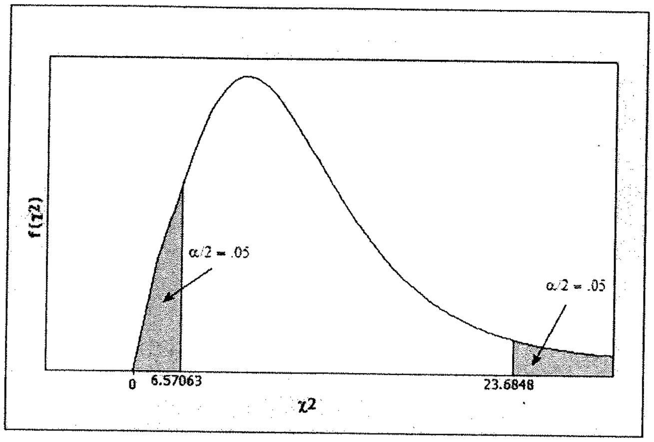
MATH 20340, Fall 2009 Homework 11 Solutions

10.49 For this exercise, $s^2 = .3214$ and $n = 15$.



A 90% confidence interval for σ^2 will be

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{(1-\alpha/2)}}$$

where $\chi^2_{\alpha/2}$ represents the value of χ^2 such that 5% of the area under the curve (shown in the figure above) lies to its right. Similarly, $\chi^2_{(1-\alpha/2)}$ will be the χ^2 value such that an area .95 lies to its right.

Hence, we have located one-half of α in each tail of the distribution. Indexing $\chi^2_{.05}$ and $\chi^2_{.95}$ with $n-1=14$ degrees of freedom in Table 5 yields

$$\chi^2_{.05} = 23.6848 \quad \text{and} \quad \chi^2_{.95} = 6.57063$$

and the confidence interval is

$$\frac{14(.3214)}{23.6848} < \sigma^2 < \frac{14(.3214)}{6.57063} \quad \text{or} \quad .190 < \sigma^2 < .685$$

10.50 **a** Calculate $\sum x_i = 17.7$, $\sum x_i^2 = 48.95$ and $n = 7$. Then

$$s^2 = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1} = \frac{48.95 - \frac{(17.7)^2}{7}}{6} = .6990476$$

b Indexing $\chi^2_{.025}$ and $\chi^2_{.975}$ with $n-1=6$ degrees of freedom in Table 5 yields

$$\chi^2_{.025} = 14.4494 \quad \text{and} \quad \chi^2_{.975} = 1.237347$$

and the 95% confidence interval is

$$\frac{6(.6990476)}{14.4494} < \sigma^2 < \frac{6(.6990476)}{1.237347} \quad \text{or} \quad .291 < \sigma^2 < 3.390$$

c It is necessary to test

$$H_0 : \sigma^2 = .8 \quad \text{versus} \quad H_a : \sigma^2 \neq .8$$

and the test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{6(.6990476)}{.8} = 5.24$$

The two-tailed rejection region with $\alpha = .05$ and $n-1=6$ degrees of freedom is

$$\chi^2 > \chi^2_{.025} = 14.4494 \quad \text{or} \quad \chi^2 < \chi^2_{.975} = 1.237347$$

and H_0 is not rejected. There is insufficient evidence to indicate that σ^2 is different from .8.

d The p-value is found by approximating $P(\chi^2 > 5.24)$ and then doubling that value to account for an equally small value of s^2 which might have produced a value of the test statistic in the lower tail of the chi-square distribution. The observed value, $\chi^2 = 5.24$, is smaller than $\chi^2_{.10} = 10.6646$ in Table 5. Hence,

$$p\text{-value} > 2(.10) = .20$$

10.51 The hypothesis of interest is

$$H_0 : \sigma = .7 \quad \text{versus} \quad H_a : \sigma > .7$$

or equivalently

$$H_0 : \sigma^2 = .49 \quad \text{versus} \quad H_a : \sigma^2 > .49$$

Calculate

$$s^2 = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1} = \frac{36 - \frac{(10)^2}{4}}{3} = 3.6667$$

The test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{3(3.6667)}{.49} = 22.449$$

The one-tailed rejection region with $\alpha = .05$ and $n-1=3$ degrees of freedom is $\chi^2 > \chi^2_{.05} = 7.81$ and H_0 is rejected. There is sufficient evidence to indicate that σ^2 is greater than .49.

10.55 **a** The force transmitted to a wearer, x, is known to be normally distributed with $\mu = 800$ and $\sigma = 40$. Hence,

$$P(x > 1000) = P\left(z > \frac{1000-8000}{40}\right) = P(z > 5) \approx 0$$

It is highly improbable that any particular helmet will transmit a force in excess of 1000 pounds.

b Since $n = 40$, a large sample test will be used to test

$$H_0 : \mu = 800 \qquad H_a : \mu > 800$$

The test statistic is

$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{825-800}{\sqrt{\frac{2350}{40}}} = 3.262$$

and the rejection region with $\alpha = .05$ is $z > 1.645$. H_0 is rejected and we conclude that $\mu > 800$.

