Math 10850, Honors Calculus 1

Quiz 3, Thursday September 19

Solutions

1. State the trichotomy axiom of positivity for real numbers (P10). I'll get things started for you: "There is a set of numbers, $\mathbb{P}$, called the *positive numbers*, that satisfies the following:

   Solution: For every number a **exactly one** of the following holds:

   (a) a is positive (or: $a \in \mathbb{P}$, or: $a \in \mathbb{R}^+$)

   (b) $-a$ is positive (or: $-a \in \mathbb{P}$, or: $-a \in \mathbb{R}^+$)

   (c) $a = 0$.

   It's ok (not good, but ok), to leave out the "for every number a"; in class it has been implicit that unless otherwise specifically stated, the universe of discourse is the set of all real numbers.

   It is **not** acceptable to replace "$-a$ is positive" with "a is negative". We *defined* the negative numbers to be those which are neither 0 nor positive, so the axiom would be circular/have no content in this case.

   It is **not** acceptable to say "for every number a either a is positive or $-a$ is positive or $a = 0$." In mathematics "or" on its own (unqualified) *always* means *inclusive* or, and the trichotomy law requires an *exclusive* or.

   The specific order in which you write the three parts of the trichotomy is, of course, irrelevant!

2. Express

   $$|(|x| - 1)|$$

   without absolute value signs, treating various cases separately where necessary. Remember that it is ok to have cases overlapping, if they agree at the point of overlap (practically this means that you can think of $|a|$ as having two clauses in its definition: one for $a \geq 0$, one for $a \leq 0$). Write your final solution using the *brace notation*, for example

   $$\text{this thing} = \begin{cases} \text{something} & \text{if condition/case 1} \\ \text{something else} & \text{if condition 2} \\ \text{something else again} & \text{if condition 3} \end{cases}$$

   Solution: Some change in the final answer will probably happen as x goes from below to above 0 (because of the $|x|$). Also some change in the final answer will probably happen as $|x|$ crosses from below to above 1 (because of the $|(|x| - 1)|$); this is the same as saying that either x crosses from below to above 1, or from below to above -1.

   This suggests 4 cases:

   - $x \leq -1$: here $|x| = -x$, and so $|x| - 1 = -x - 1$, which is positive ($x \leq -1$, so $-x \geq 1$, so $-x - 1 \geq 0$). So $||x| - 1| = -x - 1$ in this case.
   - $-1 \leq x \leq 0$: here $|x| = -x$, and so $|x| - 1 = -x - 1$, which is negative ($-1 \leq x \leq 0$, so $1 \geq -x \geq 0$, so $0 \geq -x - 1 \geq -1$). So $||x| - 1| = x + 1$ in this case.
   - $0 \leq x \leq 1$: here $|x| = x$, and so $|x| - 1 = x - 1$, which is negative ($0 \leq x \leq 1$, so $-1 \leq x - 1 \leq 0$). So $||x| - 1| = -x + 1$ in this case.
   - $x \geq 1$: here $|x| = x$, and so $|x| - 1 = x - 1$, which is positive ($x \geq 1$, so $x - 1 \geq 0$). So $||x| - 1| = x - 1$ in this case.

   In brace notation:

   $$||x| - 1| = \begin{cases} -x - 1 & \text{if } x \leq -1 \\ x + 1 & \text{if } -1 \leq x \leq 0 \\ -x + 1 & \text{if } 0 \leq x \leq 1 \\ x - 1 & \text{if } x \geq 1. \end{cases}$$