = \frac{1}{3}\int_1^\infty x^{-3}\, dx$

$$= \frac{1}{3}\left[-\frac{1}{2}x^{-2}\right]_1^\infty = \frac{1}{6}$$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \int_1^\infty x^2 \frac{1}{3x^4}\, dx = \frac{1}{3}\int_1^\infty x^{-2}\, dx$

$$= \frac{1}{3}\left[-x^{-1}\right]_1^\infty = \frac{1}{3}$$

So $Var(X) = \frac{1}{3} - \left(\frac{1}{6}\right)^2 = \frac{11}{36}$

4) a) Distribution of total weight (if null hypothesis that teacher has no effect on weight is true) is sum of 14 $N(160, (80)^2)$ and 2 $N(240, (120)^2)$.

So it is $N(2720, 118400) = N(2720, (344)^2)$

b) We want to test the null hypothesis that 2100 is a reading from $N(2720, (344)^2)$, against the alternative that it is a reading from a normal with a different mean.

p-value: P(reading as ~~or~~ more extreme than 2100)

Since s.d. of $N(2720, (344)^2)$ is ~~[illegible]~~ 344, a reading of 2100 is $\approx$ ~~[illegible]~~ 1.8 s.d.'s away from mean, so p-value is ~~[illegible]~~ 7%; null ~~rejected~~ accepted

5) $\bar{X} = 3300$ $n = 10$

$S = 176.64$

95% confidence interval:

$$3300 \pm t_{.025,9} \frac{176.64}{\sqrt{10}}$$

6) At moment that roommate puts in his "backup", its life is exponentially distributed with mean 6. At same time, yours is still working, so by memorylessness, its life is also exponentially distributed with mean 6 (it doesn't matter

how long it's been working for up to this point). By ~~symmetry~~ symmetry, each bulb is equally likely to blow first.

So $P(\text{You buy next package}) = \frac{1}{2}$

7) a) Rejecting null when it's true

b) Accepting " " " false

c) Accepted at 2% => p-value > .02

will it be accepted at 5%? It ~~depends~~ depends on whether p-value is > .05 or not

d) Because Mary needs to be more confident, she needs a longer interval

8) a) If null is true, $z = \frac{\bar{X} - 16}{\frac{2}{\sqrt{25}}}$ is a standard normal

95% of time, I want to accept true null, so want to accept when

$$-1.96 \le \frac{\bar{X} - 16}{\frac{2}{5}} \le 1.96,$$

$$\bar{X} = 16 \pm .784 = (15.216, 16.784)$$

b) If true mean is 20, $\dfrac{\bar{X} - 20}{2/5} = Z$

I accept null if

$$\bar{X} \in (15.216, 16.784)$$

$$\frac{\bar{X} - 20}{2/5} \in (-11.96, -8.04)$$

$$Z \in (-11.96, -8.04)$$

This happens with probability ≈ 0

c) Necessary sample size is

$$n \approx \frac{(z_{.025} + z_{.05})^2 (4)}{(20-16)^2} = 3.24$$

So just need sample of size 4

9) a) Readings are normally distributed

b) 90% one-sided confidence interval:

$$\left(0, \frac{(n-1)S^2}{\chi^2_{.90,n-1}}\right) = \left(0, \frac{19 \times .8}{\chi^2_{.9,19}}\right)$$

So $C = \dfrac{19 \times .8}{\chi^2_{.9,19}}$

10) Joint density of $X_1, \dots, X_n$ at readings $x_1, \dots, x_n$ is

$$f(\theta) = \prod_{i=1}^{n} \frac{4x_i^3}{\theta} e^{-\frac{x_i^4}{\theta}}$$

$$= \frac{4^n}{\theta^n} \prod_{i=1}^{n} x_i^3 \, e^{-\frac{1}{\theta}\sum x_i^4}$$

if all $x_i > 0$; it's 0 if any of the $x_i \le 0$

$$\log f(\theta) = n \log 4 - n \log \theta + \log \prod x_i^3 - \frac{1}{\theta} \sum x_i^4$$

$$\log' f(\theta) = \frac{-n}{\theta} + \frac{\sum x_i^4}{\theta^2}$$

$$= 0 \text{ when } \theta = \frac{\sum x_i^4}{n}$$

11) Joint density at particular readings

$$f(\theta) = \begin{cases} \frac{1}{(2\theta)^5} & \text{if all readings between } -\theta \text{ and } \theta \\ 0 & \text{otherwise} \end{cases}$$

To make $f(\theta)$ large, need θ as small as possible, and still satisfying all readings between $-\theta$ and θ; take $\theta = 3$

12) a) X_M = # errors in Morning

X_M = Poisson (4) (Average # errors = 4)