10.56 Refer to Exercise 10.55. The hypothesis of interest is

$$H_0 : \sigma = 40 \qquad H_a : \sigma > 40$$

and the test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{39(2350)}{40^2} = 57.281$$

The one-tailed rejection region with $\alpha = .05$ and $n-1 = 39$ degrees of freedom (approximated with 40 degrees of freedom) is $\chi^2 > \chi^2_{.05} = 55.7585$, and H_0 is rejected. There is sufficient evidence to indicate that σ is greater than 40.

10.58 When the assumptions for the *F* distribution are met, then s_1^2/s_2^2 possesses an *F* distribution with $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$ degrees of freedom. Note that df_1 and df_2 are the degrees of freedom associated with s_1^2 and s_2^2, respectively. The *F* distribution is non-symmetrical with the degree of skewness dependent on the above-mentioned degrees of freedom. Table 6 presents the critical values of *F* (depending on the degrees of freedom) such that $P(F > F_a) = a$ for $a = .10, .05, .025, .01$ and $.005$, respectively. Because right-hand tail areas correspond to an upper-tailed test of an hypothesis, we will always identify the larger sample variance as s_1^2 (that is, we will always place the larger sample variance in the numerator of $F = s_1^2/s_2^2$). Hence, an upper-tailed test is implied and the critical values of *F* will determine the rejection region. If we wish to test the hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ versus } H_a : \sigma_1^2 \neq \sigma_2^2$$

There will be another portion of the rejection region in the lower tail of the distribution. The area to the right of the critical value will represent only $\alpha/2$, and the probability of a Type I error is $2(\alpha/2) = \alpha$.

a In this exercise, the hypothesis of interest is

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ versus } H_a : \sigma_1^2 \neq \sigma_2^2$$

and the test statistic is $F = \frac{s_1^2}{s_2^2} = \frac{55.7}{31.4} = 1.774$.

The rejection region (two-tailed) will be determined by a critical value of *F* based on $df_1 = n_1 - 1 = 15$ and $df_2 = n_2 - 1 = 19$ degrees of freedom with area .025 to its right. That is, from Table 6, $F > 2.62$. The observed value of *F* does not fall in the rejection region, and we cannot conclude that the variances are different.

b The student will need to find critical values of *F* for various levels of α in order to find the approximate *p*-value. The critical values with $df_1 = 15$ and $df_2 = 19$ are shown below from Table 6.

α	.10	.05	.025	.01	.005
F_α	1.86	2.23	2.62	3.15	3.59

Hence,

$$p\text{-value} = 2P(F > 1.774) > 2(.10) = .20$$

10.59 Refer to Exercise 10.58. From Table 6, $F_{df_1,df_2} = 2.62$ and $F_{df_2,df_1} \approx 2.76$. The 95% confidence interval for σ_1^2/σ_2^2 is

$$\frac{s_1^2}{s_2^2}\frac{1}{F_{df_1,df_2}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}F_{df_2,df_1}$$

$$\frac{55.7}{31.4}\left(\frac{1}{2.62}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \frac{55.7}{31.4}(2.76) \quad \text{or} \quad .667 < \frac{\sigma_1^2}{\sigma_2^2} < 4.896$$

10.61 The hypothesis of interest is $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_a : \sigma_1^2 \neq \sigma_2^2$
and the test statistic is

$$F = \frac{s_1^2}{s_2^2} = \frac{110^2}{107^2} = 1.057.$$

The critical values of F for various values of α are given below using $df_1 = 15$ and $df_2 = 14$.

α	.10	.05	.025	.01	.005
F_α	2.01	2.46	2.95	3.66	4.25

Hence,

$$p\text{-value} = 2P(F > 1.057) > 2(.10) = .20$$

Since the p-value is so large, H_0 is not rejected. There is no evidence to indicate that the variances are different.

10.65 For each of the three tests, the hypothesis of interest is

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ versus } H_a : \sigma_1^2 \neq \sigma_2^2$$

and the test statistics are

$$F = \frac{s_1^2}{s_2^2} = \frac{3.98^2}{3.92^2} = 1.03 \qquad F = \frac{s_1^2}{s_2^2} = \frac{4.95^2}{3.49^2} = 2.01 \qquad \text{and} \qquad F = \frac{s_1^2}{s_2^2} = \frac{16.9^2}{4.47^2} = 14.29$$

The critical values of F for various values of α are given below using $df_1 = 9$ and $df_2 = 9$.

α	.10	.05	.025	.01	.005
F_α	2.44	3.18	4.03	5.35	6.54

Hence, for the first two tests,

$$p\text{-value} > 2(.10) = .20$$

while for the last test,

$$p\text{-value} < 2(.005) = .01$$

There is no evidence to indicate that the variances are different for the first two tests, but H_0 is rejected for the third variable. The two-sample t-test with a pooled estimate of σ^2 cannot be used for the third variable.