Math 10860, Honors Calculus 2

Quiz 3, Thursday February 6

Solutions

1. (a) Give a clear & correct statement (including all hypotheses) of the fundamental theorem of calculus, part 1.

   Solution: If I is an interval, and $f : I \to \mathbb{R}$ is integrable on (every closed interval contained in) I, and the function $F : I \to \mathbb{R}$ is defined by $F(x) = \int_a^x f$ (or, $F(x) = \int_a^x f(t)\ dt$), *and if f is continuous at c*, then F is differentiable at c, and $F'(c) = f(c)$.

   A *corollary* of this is that if f is continuous on I then F is differentiable on I, and $F' = f$ on I, but this corollary doesn't imply FTOC1 — knowing something about what happens if f is continuous everywhere doesn't allow us to deduce anything about what happens if f is continuous at just one point. So this cannot be considered a correct answer to the question.

   Also, FTOC 1 has nothing to do with the integrals of integrable functions being continuous — that was something we proved before introducing FTOC1, as a first indication that integration is a "smoothing" operation, and is not needed for FTOC1.

   (b) Give a clear & correct statement (including all hypotheses) of the fundamental theorem of calculus, part 2.

   Solution: If $f : [a, b] \to \mathbb{R}$ is integrable, and if there is a function $g : [a, b] \to \mathbb{R}$ satisfying $g' = f$ on $[a, b]$, then $\int_a^b f = g(b) - g(a)$.

   The weaker statement that replaces "f is integrable" with "f is *continuous*" (in which case the existence of a g does not need to by hypothesized) is not FTOC2; it is just a corollary of FTOC1.

2. Determine directly (without comparison to other, known, integrals) whether the improper integral $\int_0^\infty \frac{dx}{\sqrt{x+1}}$ exists. (You may use FTOC if you wish.)

   Solution: The easiest approach uses FTOC2. On $[0, N]$ the function $f(x) = 1/\sqrt{x+1}$ is bounded and integrable, and has an antiderivative $g(x) = 2\sqrt{x+1}$, so $\int_0^N f = 2\sqrt{N+1} - 2\sqrt{1}$. Since $2\sqrt{N+1} - 2\sqrt{1}$ can be made arbitrarily large by choosing N large enough, we get that $\lim_{N\to\infty} \int_0^N dx/\sqrt{x+1}$ does not exist, so neither does $\int_0^\infty \frac{dx}{\sqrt{x+1}}$.