Math 10860, Honors Calculus 2

Homework 7 **NAME**:

Due in class Friday March 20

Instructions

Same as always, plus: some of this homework is a list of integrals to compute. You could in principle do each of these by entering the integrand into `Mathematica` (or some similar program), noting the result, and then verifying it by differentiation. This is **not** what I'm intending. I want to see you tackle these integrals using integration by parts, or integration by an appropriate substitution. For each integral, you should say clearly what method/substitution you are using in each step; other than that, no great explanation is need.

Nota bene: I promise that unless explicitly stated otherwise, all the integrals below *have elementary primitives*. I don't promise that the homework is typo-free, and unfortunately even a tiny typo can turn a do-able integration into an impossible one; so alert me if you think that there is a problem with any of these!

Reading for this homework

Class notes, Chapter 13 (Spivak Chapter 19).

Assignment

1. Here are a few "standard" integration formulae, randomly culled from the back page of a calculus textbook. Verify that each of them is correct. Part of this entails checking that both the function being integrated and the proposed antiderivative have the same domain (part of the full answer will be a statement of that domain); the other part entails checking that at each point in the domain of the proposed antiderivative, the the derivative of the proposed antiderivative is the function being integrated. These should be very easy, but a little care might be required for the antiderivatives that involve absolute values.

   (a) $\int \cot x \, dx = \log|\sin x|$.

   (b) $\int \sec x \, dx = \log|\sec x + \tan x|$.

   (c) For an arbitrary real a (positive or negative), $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$.

2. (a) In class we found $\int \frac{\log x}{x} \, dx$. Now find $\int \frac{1}{x \log x} \, dx$. (A little care might be needed.)

(b) We've seen that $\int_e^\infty \frac{dx}{x}$ goes off to infinity, but not $\int_e^\infty \frac{dx}{x^{1+\varepsilon}}$, for any $\varepsilon > 0$. In other words, increasing the denominator from x to $x^{1+\varepsilon}$ causes the area trapped under the curve (between e and ∞) to go from being infinite to being finite. What about increasing the denominator from x to $x \log x$? I.e., does $\int_e^\infty \frac{1}{x \log x}\, dx$ exist?

(c) What about $\int_1^e \frac{1}{x \log x}\, dx$?

3. This problem concerns a very important non-elementary function, called the *gamma function*.

   (a) Show that for $x > 0$,

   $$\int_0^\infty e^{-t} t^{x-1} dt$$

   is finite. The value of this integral, for each such x, is denoted $\Gamma(x)$. (The gamma function can also be defined for $x \leq 0$, as long as x is not an integer — see the graph at `https://en.wikipedia.org/wiki/Gamma_function`.)

   (b) Prove that for all $x > 0$ in the domain of Γ, $\Gamma(x+1) = x\Gamma(x)$.

   (c) Prove that $\Gamma(n) = (n-1)!$ for all natural numbers n. (So, the gamma function is a continuous function that extends the factorial function to (almost) all reals).

4. Explain *precisely* what is wrong with the following "proof" that black is white:

   Evaluating $\int \frac{dx}{x}$ using integration by parts, taking $u = 1/x$ and $dv = dx$ (so $du = -dx/x^2$ and $v = x$), yields

   $$\int \frac{dx}{x} = \left(\frac{1}{x}\right) x - \int x \left(\frac{-1}{x^2}\right) dx = 1 + \int \frac{dx}{x}.$$

   Subtracting $\int \frac{dx}{x}$ from both sides yields $0 = 1$.

   No credit for just mumbling something vague about the constant of integration — pinpoint *exactly* what is wrong, and say what the argument actually proves.

5. (Not to be turned in) As a warm-up for the coming integrals, you could do Spivak, Chapter 19, Questions 1 and 2, 3rd ed. (twenty integrals involving simple algebraic manipulation and quick substitutions).

6. First, some problems that are best suited to integration by parts. Do any two of these:

   (a)

   $$\int x^2 \sin x \; dx.$$

   (b)

   $$\int x (\log x)^2 \; dx.$$

(c)

$$\int \sec^3 x \; dx.$$

(Remember that you know $\int \sec x dx$.)

7. None of $\log(\log x)$, $1/(\log x)$, $x^2 e^{-x^2}$ or e^{-x^2} have elementary primitives. However, we can still say things about their primitives. Do any two of these:

   (a) Express $\int \log(\log x) dx$ in terms of $\int dx/\log x$.

   (b) Express $\int x^2 e^{-x^2} \; dx$ in terms of $\int e^{-x^2} \; dx$.

   (c) Find a reduction formula for $\int (\log x)^n \; dx$, and use it to calculate $\int (\log x)^3 \; dx$.

8. Remember that there are no silver-bullet rules for substitution. Just try to substitute for an expression that appears frequently or prominently. If two different troublesome expressions appear, try to express them both in terms of some new expression. Do any two of these:

   (a)

$$\int \frac{dx}{\sqrt{1+e^x}}.$$

   (b)

$$\int \frac{4^x + 1}{2^x + 1} \; dx.$$

   (c)

$$\int \frac{1}{x^2} \sqrt{\frac{x-1}{x+1}} \; dx.$$

9. Some problems involving substitutions such as $x = \sin u$, $x = \cos u$: (As well as knowing $\int \sec dx$, it *might* be helpful here to know

$$\int \csc x \; dx = -\log|\csc x + \cot x|,$$

   which can also be verified easily by differentiation.) Do any two of these:

   (a)

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

   (b)

$$\int \frac{dx}{x\sqrt{x^2-1}}.$$

   (c)

$$\int x^3 \sqrt{1-x^2} \; dx.$$

   This will also involve the integration of powers of sin and cos.