Math 10860, Honors Calculus 2

Homework 3 **NAME**:

Due in class Friday February 7

Instructions

Please present your answers neatly and clearly. Make use of space to increase the clarity of your presentation.

> *I strongly encourage you to leave wide margins, leave at least an inch of space at the end of each answer, and write large! Remember that the grader is older than you (by a year or two) and may already be suffering from eyestrain!*

Justify your non-obvious assertions —

> *the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.*

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like ”from this it follows that”, or “it is now enough to prove ..., which we now do”, etc), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I'll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Class notes Sections 10.4, 10.5 and 10.6, and/or Spivak Chapter 14, and Appendix to Chapter 8.

Assignment

1. Some questions on uniform continuity.

   (a) Recall that we argued in class that the function $f : (0, 1] \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous, and we further argued that the issue was what was happening near 0 (the function is “blowing up”, with unboundedly increasing slope). Find a function $f : (0, 1] \to \mathbb{R}$ that is continuous but not uniformly continuous, *and is bounded on* $(0, 1]$.

(b) Show that if $f, g : A \to \mathbb{R}$ are both uniformly continuous on A (some interval in $\mathbb{R}$), *and both bounded*, then fg is uniformly continuous on A.

(c) Give an example of an interval A, and functions $f, g : A \to \mathbb{R}$ that are both uniformly continuous on A, with f *not* bounded on A, g bounded on A, such that fg is not uniformly continuous on A.

2. Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Prove that there does not exist a function $g : [0, 2] \to \mathbb{R}$ with the property that $g' = f$.

Comment: It's tempting to think of *defining* the integral via

"$\int_a^b f = g(b) - g(a)$ where g is a function whose derivative is f."

One issue with this definition is that the function f may well be integrable on $[a, b]$, without there being any function g whose derivative is f. The function in this question (which certainly is integrable on $[0, 2]$) furnishes an example.

3. Find the derivatives of the following functions.

(a) $F(x) = \int_a^{x^3} \sin^3 t \, dt$

(b) $F(x) = \int_x^{15} \left(\int_8^y \frac{dt}{1+t^2+\sin t} \right) dy$

(c) $F(x) = \int_a^b \frac{x \, dt}{1+t^2+\sin^2 t}$

4. For each of the following functions f, consider $F(x) = \int_0^x f$, and determine at which points x is $F'(x) = f(x)$. Caution: there may be some x for which $F'(x) = f(x)$ even though the hypotheses of the obvious theorem do not apply.

(a) $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}.$

(b) $f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}.$

(c) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}.$

5. Let f be integrable on $[a, b]$, let c be in (a, b) and let

$$F(x) = \int_a^x f, \qquad a \leq x \leq b.$$

For each of the following statements, either give a proof or a counter-example.

(a) If f is differentiable at c then F is differentiable at c.

(b) If f is differentiable at c then F' is continuous at c.

(c) If f' is continuous at c, then F' is continuous at c.

6. Two unrelated, but hopefully quick, parts.

   (a) Show that, as x ranges over the interval $(0, \infty)$, the value of the following expression does not depend on x:

   $$\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2},$$

   and then (using this fact, or otherwise) deduce that

   $$\int_0^1 \frac{dt}{1+t^2} = \int_1^\infty \frac{dt}{1+t^2}.$$

   (b) Find $F'(x)$ if $F(x) = \int_0^x xf(t)\ dt$. **Hint**: the answer is *not* $xf(x)$.

7. Define $F(x) = \int_1^x \frac{dt}{t}$ and $G(x) = \int_b^{bx} \frac{dt}{t}$ (for $b \geq 1$).

   (a) Find $F'(x)$ and $G'(x)$.

   (b) Use the result of the last part to answer the extra credit problem from Homework 1: for $a, b \geq 1$, prove that

   $$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$

8. Prove that if h is continuous, f and g are differentiable, and

   $$F(x) = \int_{f(x)}^{g(x)} h(t)\ dt$$

   then

   $$F'(x) = h(g(x))g'(x) - h(f(x))f'(x).$$

An extra credit problem: Let I, J and K be intervals. Suppose that $g : I \to J$ and $f : J \to K$ are both integrable (f on J and g on I). What can you say about the composition function $f \circ g : I \to K$? (It will be one of three things: exactly one of

A $f \circ g$ is integrable (on I)

B $f \circ g$ is not integrable

C $f \circ g$ is sometimes integrable, sometimes not, depending on the specific choices of f and g

is true. Which one? If **A** or **B**, give a proof; if **C**, give examples to show that both behaviors are possible.)