Math 10860, Honors Calculus 2

Homework 2 **NAME**:

Solutions

1. Decide which of the following functions are integrable on $[0, 2]$, and calculate the integral when the function is integrable. You can use that $\int_a^b x\,dx$ exists and equals $(b^2 - a^2)/2$; but don't assume anything else other than the definition of the integral, and the basic facts that we have proven in class or in the notes.

   (a) $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ x - 2 & \text{if } 1 \le x \le 2 \end{cases}$

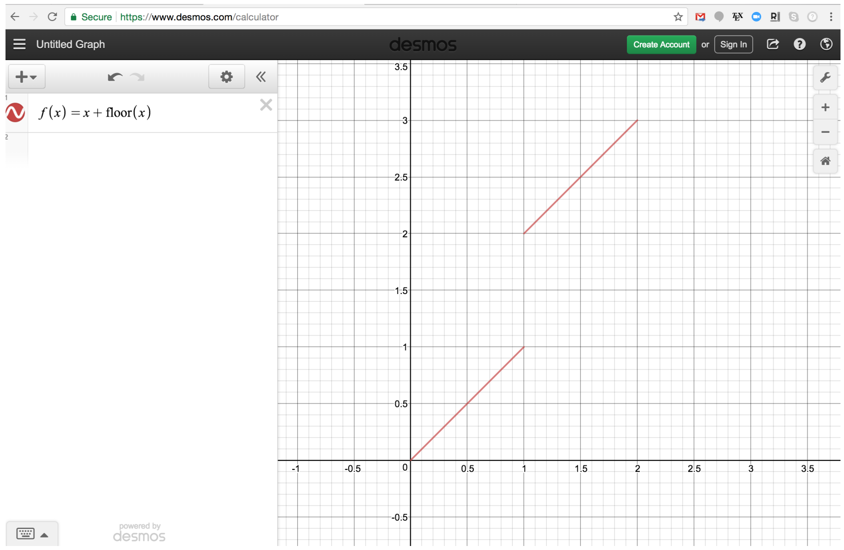
   Solution: This is integrable: using results that we or Spivak have derived,

$$\begin{aligned} \int_0^2 f &= \int_0^1 x\,dx + \int_1^2 (x-2)\,dx \\ &= \int_0^1 x\,dx + \int_1^2 x\,dx - \int_1^2 2\,dx \\ &= \int_0^2 x\,dx - 2(2-1) \\ &= (2^2)/2 - 2 \\ &= 0. \end{aligned}$$

   Note that one of the basic facts used here is that if f and g differ at only finitely many places then $\int_a^b f = \int_a^b g$ (as long as at least one of the integrals is known to exist). This allows us, for example, to replace the function defined on $[0, 1]$ that is x is $x < 1$ and $x - 2$ if $x = 1$, with the easier-to-integrate function defined on $[0, 1]$ that is x for every input x, without changing the value of the integral.

   (b) $f(x) = x + [x]$ (recall $[x]$ is the largest integer that is less than or equal to x).

   Solution: This function is integrable. It's non-negative on $[0, 2]$, and it's graph (shown below) traps a region over the x-axis that is a union of finitely many rectangles and triangles, so we can easily calculate the integral by computing areas, getting that the value is 3.



We could also argue that f can be re-written as

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ x+1 & \text{if } 1 \leq x < 2 \\ 3 & \text{if } x = 2 \end{cases}$$

from which we can easily calculate, using basic results, that $\int_0^2 f = 3$.

(c) $f(x) = \begin{cases} 1 & \text{if } x \text{ is of the form } a + b\sqrt{2} \text{ for rational } a, b \\ 0 & \text{otherwise} \end{cases}$

Solution: This function is not integrable; but there's a slight subtlety in the solution. It's tempting to say: the rationals are dense in the reals, so the numbers *not* of the form $a + b\sqrt{2}$ for rational a, b are dense (since this set of numbers includes rationals), so in any interval there is at least one such number, so all lower Darboux sums are 0, so the supremum over all lower Darboux sums is 0; on the other hand, we showed in class last semester that rational multiples of $\sqrt{2}$ are dense in the reals (that's how we showed that irrationals are dense in the reals), so it follows that numbers of the form $a + b\sqrt{2}$ for rational a, b are dense, so in any interval there is at least one such number, so all upper Darboux sums are 1, so the infimum over all upper Darboux sums is 1.

