Math 10860, Honors Calculus 2

Homework 1 **NAME**:

Due in class Friday January 24

Instructions

Please present your answers neatly and clearly. Make use of space to increase the clarity of your presentation.

> *I strongly encourage you to leave wide margins, leave at least an inch of space at the end of each answer, and write large! Remember that the grader is older than you (by a year or two) and may already be suffering from eyestrain!*

Justify your non-obvious assertions —

> *the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.*

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like "from this it follows that", or "it is now enough to prove ..., which we now do", etc), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I'll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Class notes Sections 10.1, 10.2 and 10.3, and/or Spivak Chapter 13.

Assignment

1. Without doing any serious computations, evaluate the following integrals. You can be informal here; I'm not looking for a watertight ε-δ justification, but rather an explanation that shows me that you know what is going on with the integral, and its interpretation as an area.
   (a) $\int_{-1}^{1} x^3\sqrt{1-x^2}\ dx$
   (b) $\int_{-1}^{1} (x^5+3)\sqrt{1-x^2}\ dx$.

2. Let $f, g : [a, b] \to \mathbb{R}$ both be bounded, and let m, m^f and m^g be given by

- $m = \inf\{f(x) + g(x) : x \in [a, b]\}$
- $m^f = \inf\{f(x) : x \in [a, b]\}$
- $m^g = \inf\{g(x) : x \in [a, b]\}$.

(a) Show that $m^f + m^g \leq m$ (we used this fact in the proof that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$).

(b) Show, by way of an example, that it is possible to have $m^f + m^g < m$.

3. (a) Which functions have the property that *every* lower Darboux sum equals *every* upper Darboux sum?

(b) Which functions have the property that there is *some* lower Darboux sum that equals *some* upper Darboux sum?

(c) Which *continuous* functions have the property that all lower Darboux sums are equal?

4. (a) Suppose f is bounded and integrable on $[a, b]$, and that m is a lower bound for f on $[a, b]$ and M an upper bound. Show that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

(b) With the same hypotheses as for the last part, show that there exists a number μ, satisfying $m \leq \mu \leq M$, such that

$$\int_a^b f(x)\, dx = \mu(b-a).$$

(c) Show that if f is integrable on $[a, b]$, and if $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(d) Prove that if f and g are both integrable on $[a, b]$, and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

5. Suppose that f is weakly increasing (a.k.a non-decreasing) on $[a, b]$. The aim of this question is to show that f is integrable on $[a, b]$ *without making any assumption on the continuity or otherwise of* f.

(a) Prove that f is bounded on $[a, b]$.

(b) If $P = t_0, t_1, \ldots, t_n$ is a partition of $[a, b]$, what are $L(f, P)$ and $U(f, P)$?

(c) Suppose that P_n is the equipartition of $[a, b]$ into n subintervals (i.e., $P = (t_0, t_1, \ldots, t_n)$ with $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \cdots = t_n - t_{n-1}$). Calculate $U(f, P) - L(f, P)$ as a short, explicit expression, involving n, a and b, that doesn't involve a summation.

(d) Prove that f is integrable on $[a, b]$.

(e) Give an example of a bounded weakly increasing function on $[0,1]$ which is discontinuous at infinitely many points (such a function is still integrable, by the last part of the question).

6. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

(a) Compute $L(f,P)$ for every partition P of $[0,1]$.

(b) Find $\inf\{U(f,P) : P \text{ a partition of } [0,1]\}$.

(c) Does $\int_0^1 f$ exist, and if so what is its value?

7. (Exercise 6 from the first tutorial) Recall the "stars over Babylon" function $s : [0,1] \to \mathbb{R}$ defined by

$$s(x) = \begin{cases} 0 & \text{if } x = 0, 1, \text{ or is irrational} \\ 1/q & \text{if } x = p/q \text{ with } p, q \in \mathbb{N} \text{ with } p \text{ and } q \text{ having no factors in common.} \end{cases}$$

Is s integrable on $[0,1]$? *Carefully* justify!

An extra credit problem:

A (No extra credit for this, but it is necessary to understand this part before doing part **B**) Prove that for all $a \geq 1$, $\int_1^a \frac{dt}{t}$ exists. (It goes without saying, but I'll say it anyway: I'm expecting you to use the basic properties of the integral here, not any witchy properties of and weird and exotic functions that you might learned about in a previous life!)

B For $a, b \geq 1$, prove that

$$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$

(Notice that by part **A**, all the integrals here exist.)