So $P(X_M \le 2) = \frac{4^0}{0!}e^{-4} + \frac{4^1}{1!}e^{-4} + \frac{4^2}{2!}e^{-4}$

b) X_a = # errors in afternoon

= Poisson (2.5)

X_t = # errors in total

$= X_M + X_a$

= Poisson (4) + Poisson (2.5)

= Poisson (6.5)

$P(X_t \le 2) = \frac{6.5^0}{0!}e^{-6.5} + \frac{6.5^1}{1!}e^{-6.5} + \frac{(6.5)^2}{2!}e^{-6.5}$

13) D = distance from actual to intended

$$D = \sqrt{(x\text{-error})^2 + (y\text{-error})^2}$$

$$= \sqrt{N(0,\sigma^2)^2 + N(0,\sigma^2)^2}$$

$$D^2 = N(0,\sigma^2)^2 + N(0,\sigma^2)^2$$

$$\frac{D^2}{\sigma^2} = \left(\frac{N(0,\sigma^2)}{\sigma}\right)^2 + \left(\frac{N(0,\sigma^2)}{\sigma}\right)^2$$

$$= Z_1^2 + Z_2^2$$

$$= \chi^2 \text{ with 2 d.o.f.}$$

a) $P(D \le 1) = P\left(\frac{D^2}{\sigma^2} \le \frac{1}{\sigma^2}\right)$

$= P\left(\chi^2_2 \le \frac{1}{9}\right) \approx .005$

b) $P(D \le 1) = P\left(\chi^2_2 \le \frac{1}{\sigma^2}\right)$

From table, $P(\chi^2_2 \le 9.2) = .99$

So want $\frac{1}{\sigma^2} = 9.2$, $\sigma \approx \frac{1}{3}$

14) a) $P(\text{Temp} > 100) = P(N(75, (25)^2) > 100)$

$= P\left(Z \geq \frac{100 - 75}{25}\right)$

$= P(Z \geq 1) = .16$

b) $P(\text{Can use at least 6 of 8})$

$= P\left(\begin{array}{l}\text{Binomial RV with } n=8, \text{ takes value} \\ p = .84 \qquad\qquad\qquad\qquad \geq 6\end{array}\right)$

$$= \sum_{k=6}^{8} \binom{8}{k} (.84)^k (.16)^{8-k}$$

15) a) $\frac{1}{10}$

b) Let $\bar{X}$ be average of 4 readings of impurity %

Want: $P(\text{Medium} \mid \bar{X} = 1.4)$

Know: $P(\text{Med}) = .1, \quad P(\text{Low}) = .9$

By Bayes, $P(\text{Med} \mid 1.4) = \dfrac{P(1.4 \mid \text{Med})\, P(\text{Med})}{P(1.4 \mid \text{Med})\, P(\text{Med}) + P(1.4 \mid \text{Low})\, P(\text{Low})}$

So need to know

$P(1.4 \mid \text{Med}), \quad P(1.4 \mid \text{Low})$

Problem: Both $= 0$!

Solution: Compute $P(\bar{X} \in (1.4 - \varepsilon, 1.4 + \varepsilon) \mid \text{Med})$,

and take limit as $\varepsilon \to 0$

If we choose crate of Medium grade,

$$\bar{X} = N(1, .04)$$

So $P(\bar{X} = 1.4 \pm \varepsilon) \simeq 2\varepsilon \times$ value of density function of $N(1, .04)$ at 1.4

$$= 2\varepsilon \frac{1}{\sqrt{2\pi} \cdot .2} e^{-\frac{(1.4-1)^2}{2 \times .04}}$$

$$\simeq .54\varepsilon$$

Similarly,

$$P(\bar{X} \in (1.4-\varepsilon, 1.4+\varepsilon) \mid \text{Low})$$

$$\approx 2\varepsilon \frac{1}{\sqrt{2\pi} \cdot .5} e^{-\frac{(2.1-1.4)^2}{2 \times .25}}$$

$$\approx .6\varepsilon$$

So by Bayes, $P(\text{Med} \mid \bar{X} \in (1.4-\varepsilon, 1.4+\varepsilon))$

$$\approx \frac{.54\varepsilon \times .1}{.54\varepsilon \times .1 + .6\varepsilon \times .9} = .09$$

So we estimate that the probability of have chosen a crate of medium ingots is approximately $.09$.

16) $H_0 : \mu = .5$

$H_1 : \mu > .5$ (One sided test since we are testing the theory that friend is doing something to increase his mean)

If null is true, our reading of $\log 2.5 = .916$ is a reading from a normal distribution with mean $.5$, variance $.1$

p-value: $P(N(.5, .1) \geq .916)$

$$= P\left(Z \geq \frac{.916 - .5}{\sqrt{.1}}\right)$$

$$= P(Z \geq 1.31)$$

$$= .095$$

Since p-value $> 5\%$, can't reject null.