The subtle issue is that actually the function is 1 on all rationals, since in the rule for the function we are allowed to take $b = 0$. We can still argue as before that the infimum over all upper Darboux sums is 1; but to see that the supremum over all lower Darboux sums is 0, we have to be more careful. One way to proceed is to observe that rational multiples of $\sqrt{3}$ are dense in the reals (via the same proof that shows that rational multiples of $\sqrt{2}$ are dense). The only rational multiple of $\sqrt{3}$ that is of the form $a + b\sqrt{2}$ for rational a, b is 0; indeed, if $c\sqrt{3} = a + b\sqrt{2}$ for rational a, b, c, then by squaring both sides we get

$$3c^2 - 2b^2 - a^2 = 2ab\sqrt{2}$$

and so, since $\sqrt{2}$ is irrational, either a or b must be 0. If $b = 0$ we have $c\sqrt{3} = a$, which can only hold if $c = a = 0$, since $\sqrt{3}$ is irrational, and if $a = 0$ we have $c\sqrt{3} = b\sqrt{2}$. Multiplying both sides by $\sqrt{2}$ yields $c\sqrt{6} = 2b$, implying $b = c = 0$, since $\sqrt{6}$ is irrational.

It's easy to check that if we remove one element from a dense subset of the reals, the result is still dense, so the set of non-zero multiples of $\sqrt{3}$ is dense in the reals; since no number of this form is of the form $a + b\sqrt{2}$ for rational a, b, we can use this dense set to show that the supremum over all lower Darboux sums is 0.

2. Let $f : [-b, b] \to \mathbb{R}$ be a function that is integrable on the interval $[0, b]$, and that is an odd function ($f(-x) = -f(x)$). Show that $\int_{-b}^{b} f$ exists, and that it equals 0.

Comment: Note the similarity to question 1 of the first homework. There I was looking for an informal explanation, based on area considerations. Here I'm looking for a *formal* proof, from the definition of the integral.

Solution: Write I for $\int_0^b f$. We will attempt to prove that $\int_{-b}^{0} f = -I$, from which the claimed result immediately follows, from some of our basic properties of integrals.

Note first that f is bounded on $[-b, 0]$ — this follows from f being odd and being bounded on $[0, b]$.

Fix $\varepsilon > 0$. There is a partition P of $[0, b]$ with $I - \varepsilon < L(f, P) \leq I$ and $I \leq U(f, P) < I + \varepsilon$. Let this partition be $0 = t_0 < t_1 < \cdots < t_n = b$.

Consider the partition P' of $[-b, 0]$ given by $-b = -t_n < -t_{n-1} < \cdots < -t_t < -t_0 = 0$. We have

$$L(f, P') = \sum_{i=1}^{n} \inf\{f(x) : x \in [-t_i, -t_{i-1}]\}(t_i - t_{i-1}).$$

(Note that in this summation, the intervals of the partition are being scanned from right to left rather than left to right).

Now for any set A, write $-A$ for $\{-x : x \in A\}$. It is clear (and easy to prove formally if we wished) that for bounded, non-empty A, $\inf -A = -\sup A$. If $A = \{f(x) : x \in [t_{i-1}, t_i]\}$ then from the oddness of f it follows that $-A = \{f(x) : x \in [-t_i, -t_{i-1}]\}$, and so

$$\inf\{f(x) : x \in [-t_i, -t_{i-1}]\} = -\sup\{f(x) : x \in [t_{i-1}, t_i]\}.$$

This gives us that

$$L(f, P') = -\sum_{i=1}^{n} \sup\{f(x) : x \in [t_{i-1}, t_i]\}(t_i - t_{i-1}) = -U(f, P).$$

By a very similar argument we get that

$$U(f, P') = -L(f, P)$$

and so

$$-I - \varepsilon < L(f, P') \leq -I$$

and

$$-I \leq U(f, P') < -I + \varepsilon.$$

Since ε was arbitrary, this shows that the lower Darboux sums for f on $[-b, 0]$ can be made arbitrarily close to $-I$, as can the upper Darboux sums, which establishes that $\int_{-b}^{0} f$ exists and is $-I$, as we needed.

