Math 10860, Honors Calculus 2

Homework 5 **NAME**:

Due in class Friday February 21

Instructions

The usual.

Reading for this homework

Sections 12.1 and 12.2 of the course notes.

Assignment

1. Differentiate these functions: (Convention: a^{b^c} always means $a^{(b^c)}$.)

   (a)
   $$f(x) = e^{e^{e^{e^x}}}.$$

   (b)
   $$f(x) = e^{\left(\int_0^x e^{-t^2}\,dt\right)}.$$

   (c)
   $$f(x) = (\log x)^{\log x}.$$

2. The *logarithmic derivative* of f is the expression f'/f. It's called "logarithmic derivative" because it is the derivative of $\log \circ f$. It is often easier to compute the derivative of log of a function than it is to compute the derivative of the function directly, because taking logs turns products into (simpler to differentiate) sums, and turns powers into (simpler to differentiate) products. The derivate of the original function can then be recovered by multiplying by the original function.

   Compute the logarithmic derivatives of these functions:

   (a)
   $$f(x) = x^x.$$

   (b)
   $$f(x) = \frac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}.$$

(c)

$$f(x) = \frac{e^x - e^{-x}}{e^{2x}(1+x^3)}.$$

3. Compute these limits:

   (a)

   $$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^2}.$$

   (b)

   $$\lim_{x\to\infty} \frac{x}{(\log x)^n} \qquad (n \text{ a natural number}).$$

   (c)

   $$\lim_{x\to 0^+} \frac{x}{(\log x)^n} \qquad (n \text{ a natural number}).$$

   (d)

   $$\lim_{x\to 0^+} x^x.$$

4. Which number is bigger: e^π or π^e? (Rigorously justify your answer!)

5. Prove that $F(x) = \int_2^x \frac{dt}{\log t}$ is not a bounded function on $[2, \infty)$.

   Meta-question: Why am I asking this question? There is an important mathematical concept, one that you've been familiar with for many years, and one that most non-mathematics are familiar with, that this integral is intimately related to. What is the concept, and what is the connection?

6. This question guides you to an alternate expression for e.

   (a) Find $\lim_{y\to 0} \frac{\log(1+y)}{y}$.

   (b) Find $\lim_{x\to\infty} x \log(1 + 1/x)$.

   (c) Prove that

   $$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x.$$

   (d) Go though the same process to argue that for all real a

   $$e^a = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x.$$

   Specifically:

   - First, argue $\lim_{y\to 0} \frac{\log(1+ay)}{y} = a$.
   - Next, argue $\lim_{x\to\infty} x \log(1 + a/x) = a$.
   - Finally, argue $e^a = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x$.

7. This gives an alternate proof of a basic estimate we proved in class.

(a) Prove that for all natural numbers $n \geq 1$ and for all real $x \geq 0$ we have

$$\sum_{k=0}^{n} \frac{x^k}{k!} \leq e^x.$$

(b) Deduce

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty.$$

8. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between its temperature and the temperature of the surrounding medium. Suppose that an object has temperature T_0 at time $t = 0$, and that the temperature of the surrounding medium remains at a constant M throughout time.

   Find the temperature of the object at time t (in terms of T_0, M, and k, the implicit constant of proportionality in Newton's law).

An extra credit problem: Let f be a function that is integrable (but not necessarily continuous) on any closed bounded interval, and that satisfies the identity

$$f(x) = \int_0^x f(t)\, dt$$

for all x. What are the possible values for $f(1)$?