Math 10860, Honors Calculus 2

Quiz 5, Thursday February 20

Solutions

1. Give the *definition* of the number e. (Not some properties that e satisfies, but rather the definition that we had in class. If your answer involves any functions that were not discussed in the fall (it doesn't need to), you need to say what those functions are).

   Solution: e is the unique real number satisfying
   $$\int_1^e \frac{dt}{t} = 1.$$
   (Or: e is the unique real number satisfying $\log e = 1$, where $\log : (0,\infty) \to \mathbb{R}$ is defined by $\log x = \int_1^x \frac{dt}{t}$; or: e is the number $\exp 1$, where $\exp : \mathbb{R} \to (0,\infty)$ is the inverse of the function $\log : (0,\infty) \to \mathbb{R}$ defined by $\log x = \int_1^x \frac{dt}{t}$.)

2. Let $f(x) = (1-x)^{1/x}$. Does f approach a limit near 0, and if so, what is it?

   Solution: We have
   $$(1-x)^{1/x} = e^{\log\left((1-x)^{1/x}\right)} = e^{\frac{\log(1-x)}{x}}.$$
   A quick application of L'Hôpital's rule shows that
   $$\lim_{x\to 0} \frac{\log(1-x)}{x} = \lim_{x\to 0} \frac{-1}{1-x} = -1,$$
   and since exp is a continuous function (and -1 is in its domain), it follows that
   $$\lim_{x\to 0}(1-x)^{1/x} = \lim_{x\to 0} e^{\frac{\log(1-x)}{x}} = e^{-1} = \frac{1}{e}.$$

3. Is $\log_{10} 2$ rational or irrational? (Justify!)

   Solution: $\log_{10} 2$ is irrational. Suppose, for a contradiction, that $\log_{10} 2 = p/q$ for some natural numbers p, q[1]. Applying $\exp_{10}$ to both sides of the equation, we get $2 = 10^{p/q}$, and raising both sides to the power q we get $2^q = 10^p$. Now there are two possible ways to proceed.

   Slick way Appeal to unique factorization of natural numbers into prime factors. The number 10^p has 5 as a prime factor, but the number 2^q does not, so it cannot possibly be that $2^q = 10^p$. This contradiction shows that $\log_{10} 2$ is irrational.

   Prosaic way We have $\log_{10} 2 < 1$ because $2 < 10$, $\log_{10} 10 = 1$, and $\log_{10}$ is increasing, so $p < q$. Dividing both sides of $2^q = 10^p$ by 2^p we get $2^{q-p} = 5^p$. This cannot happen, since the right-hand side is even while the left-hand side is odd. This contradiction shows that $\log_{10} 2$ is irrational.

[1] Note $\log_{10} 2 > 0$ (because $2 > 1$, $\log_{10} 1 = 0$, and $\log_{10}$ is increasing) so if it is rational then it can be represented in this way, with positive numerator and positive denominator.