3. Let A be a bounded, non-empty set of real numbers, and let $|A| = \{|a| : a \in A\}$. Prove that

$$\sup |A| - \inf |A| \leq \sup A - \inf A.$$

Solution: Let $\sup |A| = \alpha$ and $\inf |A| = \beta$. We have $0 \leq \beta \leq \alpha$. Suppose $\beta = \alpha$; then $\sup |A| - \inf |A| = 0 \leq \sup A - \inf A$. So assume $\beta < \alpha$.

Choose any ε with $\beta + \varepsilon/2 < \alpha - \varepsilon/2$. There are numbers $x, y \in |A|$ such that $x \in [\beta, \beta + \varepsilon/2)$ and $y \in (\alpha - \varepsilon/2, \alpha]$. We consider cases, according to whether $x \in |A|$ because $x \in A$ or $-x \in A$, and similarly for y:

Case 1, $x \in A, y \in A$: In this case, $\sup A \geq x$, $\inf A \leq y$, so

$$\sup A - \inf A \geq x - y \geq \alpha - \beta - \varepsilon.$$

Case 2, $x \in A, -y \in A$: In this case, $\sup A \geq x$, $\inf A \leq -y \leq y$, so again

$$\sup A - \inf A \geq x - y \geq \alpha - \beta - \varepsilon.$$

Case 3, $-x \in A, y \in A$: In this case, $\sup A \geq y \geq -y$, $\inf A \leq -x$, so again

$$\sup A - \inf A \geq x - y \geq \alpha - \beta - \varepsilon.$$

Case 4, $-x \in A, -y \in A$: In this case, $\sup A \geq -y$, $\inf A \leq -x$, so again

$$\sup A - \inf A \geq x - y \geq \alpha - \beta - \varepsilon.$$

In all four cases, $\sup A - \inf A \geq \sup |A| - \inf |A| - \varepsilon$. Since this is true for all (sufficiently small) $\varepsilon > 0$, it follows that $\sup A - \inf A \geq \sup |A| - \inf |A|$, as claimed.

4. The goal of this multi-part question is to establish some properties of integrability that we discussed in class, but did not prove.

   (a) Prove that if f is integrable on $[a, b]$ then so is $|f|$

   Comment: You will most likely need to use the result of the last question.

   Solution: Fix $\varepsilon > 0$. There is a partition P of $[a, b]$, $a = t + 0 < t_1 < \ldots < t_n = b$, with

$$U(f, P) - L(f, P) < \varepsilon.$$

Now

$$\begin{aligned} U(|f|,P)-L(|f|,P) &= \sum_{i=1}^{n}\left(\sup\{|f(x)|:x\in[t_{i-1},t_i]\}-\inf\{|f(x)|:x\in[t_{i-1},t_i]\}\right)(t_i-t_{i-1}) \\ &\le \sum_{i=1}^{n}\left(\sup\{f(x):x\in[t_{i-1},t_i]\}-\inf\{f(x):x\in[t_{i-1},t_i]\}\right)(t_i-t_{i-1}) \\ &< \varepsilon \end{aligned}$$

with the first inequality coming directly from the result of question 3. Since $\varepsilon > 0$ was arbitrary, this establishes the integrability of $|f|$.

(b) Deduce from the result of part (a) that if f is integrable on $[a,b]$ then so are both of

- $\max\{f,0\}$ (the function which at input x takes the value $f(x)$ if $f(x) \geq 0$, and takes value 0 otherwise) and
- $\min\{f,0\}$.

