Math 10860, Honors Calculus 2

Homework 1 solutions

Due in class Friday January 25

1. Without doing any serious computations, evaluate the following integrals. You can be informal here; I'm not looking for a watertight ε-δ justification, but rather an explanation that shows me that you know what is going on with the integral, and its interpretation as an area.

   (a) $\int_{-1}^{1} x^3\sqrt{1-x^2}\, dx$

   Solution: We have

   $$\int_{-1}^{1} x^3\sqrt{1-x^2}\, dx = \int_{-1}^{0} x^3\sqrt{1-x^2}\, dx + \int_{0}^{1} x^3\sqrt{1-x^2}\, dx.$$

   Now $x^3\sqrt{1-x^2}$ is non-negative on $[0,1]$, so the second integral on the right above is the area under $y = x^3\sqrt{1-x^2}$, above the x-axis, between $x=0$ and $x=1$; call this region R, with area $a(R)$.
   Also, $x^3\sqrt{1-x^2}$ is non-positive on $[0,1]$, so the first integral on the right above is the negative of the area above $y = x^3\sqrt{1-x^2}$, below the x-axis, between $x=-1$ and $x=0$; call this this region S, with area $a(S)$, so that the value of the integral is $a(R) - a(S)$.
   The function being integrated is odd ($f(-x) = -f(x)$), and so, by reflection though the point $(0,0)$, the region R is mapped to the region S. That means that they have the same areas, and so $a(R) - a(S) = 0$. This is the value of the integral.

   (b) $\int_{-1}^{1} (x^5+3)\sqrt{1-x^2}\, dx$.

   Solution: We have, using basic properties of the integral,

   $$\int_{-1}^{1} (x^5+3)\sqrt{1-x^2}\, dx = \int_{-1}^{1} x^5\sqrt{1-x^2}\, dx + 3\int_{-1}^{1} \sqrt{1-x^2}\, dx.$$

   By the same reasoning as in the first part of the question, the value of the first integral on the right-hand side above is 0. The value of the second term is three times the area under $y = \sqrt{1-x^2}$ (non-negative on $[-1,1]$) above the x-axis between $x=-1$ and $x=1$. This is three times the area of a semi-circle of radius 1, or $3 \cdot (\pi/2) = (3/2)\pi$. This is the value of the integral.

2. Let $f, g : [a, b] \to \mathbb{R}$ both be bounded, and let m, m^f and m^g be given by

- $m = \inf\{f(x) + g(x) : x \in [a, b]\}$
- $m^f = \inf\{f(x) : x \in [a, b]\}$
- $m^g = \inf\{g(x) : x \in [a, b]\}$.

(a) Show that $m^f + m^g \leq m$ (we used this fact in the proof that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$).

Solution: We have that for all $x \in [a, b]$, $f(x) \geq m^f$ (if there was an $x \in [a, b]$ with $f(x) < m^f$, then m^f would not be a lower bound on $\{f(x) : x \in [a, b]\}$), and for the same reason $g(x) \geq m^g$. Combining, we see that for all $x \in [a, b]$, $f(x) + g(x) \geq m^f + m^g$. This says that $m^f + m^g$ is a lower bound for $\{f(x) + g(x) : x \in [a, b]\}$. It follows that m, the greatest lower bound, satisfies $m \geq m^f + m^g$, as claimed.

(b) Show, by way of an example, that it is possible to have $m^f + m^g < m$.

Solution: There are many examples that work. On $[0, 1]$, let $f(x) = x$ and $g(x) = 1 - x$, so that $(f + g)(x) = 1$. Then $m^f = m^g = 0$ and $m = 1$.

3. (a) Which functions have the property that *every* lower Darboux sum equals *every* upper Darboux sum?

Solution: For this to happen, we must have that on every interval the infimum of the function equals the supremum. This can only happen if the function is constant.

(b) Which functions have the property that there is *some* lower Darboux sum that equals *some* upper Darboux sum?

