Math 10850, Honors Calculus 1

Quiz 1, Thursday September 5

Solutions

1. Let $p(x,y)$ be the predicate "$x \cdot y = 1$", where the universe of discourse for x is the natural numbers $\{1, 2, 3, \ldots\}$, the universe of discourse for y is the real numbers, and "$\cdot$" is ordinary multiplication. Which of the following statements is true, and which is false? For each one, *briefly* explain your reasoning.

   (a) $(\forall x)(\forall y)p(x,y)$.

   Solution: This is false. It asserts that for every natural number x and every real y, we have $xy = 1$; but there are many examples of pairs x, y that don't work, e.g., $x = 1$ and $y = \sqrt{\pi}/5$.

   (b) $(\forall x)(\exists y)p(x,y)$.

   Solution: This is true. It asserts that for every natural number x there is a real y with $xy = 1$ or $y = 1/x$. This is true since for each x we can take $1/x$ as the value of y that works.

   (c) $(\exists y)(\forall x)p(x,y)$.

   Solution: This is false. If says that there is a special real number y, such that for every natural number x, $xy = 1$. If y is different from 0, this is clearly nonsense, as the value of xy changes as x changes, so won't always be equal to 1; and if $y = 0$ then $xy = 0$ for all x and so is never equal to 1. (Note that to completely answer this part, you do need to make some comment about the case $y = 0$.)

2. We defined $\Leftrightarrow$ in terms of $\Rightarrow$ and $\wedge$, and we can express $\Rightarrow$ as a combination of $\vee$ and $\neg$. So:

   (a) Write down an expression involving $\wedge$, $\vee$ and $\neg$ that is equivalent to $p \Leftrightarrow q$.

   Solution: $p \Leftrightarrow q$ is equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$, and using that $A \Rightarrow B$ is equivalent to $\neg A \vee B$, this is equivalent to

   $$(\neg p \vee q) \wedge (\neg q \vee p).$$

   (b) Go further: write down an expression involving *only* $\wedge$ and $\neg$ that is equivalent to $p \Leftrightarrow q$.

   Solution: From the last part, $p \Leftrightarrow q$ is equivalent to $(\neg p \vee q) \wedge (\neg q \vee p)$. One of De Morgan's laws says that $\neg(A \vee B)$ is equivalent to $\neg A \wedge \neg B$, so $A \vee B$ is equivalent to $\neg(\neg A \wedge \neg B)$. Using this on the result of the last part we get that $p \Leftrightarrow q$ is equivalent to

   $$\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p).$$