Comment: This should follow *very quickly*, and without any real technical work, from the result of the last part, if you also use some of the basic properties of the integral that we have previously established.

Solution: We have

$$\max\{f,0\} = \frac{f+|f|}{2}$$

and

$$\min\{f,0\} = \frac{f-|f|}{2}$$

so by the part b) and the usual properties of the integral, both $\max\{f,0\}$ and $\min\{f,0\}$ are integrable.

(c) The *positive part of* f is the function $f^+ = \max\{f,0\}$. Informally, think of the positive part of f as being obtained from f by pushing all parts of the graph of f that lie below the x-axis, up to the x-axis. The *negative part of* f is the function $f^- = -\min\{f,0\}$. Note that $f = f^+ - f^-$ is a representation of f as a linear combination of non-negative functions.

Deduce from the previous parts of this question that f is integrable on $[a,b]$ if and only if f^+ and f^- are both integrable on $[a,b]$.

Comment: As with the last part, this should be quick.

Solution: If f is integrable on $[a,b]$ then so are $\max\{f,0\}$ and $\min\{f,0\}$ (by part c)), and so f^+ and f^- (by the usual properties of the integral).

If f^+ and f^- are both integrable on $[a,b]$, then since $f = f^+ - f^-$ we get the integrability of f by the usual properties of the integral.

5. Prove the *triangle inequality* for integrals: if f is integrable on $[a,b]$ then

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Solution: For all x in $[a,b]$ we have

$$-|f(x)| \le f(x) \le |f(x)|.$$

From the result of Question 4(a), the functions at either end of this inequality are integrable, and from a result proved in the last homework ($f \le g$ for integrable f, g implies $\int f \le \int g$) it follows that

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$$

which is equivalent to the claimed result.

An alternate solution:

$$\begin{aligned}
\left|\int_a^b f\right| &= \left|\int_a^b (f^+ - f^-)\right| \\
&= \left|\int_a^b f^+ - \int_a^b f^-\right| \\
&\le \left|\int_a^b f^+\right| + \left|\int_a^b f^-\right| \quad \text{(ordinary triangle inequality)} \\
&= \int_a^b f^+ + \int_a^b f^- \quad (f^+,\ f^- \text{ both non-negative}) \\
&= \int_a^b (f^+ + f^-) \\
&= \int_a^b |f|.
\end{aligned}$$

6. The goal of this question is to establish that if f and g are integrable on $[a,b]$, then so is fg.

   (a) Suppose that f and g are both non-negative on $[a,b]$. With the notation as in the proof that $f+g$ is integrable (Lemma 11.8 of the class notes), argue that

   $$M_i \le M_i^f M_i^g, \qquad m_i^f m_i^g \le m_i.$$

   Solution: Fix $\varepsilon > 0$. There is $x \in [t_{i-1}, t_i]$ with $M_i - \varepsilon < f(x)g(x)$. Now $f(x) \le M_i^f$ and $g(x) \le M_i^g$, so we have $M_i - \varepsilon < M_i^f M_i^g$ or

   $$M_i < M_i^f M_i^g + \varepsilon.$$

   Since this is true for all $\varepsilon > 0$, it follows that $M_i \le M_i^f M_i^g$.
   The proof that $m_i^f m_i^g \le m_i$ is almost identical.

(b) By using the trick

$$M_i^f M_i^g - m_i^f m_i^g = M_i^f M_i^g - m_i^f M_i^g + m_i^f M_i^g - m_i^f m_i^g,$$

together with the result of part (a), show that fg is integrable.

Comment: For this part it might be helpful to remember that f and g are bounded.