Solution: Suppose $L(f, P_1) = U(f, P_2)$ for some partitions P_1 and P_2. Consider the partition $P = P_1 \cup P_2$. We have

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2) = L(f, P_1),$$

so that

$$L(f, P_1 \cup P_2) = U(f, P_1 \cup P_2).$$

Thus there is at least one partition for which the lower Darboux sum equals the upper Darboux sum (*for that single partition*).

Let P be such a partition, with $L(f, P) = U(f, P)$. Then on each subinterval $[t_{i-1}, t_i]$, f is constant (since $\inf\{f(x) : x \in [t_{i-1}, t_i]\} = \sup\{f(x) : x \in [t_{i-1}, t_i]\}$). But the intervals $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$ overlap and cover $[a, b]$, so in fact f is constant on $[a, b]$.

(c) Which *continuous* functions have the property that all lower Darboux sums are equal?

Solution: Only constant functions have this property. That all constant functions do is evident. In the other direction, suppose that f is not constant on $[a, b]$. Let m be the minimum value of f on $[a, b]$ (minimum, not infimum: extreme value theorem!), and let M be the maximum, achieved at c. Let ε be smaller than $M - m$, but still positive. There is an interval that includes c with the property that for x in that interval, $f(x) > M - \varepsilon > m$ (this uses continuity of f), so the infimum of $f(x)$ for x in that interval is greater than m. If P is any partition of $[a, b]$ that has this interval as one of its intervals, then $L(f, P) > m(b - a)$. But if Q is the partition a, b, then $L(f, Q) = m(b - a)$, showing that for non-constant continuous f, it cannot be the case that all lower Darboux sums are equal.

4. (a) Suppose f is bounded and integrable on $[a, b]$, and that m is a lower bound for f on $[a, b]$ and M an upper bound. Show that

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Solution: If m is a lower bound for f on $[a, b]$, then it is also a lower bound for f on any interval $[t_{i-1}, t_i]$ contained in $[a, b]$. So for any partition P we have $m(t_i - t_{i-1}) \leq m_i(t_i - t_{i-1})$ and so

$$m(b - a) = m \sum_{i=1}^{n} (t_i - t_{i-1}) = \sum_{i=1}^{n} m(t_i - t_{i-1}) \leq \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \leq \int_a^b f.$$

A similar argument gives $\int_a^b f \leq M(b - a)$.

(b) With the same hypotheses as for the last part, show that there exists a number μ, satisfying $m \leq \mu \leq M$, such that

$$\int_a^b f(x)\ dx = \mu(b - a).$$

Solution: When $a = b$ the result is trivial. For $a < b$, this is immediate from the previous part, which asserts so that (as long as $b > a$; when $b = a$ the result is trivial)

$$m \leq \frac{1}{b - a} \int_a^b f(x)\ dx \leq M.$$

We take $\mu = \left(\int_a^b f(x)\ dx \right) / (b - a)$, so that certainly $\int_a^b f(x)\ dx = \mu(a - b)$, and the above inequality shows that $m \leq \mu \leq M$.

(c) Show that if f is integrable on $[a, b]$, and if $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

Solution: If $f(x) \geq 0$ for all $x \in [a, b]$, then for any subinterval of the form $[t_{i-1}, t_i]$ we have $f(x) \geq 0$ on the whole subinterval, so 0 is a lower bound for $\{f(x) : x \in [t_{i-1}, t_i]\}$, so the greatest lower bound is non-negative. In the standard notation, then, the lower Darboux sum for any partition P, $L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$, has $m_i \geq 0$ for all i, so $L(f, P) \geq 0$. Hence $\int_a^b f$, being the supremum over all lower Darboux sums, is non-negative.

(d) Prove that if f and g are both integrable on $[a, b]$, and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

Solution: If f and g are both integrable on $[a, b]$ then so is $f - g$, and since $f - g \geq 0$ on $[a, b]$ we have from the first part that $\int_a^b (f - g) \geq 0$. But then from the basic properties of the integral it follows that $\int_a^b f \geq \int_a^b g$.

5. Suppose that f is weakly increasing (a.k.a non-decreasing) on $[a, b]$. The aim of this question is to show that f is integrable on $[a, b]$ *without making any assumption on the continuity or otherwise of f*.

