Math 10850, Honors Calculus 1

Quiz 5, Thursday October 10

Solutions

1. Give the *formal* definition of a function, and say, in terms of this definition, what are the domain and the range of a function.

   Solution: A function is a set of ordered pairs, with the property that each element a that appears as the first coordinate of a pair, appears as the first coordinate of exactly one pair. The domain is the set of first coordinates of the pairs, and the range is the set of second coordinates.

   Note that this definition of a function is not the same as "a set of ordered pairs with the property that each element a that appears as the first coordinate of a pair, appears only once" — a might be the *second* coordinate of a whole lot of pairs.

   Note also that the formal definition doesn't make any mention of vague terms such as "input" and "output".

   The definition of a function can be expressed quite succinctly: a function is a set of ordered pairs, with the property that if (a, b) and (a, c) are pairs in the set, then $b = c$. The domain is the set of all a such that (a, b) is a pair in the set for some b, and the range is the set of all b such that (a, b) is a pair in the set for some a.

2. Let $f(x) = \sqrt{x^3 - x}$. Find the domain of f — and as you do so, please make clear the process that you are using. Write your final answer using interval notation (by which I mean $(a, b]$, (c, ∞), etc. You may need to express your final answer as a union of intervals, for example $(-3, 0] \cup [2, 5]$).

   Solution: Because no domain is specified, we take as the domain the largest set of reals for which the expression $\sqrt{x^3 - x}$ makes sense. For $\sqrt{x^3 - x}$ to make sense, we require just that $x^3 - x \geq 0$ or $x^3 \geq x$.

   One solution to this inequality is $x = 0$. If $x > 0$, then dividing through by x the inequality is equivalent to $x^2 \geq 1$, which (in the regime $x > 0$) is true for all $x \geq 1$. If $x < 0$, then dividing through by x the inequality is equivalent to $x^2 \leq 1$, which (in the regime $x < 0$) is true for all x, $-1 \leq x < 0$.

   So the domain of f is $x = 0$, together with all $x \geq 1$, together with all x satisfying $-1 \leq x < 0$. Putting this in interval notation, we have

   $$\text{Domain}(f) = [-1, 0] \cup [1, \infty).$$