Solution: Let M be any number such that $f(x), g(x) \leq M$ for all $x \in [a,b]$ (such an M exists because f, g bounded). For *any* partition P,

$$\begin{aligned}
U(fg,P) - L(fg,P) &= \sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) \\
&\leq \sum_{i}(M_i^f M_i^g - m_i^f m_i^g)(t_i - t_{i-1}) \quad \text{(using part (a))} \\
&= \sum_{i=1}^{n}(M_i^f M_i^g - m_i^f M_i^g + m_i^f M_i^g - m_i^f m_i^g)(t_i - t_{i-1}) \quad \text{(using trick)} \\
&= \sum_{i=1}^{n}\left(M_i^g(M_i^f - m_i^f)(t_i - t_{i-1}) + m_i^f(M_i^g - m_i^g)(t_i - t_{i-1})\right) \\
&\leq M\left(\sum_{i=1}^{n}(M_i^f - m_i^f)(t_i - t_{i-1}) + \sum_{i=1}^{n}(M_i^g - m_i^g)(t_i - t_{i-1})\right),
\end{aligned}$$

the last line using that $M_i^g, m_i^f \leq M$.
By the integrability of f and g, for each $\varepsilon > 0$ there is a partition that makes both

$$\sum_{i=1}^{n}(M_i^f - m_i^f)(t_i - t_{i-1}) < \frac{\varepsilon}{2M}$$

and

$$\sum_{i=1}^{n}(M_i^g - m_i^g)(t_i - t_{i-1}) < \frac{\varepsilon}{2M}.$$

For this P, $U(fg,P) - L(fg,P) < \varepsilon$, so fg is integrable.

(c) Use the result of Question 4, part (c) (together with some basic properties of the integral) to show that if f and g are both arbitrary (not necessarily non-negative) integrable functions on $[a,b]$, then fg is integrable on $[a,b]$.

Solution: Write $f = f^+ - f^-$ and $g = g^+ - g^-$. We have

$$fg = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-.$$

Each of f^+, f^-, g^+, g^- are non-negative and integrable (by Question 4(c)), so pairwise products of them are integrable also (by part (b)). Hence fg, a linear combination of integrable functions, is integrable.

7. Suppose that f is integrable on $[0, x]$ for all $x \geq 0$ and that $\lim_{x\to\infty} f(x) = a$. Find (with proof)

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\ dt.$$

Comment: Draw a picture to get an intuition for what the limit should be.

Solution: We claim that the limit exists and equal a. Indeed, given $\varepsilon > 0$, there exists some $N > 0$ such that $x > N$ implies $a - \varepsilon/2 < f(x) < a + \varepsilon/2$. For $x > N$ we have

$$\frac{1}{x} \int_0^x f(t)\ dt = \frac{1}{x} \int_0^N f(t)\ dt + \frac{1}{x} \int_N^x f(t)\ dt.$$

The first term on right-hand side above is equal to C/x for some constant C (depending on N). The second lies between

$$\frac{x-N}{x}(a - \varepsilon/2) \quad \text{and} \quad \frac{x-N}{x}(a + \varepsilon/2).$$

We therefor have

$$\frac{C}{x} - \frac{N}{x}(a - \varepsilon/2) + (a - \varepsilon/2) \leq \frac{1}{x} \int_0^x f(t)\ dt \leq \frac{C}{x} - \frac{N}{x}(a + \varepsilon/2) + (a + \varepsilon/2).$$

As $x \to \infty$ we have both

$$\frac{C}{x} - \frac{N}{x}(a - \varepsilon/2), \frac{C}{x} - \frac{N}{x}(a + \varepsilon/2) \to 0,$$

and so there is $N' > 0$ such that for all $x > N'$,

$$\frac{C}{x} - \frac{N}{x}(a + \varepsilon/2) < \varepsilon/2 \quad \text{and} -\varepsilon/2 < \frac{C}{x} - \frac{N}{x}(a - \varepsilon/2).$$

For $x > \max\{N, N'\}$ we thus have

$$a - \varepsilon < \frac{1}{x} \int_0^x f(t)\ dt < a + \varepsilon.$$

Since this was true for arbitrary $\varepsilon > 0$ we have

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\ dt = a,$$

as claimed.