(a) Prove that f is bounded on $[a, b]$.

Solution: Because f is non-decreasing on $[a, b]$, we have that for all $x \in [a, b]$, $f(a) \leq f(x) \leq f(b)$, so $f(a)$ is a lower bound for f on $[a, b]$ and $f(b)$ is an upper bound.

(b) If $P = t_0, t_1, \ldots, t_n$ is a partition of $[a, b]$, what are $L(f, P)$ and $U(f, P)$?

Solution: Because f is non-decreasing, on any interval $[t_{i-1}, t_i]$ we have that $f(t_{i-1})$ is a lower bound for $\{f(x) : x \in [t_{i-1}, t_i]\}$. It is the greatest lower bound, for if $\alpha > f(t_{i-1})$ then α is *not* a lower bound for the set (being larger than $f(t_{i-1})$). Similarly, $f(t_i)$ is least upper bound for $\{f(x) : x \in [t_{i-1}, t_i]\}$. If follows that

$$L(f, P) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

(c) Suppose that P_n is the equipartition of $[a, b]$ into n subintervals (i.e., $P = (t_0, t_1, \ldots, t_n)$ with $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \cdots = t_n - t_{n-1}$). Calculate $U(f, P) - L(f, P)$ as a short, explicit expression, involving n, a and b, that doesn't involve a summation.

Solution: From the last part we have

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} f(t_{i-1})(1/n) \\ &= \left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f(t_{i-1}) \\ &= \left(\frac{b-a}{n}\right)(f(a) + f(t_1) + f(t_2) + \cdots + f(t_{n-1})), \end{aligned}$$

and

$$U(f,P) = \left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f(t_i) = \left(\frac{b-a}{n}\right)(f(t_1) + f(t_2) + \cdots + f(t_{n-1}) + f(b)).$$

So in $U(f,P) - L(f,P)$ there is a lot of cancelation:

$$U(f,P) - L(f,P) = \left(\frac{b-a}{n}\right)(f(b) - f(a)).$$

(d) Prove that f is integrable on $[a,b]$.

Solution: Given $\varepsilon > 0$ there is $n \in \mathbb{N}$ large enough that $(b-a)(f(b)-f(a))/n < \varepsilon$. Consider the partition P that divides $[a,b]$ into n equal subintervals, each of length $(b-a)/n$. By the last part, for this partition we have

$$U(f,P) - L(f,P) = (b-a)(f(b) - f(a))/n < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that f is integrable on $[a,b]$.

(e) Give an example of a bounded non-decreasing function on $[0,1]$ which is discontinuous at infinitely many points (such a function is still integrable, by the last part of the question).

Solution: There are many, many such examples. Here's one:

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/2 \\ 1/2 & \text{if } 1/2 \le x < 2/3 \\ 2/3 & \text{if } 2/3 \le x < 3/4 \\ 3/4 & \text{if } 3/4 \le x < 4/5 \\ \cdots & \\ (k-1)/k & \text{if } (k-1)/k \le x < k/(k+1),\ k = 1,2,3,\ldots \\ 1 & \text{if } x = 1. \end{cases}$$

This is non-decreasing and has discontinuities at $1/2, 2/3, 3/4, 4/5, \ldots$.

6. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

   (a) Compute $L(f,P)$ for every partition P of $[0,1]$.

   Solution: The irrational numbers are dense in the reals, so there is one in every interval in any partition P. This means that $L(f,P) = 0$ for every partition P.

   (b) Find $\inf\{U(f,P) : P \text{ a partition of } [0,1]\}$.

