# Math 10850, Honors Calculus 1

Quiz 8, Thursday November 14

Solutions

1. Using the definition of the derivative, show that the function $f$ given by $f(x) = \frac{2}{1+x^2}$ is differentiable at $x = 1$, and find $f'(1)$. (You may use familiar facts about limits, but nothing about the derivative except the definition).

   **Solution**: We have

$$\begin{aligned}\frac{f(1+h)-f(1)}{h} &= \frac{\frac{2}{1+(1+h)^2}-\frac{2}{2}}{h} \\ &= \frac{2-(1+(1+h)^2)}{h(1+(1+h)^2)} \\ &= \frac{2-(2+2h+h^2)}{h(1+(1+h)^2)} \\ &= \frac{-2h-h^2}{h(1+(1+h)^2)} \\ &= \frac{-2-h}{1+(1+h)^2}.\end{aligned}$$

   If follows that

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-2-h}{1+(1+h)^2} = \frac{-2}{2} = -1.$$

   It follows that $f'(1)$ exists and equals $-1$.

2. Show that if a function $f$ is differentiable at $a$, then it must be that $\lim_{h\to 0} f(a+h) = f(a)$ (i.e., that $f$ is continuous at $a$).

   **Solution**: Here is a solution that goes into rather more detail than is probably needed.

   Since $f$ is differentiable at $a$,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

   exists and equals some finite value $f'(a)$. But also,

$$\lim_{h\to 0} h$$

   exists and equals 0. So by the sum-product-reciprocal theorem for limits,

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)h \\ &= \left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)\left(\lim_{h\to 0} h\right) \\ &= f'(a)\cdot 0 \\ &= 0.\end{aligned}$$

It follows (by the sum-product-reciprocal theorem for limits) that

$$\begin{aligned}\lim_{h\to 0} f(a+h) &= \lim_{h\to 0}(f(a+h)-f(a))+f(a)\\ &= \lim_{h\to 0}(f(a+h)-f(a))+\lim_{h\to 0} f(a)\\ &= 0+f(a) \quad (f(a) \text{ a constant})\\ &= f(a),\end{aligned}$$

as required.