   Solution: We claim that $\inf\{U(f,P) : P \text{ a partition of } [0,1]\} = 1/2$. One way to see this is to observe that for any interval $[a,b]$,

$$\sup\{f(x) : x \in [a,b]\} = \sup\{x : x \in [a,b]\} = b.$$

   That $\sup\{x : x \in [a,b]\} = b$ is obvious. That b is an upper bound for $\{f(x) : x \in [a,b]\}$ is also obvious. If $b' < b$ then there is a rational b'' with $b' < b'' < b$ and also with $a < b'' < b$ (by density of rationals in the reals); since $f(b'') = b'' > b'$, it can't be that b' is also an upper bound for $\{f(x) : x \in [a,b]\}$, from which it follows, as claimed, that $\sup\{f(x) : x \in [a,b]\} = b$.
   Now using that $\sup\{f(x) : x \in [a,b]\} = \sup\{x : x \in [a,b]\}$ for any interval, we conclude that for any partition P of $[0,1]$,

$$U(f,P) = U(\mathrm{id}, P)$$

   where $\mathrm{id} : [0,1] \to \mathbb{R}$ is the identity function $\mathrm{id}(x) = x$. We know that

$$\inf\{U(\mathrm{id},P) : P \text{ a partition of } [0,1]\} = \int_0^1 x\ dx = \frac{1}{2},$$

   so indeed

$$\inf\{f,P) : P \text{ a partition of } [0,1]\} = \frac{1}{2}.$$

   (c) Does $\int_0^1 f$ exist, and if so what is its value?

   Solution: The integral does not exist since the supremum of lower Darboux sums is 0, different from the infimum of upper Darboux sums (1/2).

7. (Exercise 6 from the first tutorial) Recall the "stars over Babylon" function $s : [0,1] \to \mathbb{R}$ defined by

$$s(x) = \begin{cases} 0 & \text{if } x = 0, 1, \text{ or is irrational} \\ 1/q & \text{if } x = p/q \text{ with } p, q \in \mathbb{N} \text{ with } p \text{ and } q \text{ having no factors in common.} \end{cases}$$

   Is s integrable on $[0,1]$? *Carefully* justify!

   Solution: We claim that s integrable on $[0,1]$, and that $\int_0^1 s = 0$.

For any partition P, we have $L(f,P)=0$, because the irrationals are dense (so in each interval of the partition there will an irrational, making $m_i=0$).

To complete the verification of $\int_0^1 s=0$, we need to show that for each n there is a partition P_n for which $U(f,P_n)<1/n$ (then $U(f)$, the infimum of all upper Darboux sums, is at most 0, so is equal to 0, since it is also at least as large as $L(f)$, the supremum of all the lower Darboux sums, which we've seen is 0).

So, fix $n\in\mathbb{N}$. There are only finitely many rationals between 0 and 1 with $s(x)\geq 1/(2n)$ — there are at most

$$1+2+3+\cdots+(2n-1)<4n^2$$

of them. List these as $a_1<a_2<\ldots<a_k$ where $k<4n^2$. This is a finite collection of distinct numbers, so it is possible to find numbers $l_1,u_1,\ldots,l_k,u_k$ with

$$0<l_1<a_1<u_1<l_2<a_2<u_2<\cdots<u_{k-1}<l_k<a_k<u_k<1,$$

and with

$$u_1-l_1=u_2-l_2=\vdots=u_k-l_k<\frac{1}{8n^3}$$

(what we are doing here is putting each a_j into its own little interval of length at most $1/8n^3$).

Consider the partition

$$P=(0,l_1,u_1,\ldots,l_k,u_k,1).$$

For each of the subintervals $[l_j,u_j]$ we have

$$\sup\{s(x):x\in[l_j,u_j]\}=a_j\leq\frac{1}{2},$$

and the sum of the lengths of all these intervals is at most

$$\frac{1}{8n^3}4n^2=\frac{1}{2n},$$

so the total contribution to the upper Darboux sum from these subintervals is at most

$$\frac{1}{2n}\frac{1}{2}<\frac{1}{2n}.$$

For all the remaining subintervals, we have that for each one, call it I,

$$\sup\{s(x):x\in I\}\leq\frac{1}{2n},$$

and the sum of the lengths of all these subintervals is at most 1. So the total contribution to the upper Darboux sum from these subintervals is at most

$$\frac{1}{2n}.$$

Combining, we get that the upper Darboux sum is less than

$$\frac{1}{2n}+\frac{1}{2n}=\frac{1}{n},$$

and we are done.